§1. Let $\kappa > \omega$ be a regular cardinal. Consider the following combinatorial principles:

($\diamondsuit_\kappa$) There is a sequence $\langle S_\nu \mid \nu < \kappa \rangle$ s.t. $S_\alpha \subset \alpha$ $(\alpha < \kappa)$ and for every $X \subset \kappa$, the set $\{\alpha \mid X \cap \alpha = S_\alpha\}$ is Mahlo in κ.

($\diamondsuit_\kappa^+$) There is a sequence $\langle S_\nu \mid \nu < \kappa \rangle$ s.t. $S_\alpha \subset \mathbb{P}(\alpha)$, $\overline{\overline{S_\alpha}} \le \overline{\overline{\alpha}}$ $(\alpha < \kappa)$ and for every $X \subset \kappa$, the set $\{\alpha \mid X \cap \alpha \in S_\alpha\}$ is Mahlo in κ.

(Note $a \subset \kappa$ is called <u>Mahlo</u> in κ iff $a \cap c \neq \emptyset$ for every closed, unbounded $c \subset \kappa$).

<u>Note</u> The designation $\diamondsuit^+_\kappa$ was originally used for the weaker principle $\diamondsuit''_\kappa$ which ~~[illegible]~~ is like $\diamondsuit^+_\kappa$ ~~[illegible]~~ except that the clause "$C \cap \alpha \in S_\alpha$" is omitted from the last line. However, Jensen has shown that ~~$\diamondsuit''_{\omega_1}$ does not imply~~ $\diamondsuit''_{\omega_1}$ does not imply Kurepa's hypothesis, whereas $\diamondsuit^+_{\omega_1}$ does, as we shall see. It seems to us that the omission of the clause "$C \cap \alpha \in S_\alpha$" should be regarded as simply an unfortunate oversight.

<u>Theorem 1</u> (Kunen) $\diamondsuit^+_\kappa \rightarrow \diamondsuit_\kappa$.

Moreover, $\diamondsuit_\kappa \longleftrightarrow \diamondsuit'_\kappa$, where $\diamondsuit'_\kappa$ is the following principle:

($\diamondsuit'_\kappa$) There is a sequence $\langle S_{\alpha<\kappa}\rangle$ s.t. $S_\alpha \subset \mathbb{P}(\alpha)$, $\overline{\overline{S}}_\alpha \leq \overline{\overline{\alpha}}$ $(\alpha<\kappa)$ and for all $X \subset \kappa$, the set $\{\alpha \mid X\cap\alpha \in S_\alpha\}$ is Mahlo in κ.

proof.

$\diamondsuit^+_\kappa \rightarrow \diamondsuit'_\kappa$; $\diamondsuit_\kappa \rightarrow \diamondsuit'_\kappa$ are trivial.

We must prove: $\diamondsuit'_\kappa \rightarrow \diamondsuit_\kappa$.

Assume $\diamondsuit'_\kappa$. Then there is a sequence $\langle S_{\alpha<\kappa}\rangle$ s.t. $S_\alpha \subset \mathbb{P}(\alpha^2)$, $\overline{\overline{S}}_\alpha \leq \overline{\overline{\alpha}}$ $(\alpha<\kappa)$ and for all $X \subset \kappa^2$, the set

$\{\alpha \mid X \cap \alpha^2 \in S_\alpha\}$ is Mahlo in κ.

Let $\langle S_\alpha^\nu \mid \nu < \alpha \rangle$ enumerate S_α $(\alpha < \kappa)$. The following ~~[illegible]~~ combinatorial fact is well known:

(*) If $C \subset \kappa$ is Mahlo in κ and $f: C \to \kappa$ s.t. $f(\gamma) < \gamma$ for $\gamma \in C$, then $\bigvee_\nu\ f^{-1}{}''\{\nu\}$ is Mahlo in κ.

