

Correction to §1 of T Robust Extenders

We gave a seriously flawed proof of Thm 4, which stated that the 1-small robust K^c model is universal if it is iterable. We also prove the same fact for the full robust K^c , assuming it to exist. An fact we shall show:

Thm A Let $K = K^c$ be the full robust K^c -model. Let P be a premouse.

There is no coiteration of P, K of length ∞ .

Thm 4 will then be a corollary of the proof of Thm A. Thm A follows from:

Thm B Let K, P be as above. Let $\bar{\theta} < \theta = (2^\beta)^+$, where $\beta \geq 2^\omega$. There is no coiteration of P, K of length $\theta + 1$.

We prove this in ZFC.

Suppose not. Set $Q = K \parallel \theta$. Then P, Q have a coiteration $\langle y^P, y^Q \rangle$ of length $\theta + 1$, where

$$y^P = \langle \langle P_i \rangle, \langle v_i \rangle, \langle \pi_{i1}^P \rangle, T^P \rangle$$

$$y^Q = \langle \langle Q_i \rangle, \langle v_i \rangle, \langle \pi_{i1}^Q \rangle, T^Q \rangle$$

Let $\Omega > \bar{H}_\theta$ be regular. Pick $X < H_\Omega$
 s.t. $y^P, y^Q \in X$, $\bar{X} < \theta$, $\bar{\theta} = X \cap \theta$ is
 transitive, $[X]^\omega \subset X$, and $V_{\omega+2} \subset X$.

(This is obviously possible by our
 assumptions on θ .) Let $\sigma: \bar{H} \xrightarrow{\sim} X$,
 where \bar{H} is transitive. Then $\sigma: \bar{H} < H$,

$\bar{\theta} = \text{crit}(\sigma)$, $\sigma(\bar{\theta}) = \theta$. Let

$\sigma(\bar{y}^P) = y^P$, $\sigma(\bar{y}^Q) = y^Q$, where

$$\bar{y}^P = \langle \langle \bar{p}_i \rangle, \langle \bar{v}_i \rangle, \langle \bar{\pi}_{i_1}^P \rangle, \bar{T}^P \rangle$$

$$\bar{y}^Q = \langle \langle \bar{q}_i \rangle, \langle \bar{v}_i \rangle, \langle \bar{\pi}_{i_1}^Q \rangle, \bar{T}^Q \rangle.$$

Set $H_{\bar{\theta}} = \text{df } (H_{\bar{\theta}}) \bar{H}$. Then $\sigma \upharpoonright H_{\bar{\theta}} = \text{id}$.

On both sides of the coiteration we
 have:

(1) (a) $i \leq_{\bar{T}} \bar{\theta} \iff i \leq_T \theta$ for $i < \hat{\theta}$

(b) $i \leq_{\bar{T}} i'$ $\iff i \leq_T i'$ for $i, i' < \bar{\theta}$,

since $\sigma \upharpoonright H_{\bar{\theta}} = \text{id}$, $\sigma(\bar{\theta}) = \theta$

(c) $\bar{\theta} \leq_T \theta$, since $\bar{\theta}$ is a limit point
 of the branch $\{i \mid i \leq_T \theta\}$.

There are at most finitely many
 truncations on the branch $\{i \mid i \leq_T \theta\}$,
 hence the last truncation lies below
 $\bar{\theta}$. Hence:

(d) There is $i_0 < \bar{\theta}$ s.t. $\bar{\pi}_{i_1}$ is a total
 fun on $P_{i_1} (Q_{i_1})$ for $i_0 < i \leq_T i' \leq_T \theta$,

(e) $\bar{\pi}_{i_1} = \sigma^{-1}(\bar{\pi}_{i_1})$ is a total function
 on $\bar{P}_{i_1} (\bar{Q}_{i_1})$ for $i_0 < i \leq_T i' \leq \bar{\theta}$.

Clearly $P_i \in H_\theta$ for $i < \theta$ and $\lambda_i \leq \text{ht}(P_i)$.