Using (*) we get:

<u>Lemma 1.1</u> Let $X \subset \kappa^2$. Then there is $\nu < \kappa$ s.t. $\{\alpha > \nu \mid X \cap \alpha^2 = S_\alpha^\nu\}$ is Mahlo in κ.

proof. Set $C = \{\alpha \mid X \cap \alpha^2 \in S_\alpha\}$. For $\alpha \in C$ set:

$$f(\alpha) = \text{the least } \nu < \alpha \text{ s.t. } X \cap \alpha^2 = S_\alpha^\nu.$$

By $(*)$, $f^{-1}``\{\nu\}$ is Mahlo in κ for some $\nu < \kappa$. QED

<u>Lemma 1.2</u> There is a $\nu_0 < \kappa$ s.t. for all $X \subset \kappa$ there is a $Y \subset \kappa^2$ s.t. $X = Y``\{\nu_0\}$ and $\{\alpha \mid Y \cap \alpha^2 = S_\alpha^{\nu_0}\}$ is Mahlo.

proof. Suppose not. Then for each $\nu < \kappa$ there is $X_\nu \subset \kappa$ s.t. for all $Y \subset \kappa^2$, if $X_\nu = Y``\{\nu\}$, then $\{\alpha \mid Y \cap \alpha^2 = S_\alpha^\nu\}$ is not Mahlo. Define $Y \subset \kappa^2$ by:

$$Y``\{\nu\} = X_\nu \quad (\nu < \kappa).$$

Then:

$$\bigwedge_\nu \ \{\alpha \mid Y \cap \alpha^2 = S_\alpha^\nu\} \text{ is not Mahlo.}$$

This contradicts Lemma 1.1.

QED (Lemma 1.2)

Now let ν_0 be as in Lemma 1.2.

Set: $\bar{S}_\alpha = S_\alpha^{\nu_0} `` \{\nu_0\}$ $(\alpha < \kappa)$.

Then:

$X \subset \kappa \rightarrow \{\alpha \mid X \cap \alpha = \bar{S}_\alpha\}$ is Mahlo.

QED

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<u>Kurepa's</u> <u>hypothesis</u>

The κ-<u>Kurepa</u> <u>hypothesis</u> (KH_κ) reads:

(KH_κ) There is a set $B \subset \mathbb{P}(\kappa)$ s.t. $\overline{\overline{B}} \geq \kappa^+$ but $\overline{\overline{B|\alpha}} \leq \bar{\bar{\alpha}}$ for $\omega \leq \alpha < \kappa$. (where $B|x =_{Df} \{b \cap x \mid b \in B\}$).

Solovay has proved that if γ is any infinite cardinal, then

$$\forall A \subset \gamma^+ \ (V = L[A] \rightarrow KH_{\gamma^+}).$$

We shall considerably extend this result. However, the basic method of proof remains Solovay's.

<u>**Theorem 2**</u> (Jensen)

$$\Diamond^+_\kappa \rightarrow KH_\kappa.$$

proof.

Let $\langle S_{\alpha<\kappa} \rangle$ be the sequence given by $\Diamond^+_\kappa$. For $\alpha < \kappa$ let M_α be a transitive model of ZF^- s.t. $\alpha \cup \{\alpha\} \subset M_\alpha$, $S_\beta \subset M_\alpha$ for $\beta \leq \alpha$ and ~~M_α max~~

Note: ZF^- is ZF without the power set axiom.

$\overline{\overline{M_\alpha}} = \max(\omega, \bar{\bar{\alpha}})$.

Set: $B = \{b \subset \kappa \mid \bigwedge \alpha < \kappa \;\; b \cap \alpha \in M_\alpha\}$.

Then $\overline{\overline{B \mid \alpha}} = \bar{\bar{\alpha}}$ for $\omega \leq \alpha < \kappa$.

<u>Claim</u> $\bar{\bar{B}} \geq \kappa^+$.

proof. Suppose not. Let $\langle b_\nu \mid \nu < \kappa \rangle$ enumerate all $b \in B$ s.t. b is unbounded in κ. We shall derive a contradiction by constructing a $c \in B$ s.t. c is unbounded in κ and $c \neq b_\nu$ for $\nu < \kappa$.

Let $a \subset \kappa$ be closed, unbounded in κ s.t. each $\alpha \in A$ is a limit ordinal and $b_\nu \cap \alpha$ is unbounded in α for $\nu < \alpha \in a$.

Let $c' \subset \kappa$ be closed, unbounded in κ s.t.

(*) $\gamma \in c' \to a \cap \gamma,\ c' \cap \gamma \in S_\gamma$.

Set: $c = a \cap c'$. Then c is closed, unbounded in κ and $c \neq b_\nu$, since, letting $\langle \gamma_{\nu < \kappa} \rangle$ be the monotone enumeration of c, $b_\nu \cap \gamma_{\nu+1}$ is unbounded in $\gamma_{\nu+1}$ but $c \cap \gamma_{\nu+1} \subset (\gamma_\nu + 1) < \gamma_{\nu+1}$.

<u>We claim</u>: $c \in B$.

Let $\alpha < \kappa$. We must show: $c \cap \alpha \in M_\alpha$. If $c \cap \alpha$ is finite, this is trivial. If not, let ~~$\gamma_\lambda = \sup(c \cap \alpha)$. Then c~~ γ_λ be the greatest limit point of $c \cap \alpha$. Then $(c \cap \alpha) \setminus \gamma_\lambda$

is finite. Hence it suffices to show: $c \cap \gamma_\lambda \in M_\alpha$.

Since $\gamma_\lambda \in C \subset c'$, we have:

$$c' \cap \gamma_\lambda \, , \, a \cap \gamma_\lambda \in S_{\gamma_\lambda} \subset M_\alpha .$$

Hence $c \cap \gamma_\lambda \in M_\alpha$, since $c \cap \gamma_\lambda = (c' \cap \gamma_\lambda) \cap (a \cap \gamma_\lambda)$ and M_α is a ZF^- model. QED

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Prikry's hypothesis

Prikry in his paper "On a problem of Keisler + Gillman" has made use of a combinatorial principle which is slightly stronger than KH. Prikry's hypothesis (PH_κ) stated for arbitrary regular κ reads:

(PH_κ) There is $F \subset \kappa^\kappa$ s.t. F dominates κ^κ but $\overline{\overline{F|\alpha}} \leq \alpha$ for $\omega \leq \alpha < \kappa$ (where $F|x = \{f \cap x^2 \mid f \in F\}$).

A slight modification of the foregoing proof yields:

<u>Theorem 3</u> (Jensen) $\Diamond^+_\kappa \to \mathrm{PH}_\kappa$

proof.

Define M_α ($\alpha < \kappa$) as before.

Set: $F = \{f \in \kappa^\kappa \mid \bigwedge \alpha < \kappa \; f \cap \alpha^2 \in M_\alpha\}$.

Then $\overline{\overline{F|\alpha}} \leq \overline{\overline{\alpha}}$ for $\omega \leq \alpha < \kappa$.

We must show that F dominates κ^κ. Let $g \in \kappa^\kappa$.

Let $a \subset \kappa$ be closed,

unbounded in κ s.t. $f \upharpoonright \alpha \in \alpha^\alpha$ for $\alpha \in a$. Let c' [redacted] be as before & set: $c = a \cap c'$ as before. Exactly as before we get: $c \cap \alpha \in M_\alpha$ for $\alpha < \kappa$.

Define $f \in \kappa^\kappa$ by:

$$f(\nu) = \min(c \setminus (\nu+1)).$$

Then f dominates g and $f \cap \alpha^2 \in M_\alpha$ for $\alpha < \kappa$. QED

Prikry shows in his paper that if PH_{ω_1} holds, then every uniform ultrafilter on $\mathcal{P}(\omega_1)$ is regular. One might hope to get the same result for ω_2. It would appear, however, PH_{ω_2} is too weak a hypothesis. The trouble is that even if $F \subset {\omega_2}^{\omega_2}$ is as in PH_{ω_2}, there may be countable $x \subset \omega_2$ s.t. $\overline{\overline{F|x}} > \omega$. This suggests the following versions of KH + PH. Let γ be a cardinal s.t. $\omega < \gamma \leq \kappa$.

($KH_{\kappa\gamma}$) There is a $B \subset \mathcal{P}(\kappa)$ s.t. $\overline{\overline{B}} \geq \kappa^+$ but $\overline{\overline{B|x}} \leq x$ for $x \subset \kappa$ s.t. $\omega \leq \overline{\overline{x}} < \gamma$.