Hence $\theta = \sup_{i < \theta} \lambda_i$ and $\bar{\theta} = \sup_{i < \bar{\theta}} \bar{\lambda}_i$,

where $\bar{\lambda}_i = \sigma^{-1}(\lambda_i) = \lambda_i$ for $i < \bar{\theta}$. We

then get:

$$(f) \bar{\theta} \leq \text{crit}(\pi_{\bar{\theta}\theta})$$

proof.

Note that $E_{\lambda_i}^P \neq \emptyset \vee E_{\lambda_i}^Q \neq \emptyset$ for all $i < \theta$. Hence on at least one side of the coiteration we have:

(*) $E_{\lambda_i} \neq \emptyset$ for arbitrarily large $i \leq_T \theta$.

But then $E_{\lambda_i} \neq \emptyset$ for arb. large $i < \bar{\theta}$.

Let $j \geq \bar{\theta}$ be least st. $\bar{\theta} \leq_T j+1 \leq_T \theta$,

and $E_j \neq \emptyset$. Then $\kappa_j = \text{crit}(\pi_{\bar{\theta}\theta})$

and $\kappa_j \geq \sup_{i < \bar{\theta}} \lambda_i = \bar{\theta}$. If (*)

does not hold, then $\pi_{i\theta} = \text{id}$ for

suff. large $i \leq_T \theta$. But then the

last place where $\pi_{i\theta} \neq \text{id}$ must

lie below $\bar{\theta}$ and $\pi_{\bar{\theta}\theta} = \text{id}$.

QED (f)

Now set: $\tau = \bar{\theta} + J_{\lambda_{\bar{\theta}}}^E$. (Note that

$$(J_{\lambda_h}^E)^{P_l} = (J_{\lambda_h}^E)^{Q_l} \text{ for all } l \geq h.)$$

Then:

$$(g) \tau = \bar{\theta} + P_{\bar{\theta}} = \bar{\theta} + Q_{\bar{\theta}}$$

prf. Suppose not.

We first note that (*) holds on the P-side, since otherwise $P_i = P_{i_0}$ for $i_0 < i \leq_T \bar{\theta}$, where $i_0 < \bar{\theta}$. Hence

$\lambda_i \in P_{i_0}$ for $i < \bar{\theta}$. Contr! Let j

again be least s.t. $\bar{\theta} \leq_T j+1 \leq_T \bar{\theta}$ and $E_{\nu_j} \neq \emptyset$ on the P-side. Let $\bar{z} = T(j+1)$.

Then \bar{z} is the least $\bar{z} \geq \bar{\theta}$ s.t. $E_{\nu_{\bar{z}}} \neq \emptyset$,

$P_{\bar{z}} = P_{\bar{\theta}}$ and $\kappa_j < \lambda_{\bar{z}}$. Since

$\bigcup_{\lambda_{\bar{z}}} E = P_{\bar{\theta}} \parallel \lambda_{\bar{z}}$ and $\bar{\theta} + P_{\bar{\theta}} > \tau$, there

is a subset of $\bar{\theta} \leq \kappa_j$ in $P_{\bar{\theta}} \setminus \bigcup_{\lambda_{\bar{z}}} E$.

Hence a truncation occurs at $j+1$, violating the fact that all truncations on the main branch occurred below $\bar{\theta}$. Contr! At (*) holds

on the Q-side the same argument goes through. Now let (*) fail on

the Q-side. Then $Q_{\theta} = Q_{\bar{\theta}}$,

$\text{ht}(Q_{\theta}) = \theta$, and J is a segment

of Q_{θ} for $i < \theta$. Hence

$$Q_{\bar{\theta}} = Q_{\theta} = \bigcup_{i < \theta} J_{\lambda_i}^E. \text{ Hence}$$

$$\tau = \bar{\theta} + Q_{\bar{\theta}}.$$

QED (g)

We now look more closely at the P -side of the coiteration.

Since $P_i \in H_\theta$ for $i < \theta$, we have $\bar{P}_i \in H_{\bar{\theta}}$ for $i < \bar{\theta}$. But $\sigma \upharpoonright H_{\bar{\theta}} = \text{id}$. Hence:

- (2) (a) $\bar{P}_i = \sigma(\bar{P}_i) = P_i$ for $i < \bar{\theta}$. Similarly:
 (b) $\bar{\pi}_{i\bar{i}}^P = \pi_{i\bar{i}}^P$ for $i \leq_T i' < \bar{\theta}$.