$(PH_{\kappa\gamma})$ There is $F \subset \kappa^\kappa$ s.t. F dominates κ^κ but $\overline{\overline{F|x}} \leq \overline{\overline{x}}$ for $x \subset \kappa$ s.t. $\omega \leq \overline{\overline{x}} < \gamma$.

There is an entirely analogous modification of $\Diamond^+$, to wit:

$(\Diamond^+_{\kappa\gamma})$ There is a sequence $\langle S_x \mid x \subset \kappa, \overline{\overline{x}} < \gamma \rangle$ s.t. $S_x \subset \mathbb{P}(x)$, $\overline{\overline{S_x}} \leq \overline{\overline{x}}$ and for every $X \subset \kappa$ there is an unbounded $C \subset \kappa$ s.t. whenever $\mathrm{lub}(x)$ is a limit point of $C \cap x$, then

$$X \cap x,\ C \cap x \in S_x .$$

(Note It is not required that C be closed)

Clearly, $\diamondsuit^+_{\kappa\kappa} \to \diamondsuit^+_{\kappa}$, $KH_{\kappa\kappa} \to KH_{\kappa}$, $PH_{\kappa\kappa} \to PH_{\kappa}$. We will show that, in L, these implications may be reversed and, moreover, $\diamondsuit^+_{\kappa}$, KH_{κ}, PH_{κ} all hold for the same ordinals κ. We shall

$2^{(\kappa)} =_{df} \bigcup_{\gamma<\kappa} 2^{\gamma}$

also show: $\diamondsuit^+_{\kappa\gamma} \wedge 2^{(\kappa)} = \kappa . \to . \to KH_{\kappa}, PH_{\kappa}$. As a preliminary we prove:

<u>Theorem 4</u> (Jensen) The following are equivalent:

(i) $\diamondsuit^+_{\kappa\gamma} \wedge 2^{(\kappa)} = \kappa$

(ii) There is a sequence $\langle S_x \mid x \subset \kappa, \bar{\bar{x}} < \gamma \rangle$ s.t. $S_x \subset \mathcal{P}(\mathrm{lub}(x))$, $\bar{\bar{S}}_x \le \bar{\bar{x}}$

and for all $X \subset \kappa$ there is an unbounded $C \subset \kappa$ s.t. whenever $\alpha = \text{lub}(x)$ is a limit point of $C \cap x$, then

$$X \cap \alpha, \ C \cap \alpha \in S_x .$$

[<u>Note</u> (ii) is the formulation of $\diamondsuit^+_{\kappa\gamma}$ which will actually be used in this paper. Hence the reader who is bored with Thm 4 can go on to the later theorems, taking (ii) as the "official" version of $\diamondsuit^+_{\kappa\gamma}$.]

proof of Thm 4.

(ii) $\rightarrow$ (i) is trivial. Now assume (i). Then there is an $A \subset \kappa$ s.t. $H_\kappa = L_\kappa[A]$ (H_κ being the collection of sets hereditarily of cardinality κ).

Let $\langle S_x \mid x \subset \kappa, \bar{\bar{x}} < \gamma \rangle$ be the sequence given by $\diamondsuit^+_{\kappa\gamma}$.

We wish to define a new sequence $\bar{S}_x$ $(x \subset \kappa, \bar{\bar{x}} < \gamma)$ which satisfies (ii). For $\bar{\bar{x}} < \omega$ set: $\bar{S}_x = \emptyset$. Otherwise set:

$$M_x = \text{the smallest } \cancel{M \prec \langle L_\kappa[A],} \; M \prec \langle L_\kappa[A], \in, A \rangle \text{ s.t. } x \cup \{x\} \in M$$

$N_x =$ the smallest $N \prec \langle L_\kappa[A], \in, A\rangle$
s.t. $M_x \cup \{M_x\} \subset N$ and
$S_y \subset N$ for $y \in N \cap \mathcal{P}(M_x)$.

$\bar{S}_x = \mathcal{P}(\operatorname{lub}(x)) \cap N_x$.

(<u>Note</u> M_x, N_x are not necessarily transitive.)

~~Now let $X \subset \kappa$ be unbounded in~~

Now let $X \subset \kappa$.

<u>Claim</u> There is $D \subset \kappa$ unbounded in κ s.t. if $\alpha = \operatorname{lub}(x)$ is a limit pt. of $D \cap x$, then

$$X \cap \alpha,\ D \cap \alpha \in N_x.$$

proof. Let $B \subset \kappa$ be unbounded s.t. if $\operatorname{lub}(x)$ is a limit pt. of B, then $X \cap x, B \cap x \in S_x$.

Define $\langle \alpha_{\nu < \kappa} \rangle$ by:

$\alpha_0 = 0$; $\alpha_\lambda = \sup_{\nu < \lambda} \alpha_\nu$ for limit λ;

$\alpha_{\nu+1} =$ the least $\alpha > \alpha_\nu$ s.t.
$B \cap \alpha$ is unbounded in α
and $F_\tau^A = \langle X \cap \alpha_\nu , B \cap \alpha_\nu \rangle$
for some $\tau < \alpha$.

(Here $\langle F_\tau^A \mid \tau < \kappa \rangle$ is the Gödel enumeration of $L_\kappa[A]$).

Set:

$\beta_\nu =$ the least $\tau \geq \alpha_\nu$ s.t.
$F_\tau^A = \langle X \cap \alpha_\nu , B \cap \alpha_\nu \rangle$.

Then $\langle \alpha_{\nu < \kappa} \rangle$ is a normal function and $\alpha_\nu \leq \beta_\nu < \alpha_{\nu+1}$.

Set: $C = \{ \beta_\nu \mid \nu < \kappa \}$.

<u>Claim</u> If $\alpha = \mathrm{lub}(X)$ is a limit point of C, then $X \cap \alpha$, $C \cap \alpha \in \overline{S}_x$.

proof. Set $\overline{x} = \alpha \cap M_x$. Then $\overline{x} \in N_x$ and $S_{\overline{x}} \subset N_x$.

(a) α is a limit pt. of $B \cap \overline{x}$

proof. Let $\tau \in \overline{x}$. Let $\beta_\nu \in X \setminus \tau$ be s.t. $\alpha_\nu > \tau$. Then:

$$\alpha_\nu \cap B = (F^A_{\beta_\nu})_1 \in M_x .$$

Let τ' = the least $\tau' > \tau$ s.t. $\tau' \in \alpha_\nu \cap B$. Then τ' is M_x - definab from τ, $\alpha_\nu \cap B$. Hence:

$$\tau \leq \tau' \in B \cap \overline{x} . \qquad \text{QED (a)}$$

As an immediate corollary of (a), we have:

(b) $X \cap \overline{x}$, $B \cap \overline{x} \in S_{\overline{x}} \subset N_x$.

Using (b) we obtain:

(c) $X \cap \alpha,\ B \cap \alpha \in N_x$.

proof. We display the proof of: $X \cap \alpha \in N_x$. Set:

$$W = \{\tau \in x \mid V\rho\,(\rho = \mathrm{lub}\,(F_\tau^A)_0 \wedge \wedge (F_\tau^A)_0 = \rho \cap X)\}.$$

Since $C \cap x \subset W$, we have:

$$X \cap \alpha = \bigcup_{\tau \in W} (F_\tau^A)_0 \ .$$

Hence it suffices to show: $W \in N_x$.

Set:

$$\overline{W} = \{\tau \in x \mid V\rho\,(\rho = \mathrm{lub}\,(F_\tau^A)_0 \wedge \wedge \bar{x} \cap (F_\tau^A)_0 = \bar{x} \cap \rho \cap X\}.$$

Then $\overline{W} \in N_x$, since $\bar{x} \cap X \in N_x$.

<u>We claim</u> $\overline{W} = W$.

$W \subset \bar{W}$ is trivial. Now suppose that $\bar{W} \not\subset W$. Let $\tau \in \bar{W} \setminus W$. ~~Let~~ $\rho = \text{lub}\,(F_\tau^A)_0$. Let $\nu =$
= the least $\nu < \rho$ s.t.

$$\nu \in (F_\tau^A)_0 \not\longleftrightarrow \nu \in X .$$

Let $\beta_i \in X$ s.t. $\alpha_i \geq \rho$. Then $\nu =$
= the least $\nu < \rho$ s.t.

$$\nu \in (F_\tau^A)_0 \not\longleftrightarrow \nu \in (F_{\beta_i}^A)_0 .$$

Hence $\nu \in \bar{x}$, since ν is M_x-definable from τ, β_i.
Hence $\bar{x} \cap (F_\tau^A)_0 \neq \rho \cap \bar{x} \cap X$, hence $\tau \notin \bar{W}$. Contradiction!

QED(c)

But $C \cap \alpha$ is definable from $X \cap \alpha$, $B \cap \alpha$ exactly the way C was defined from X, B. Since

$N_x \prec \langle L_\kappa[A], \in, A \rangle$, this definition can be carried out in N_x. Hence $C \cap \alpha \in N_x$.

QED

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~~**Theorem 5**~~ $\cancel{\diamondsuit^+_{\kappa\gamma} \wedge 2^{\omega} = \kappa. \rightarrow KH_{\kappa\gamma}}$

~~proof.~~

~~For each~~ $\cancel{x \subset \kappa}$ ~~s.t.~~ $\cancel{\bar{\bar{x}} < \gamma}$ ~~let~~ $\cancel{M_x}$ ~~be the smallest~~ $\cancel{M \prec H_\kappa}$ ~~s.t.~~ $\cancel{x \cup \{x\} \in M}$ ~~and~~ $\cancel{S_{x \cap \alpha} \in M}$ ~~for~~ $\cancel{\alpha \leq \mathrm{lub}(x)}$.

~~Then~~ $\cancel{\bar{\bar{M}}_x = \max(\omega, \bar{\bar{x}})}$.

~~Set:~~ $\cancel{B = \{b \subset \kappa \mid \bigwedge x (\bar{\bar{x}} < \gamma \rightarrow b \cap x \in M_x)\}}$

~~Then~~ $\cancel{\bar{\bar{B}} \restriction x \leq \bar{\bar{x}}}$

Theorem 5 $\underbrace{}_{}$ (Jensen) $\Diamond^{+}_{\kappa\gamma} \wedge 2^{\underset{\smile}{\kappa}} = \kappa . \longrightarrow KH_{\kappa\gamma}$

proof.

Let S_x $(x \subset \kappa, \overline{\overline{x}} < \gamma)$ be as in Thm 4 (ii). Let M_x be the smallest $M \prec H_\kappa$ s.t. $x \cup \{x\} \subset M$ and $S_{x \cap \alpha} \subset M$ for $\alpha \leq \operatorname{lub}(x)$.

Then $\overline{\overline{M}}_x = \max(\omega, \overline{\overline{x}})$.

Let B be the set of $b \subset \kappa$ s.t. $b \cap x \in M_x$ for $x \subset \kappa$, $\overline{\overline{x}} < \gamma$.

$\cancel{\omega \leq \overline{\overline{x}} < \gamma}$

Then $\overline{\overline{B|x}} \leq \overline{\overline{x}}$ for $x \subset \kappa$, $\omega \leq \overline{\overline{x}} < \gamma$.

Claim $\overline{\overline{B}} \geq \kappa^{+}$.

Suppose not. Let $\langle b_{\nu<\kappa}\rangle$ ennumerate all $b\subset\kappa$ s.t. $b\in B$ and b is unbounded in κ. We shall derive a contradiction by constructing a $c\in B$ s.t. c is unbounded in κ and $c\neq b_\nu$ for $\nu<\kappa$.

Let $a\subset\kappa$ be closed, unbounded in κ s.t., for each $\alpha\in a$, α is a limit ordinal and $b_\nu\cap\alpha$ is unbounded in α for $\nu<\alpha$

Let c' be unbounded in κ s.t.

(*) If $\beta=\text{lub}(x)$ is a limit point of $x\cap c'$, then $a\cap\beta$, $c'\cap\beta\in S_x$.

Let $\langle\alpha_{\nu<\kappa}\rangle$ ennumerate the $\alpha\in a$ s.t. α is a limit pt. of c'.

Set: $\beta_\nu = \min(c' \setminus \alpha_\nu)$. Then $\alpha_\nu \leq \beta_\nu < \alpha_{\nu+1}$. Set:

$$c = \{\beta_\nu \mid \nu < \kappa\}.$$

Then c is unbounded in κ and $c \neq b_\nu$, since $b_\nu \cap \alpha_{\nu+1}$ is unbounded in $\alpha_{\nu+1}$, whereas $c \cap \alpha_{\nu+1} \subset (\beta_\nu + 1) < \alpha_{\nu+1}$.

<u>Claim</u> $c \in \mathbb{B}$.

proof. Let $x \subset \kappa$, $\overline{\overline{x}} < \gamma$.
We must show: $c \wedge x \in M_x$.
If $c \cap x$ is finite, this is trivial.
If not, let β be the maximal limit pt. of $c \cap x$. It suffices to show: $c \wedge x \wedge \beta \in M$, since $c \cap x \setminus \beta$ is finite.

Since $\beta = \text{lub}(x \cap \beta)$ is a limit pt. of $c' \cap x \cap \beta$, we have:

$$c' \cap \beta,\ a \cap \beta \in S_{x \cap \beta} \subset M_x .$$

But ~~c' is defin~~ $c \cap \beta$ is definable from $c' \cap \beta$, $a \cap \beta$ the way c was defined from c', a. Since M_x is a ZF^- model, this definition can be carried out in M_x. Hence $c \cap \beta \in M_x$. QED

<u>**Theorem 6**</u> (Jensen) ~~$\diamondsuit^+_{\kappa\gamma} \wedge 2^{\kappa} = \kappa \rightarrow$~~

$$\diamondsuit^+_{\kappa\gamma} \wedge 2^{\underset{\smile}{\kappa}} = \kappa . \rightarrow PH_{\kappa\gamma}$$

proof. Modify the proof of Thm 5 the way the proof of Thm 2 was modified to give Thm 3.

We turn now to the problem considered by Prikry in his paper "On a problem of Gillman and Keisler".

<u>Def</u> Let κ be regular and let U be a uniform ultrafilter on $\mathbb{P}(\kappa)$ (i.e. $X \in U \to \overline{\overline{X}} = \kappa$).

U is γ-<u>regular</u> ($\omega \leq \gamma \leq \kappa$) iff there is a sequence $\langle A_\nu \mid \nu < \kappa \rangle$ s.t. $A_\nu \in U$ and for all $a \subset \kappa$: $\overline{\overline{a}} \geq \gamma \to \bigcap_{\nu \in a} A_\nu = \emptyset$.

U is <u>regular</u> iff U is ω-regular.

The following equivalence is obvious:

U is γ-regular iff there is a map $h : \kappa \longrightarrow \{u \subset \kappa \mid \bar{\bar{u}} < \gamma\}$ s.t. $\bigwedge \nu \; \{\tau \mid \nu \in h(\tau)\} \in U$.

Prikry proves: $PH_{\omega_1} \longrightarrow$
$\longrightarrow$ every uniform ultrafilter on $\mathfrak{P}(\omega_1)$ is regular.

However, though he does not explicitely state it, his main argument establishes a more general result:

Theorem 7 (Prikry) Let U be a uniform ultrafilter on κ which is γ^+-regular ($\omega \leq \gamma < \kappa$). Let $PH_{\kappa\gamma^+}$ hold. Then U is γ-regular.

proof.

Lemma 7.1. Let $F \subset \kappa^\kappa$ be s.t. $\overline{\overline{F|x}} \leq \gamma$ for $x \subset \kappa$, $\overline{\overline{x}} \leq \gamma$.

Let $\Gamma = \{x \subset \kappa \mid \overline{\overline{x}} = \gamma\}$. Then there is a sequence $\langle B_{\nu\tau} \mid \nu, \tau < \kappa \rangle$ s.t. $B_{\nu\tau} \subset \Gamma$ and:

(a) $\bigcap_{\tau \in s} B_{\nu\tau} = \emptyset$ for $s \in \Gamma$

(b) If $s \in \Gamma$ and $f \in F$, then

$$\bigcup_{\nu \in s} B_{\nu f(\nu)} \supset \{x \in \Gamma \mid s \cup f''s \subset x\}.$$

proof of Lemma 7.1.

For $x \in \Gamma$ let $\langle x_{\nu < \gamma} \rangle$ be a 1-1 ennumeration of x and let $\langle f^x_{\nu < \gamma} \rangle$ be an ennumeration of $F|x$. Set:

$$B_{\nu\tau} = \{ x \in \Gamma \mid \bigvee i < \gamma \bigvee j < i \, (\nu = x_i \wedge \tau = f^x_j(\nu)) \}.$$

We first ~~prove~~ prove (a).

Let $x, s \in \Gamma$.

<u>Claim</u> $x \notin \bigcap_{\tau \in s} B_{\nu\tau}$.

If $\nu \notin x$, this is trivial.

Otherwise, let $\nu = x_{i_\nu}$.

Then $\overline{\overline{\{ f^x_j(\nu) \mid j < i_\nu \}}} < \gamma$.

Since $\overline{\overline{s}} = \gamma$, there is $\tau \in s$ s.t. $\tau \neq f_j^x(\nu)$ for $j < i_\nu$. Hence $x \notin B_{\nu\tau}$.

QED (a).

We now turn to (b).

Let $s \in \Gamma$, $f \in F$. Let $x \in \Gamma$ s.t. $s \cup f''s \subset x$. We claim:

$$x \in \bigcup_{\nu \in s} B_{\nu f(\nu)} .$$

Let $j < \gamma$ s.t. $f \cap x^2 = f_j^x$.

For $\nu \in s$ define $i_\nu < \gamma$ by: $\nu = x_{i_\nu}$. Since $\overline{\overline{s}} = \gamma$, there is a $\nu \in s$ s.t. $i_\nu > j$. Hence $x \in B_{\nu f(\nu)}$ since:

$$\nu = x_{i_\nu} \ ; \ j < i_\nu \ ; \ f(\nu) = f_j^x(\nu).$$

QED (Lemma 7.1)

By $PH_{\kappa\gamma^+}$, we may select F s.t. F dominates κ^κ.

Since U is γ^+-regular, there is $h : \kappa \longrightarrow \Gamma$ s.t.

$$A\nu \{\tau \mid \nu \in h(\tau)\} \in U.$$

Set: $\tilde{B}_{\nu\tau} = \{\rho \mid h(\rho) \in B_{\nu\tau}\}$

$$X_\nu = \{\tau \mid \tilde{B}_{\nu\tau} \in U\}.$$

We consider two cases:

<u>Case 1</u> $\overline{\overline{X}}_\nu = \kappa$ for some ν.

Let $\langle \tau_i \mid i < \kappa \rangle$ be the monotone enumeration of X_ν. Set:

$$A_i = \tilde{B}_{\nu\tau_i}.$$

Then $A_i \in U$ and $\bigcap_{i \in s} A_i = \emptyset$ for $s \in \Gamma$ by (a).

<u>Case 2</u> $\bar{\bar{X}}_\nu < \kappa$ for $\nu < \mu$.

Then there is $f \in F$ s.t.

$$\bigwedge_{\nu < \kappa} f(\nu) > \sup X_\nu \quad .$$

Set :

$$A_\nu = \{\rho \mid \nu, f(\nu) \in h(\rho)\} \setminus \tilde{B}_{\nu\, f(\nu)} \quad .$$

Then $A_\nu \in \mathcal{U}$ and, if $s \in \Gamma$,

$$\bigcap_{\nu \in s} A_\nu = \{\rho \mid s \cup f''s \subset h(\rho)\} \setminus \bigcup_{\nu \in s} \tilde{B}_{\nu\, f(\nu)}$$

$$= \emptyset \text{ by (b).} \qquad \text{QED}$$

<u>Corollary 8</u> $PH_{\omega_n \omega_n} \longrightarrow$ Every uniform ultrafilter on $\mathbb{P}(\omega_n)$ is regular ~~...~~ $(n < \omega)$.