But then:

(c) $\bar{P}_{\bar{\theta}} = P_{\bar{\theta}}$, $\bar{\pi}_{i\bar{\theta}}^P = \pi_{i\bar{\theta}}^P$ for $i < \bar{\theta}$,
 since $\langle \bar{P}_{\bar{\theta}}, \langle \bar{\pi}_{i\bar{\theta}}^P \mid i \leq_T i' < \bar{\theta} \rangle \rangle =$
 $= \text{lim}(\langle P_i \mid i \leq_T i' < \bar{\theta} \rangle, \langle \pi_{i\bar{i}}^P \mid i \leq_T i' < \bar{\theta} \rangle)$
 $= \langle P_{\bar{\theta}}, \langle \pi_{i\bar{\theta}}^P \mid i < \bar{\theta} \rangle \rangle$ by (a), (b) and
 (1) (a), (b). QED(c)

Hence:

(d) $\sigma \upharpoonright P_{\bar{\theta}} = \pi_{\bar{\theta}\bar{\theta}}^P$,
 since, letting $x \in P_{\bar{\theta}}$, $x = \pi_{i\bar{\theta}}^P(x')$, $i < \bar{\theta}$,
 we have: $\sigma(x) = \sigma(\pi_{i\bar{\theta}}^P(x')) = \pi_{i\bar{\theta}}^P(x') = \pi_{\bar{\theta}\bar{\theta}}^P(x)$,
 QED(2)(c).

We then conclude:

(3) $\bar{\theta} \in P_{\bar{\theta}}$

pf. Suppose not.

Clearly $\bar{\theta} = \sup_{i < \bar{\theta}} \lambda_i \leq \text{ht}(P_{\bar{\theta}})$. Hence
 $\text{ht}(P_{\bar{\theta}}) = \bar{\theta}$, where $J_{\bar{\theta}}^E P_{\bar{\theta}} = J_{\bar{\theta}}^E \mathcal{Q}_{\bar{\theta}}$. Since
 $\bar{\theta}$ is a limit of cardinals in $P_{\bar{\theta}}$, $\mathcal{Q}_{\bar{\theta}}$,

it cannot index an extender in these models. Hence $P_{\bar{\theta}} = Q_{\bar{\theta}} \parallel \bar{\theta}$ and the coiteration terminates at $\bar{\theta}$. Conts!

QED(3)

We now attempt to apply this analysis to the Q -side, bearing in mind that

$ht(Q) = \theta$. Since $Q_i \subset H_{\theta}$ for $i < \theta$ we certainly have: $\bar{Q}_i \subset H_{\bar{\theta}}$ for $i < \theta$.

Since $\sigma \upharpoonright H_{\bar{\theta}} = id$ we have: $\bar{Q}_i \parallel \bar{\theta} = \sigma(\bar{Q}_i \parallel \bar{\theta}) = Q_i \parallel \bar{\theta}$ for $i, \bar{\theta} < \theta$. Hence

$J_{\bar{\theta}}^{Q_i} = J_{\bar{\theta}}^{\bar{Q}_i}$ for $i < \theta$. Since $\bar{\theta}$ is a limit of cardinals in both Q_i and \bar{Q}_i , we conclude:

(4) (a) $\bar{Q}_i = Q_i \parallel \bar{\theta}$ for $i < \bar{\theta}$. Similarly

(b) $\bar{\pi}_{i_j}^Q = \pi_{i_j}^Q \upharpoonright \bar{Q}_i$ for $i, j < \bar{\theta}$

Set: $\tilde{\theta} = ht(\bar{Q}_{\bar{\theta}})$. Then $\tilde{\theta} = \bigcup_{i \leq \bar{\theta}} \pi_{i\bar{\theta}}^{\bar{Q}_i}$.

(c) $\bar{Q}_{\bar{\theta}} = Q_{\bar{\theta}} \parallel \tilde{\theta}$ and $\pi_{i\bar{\theta}}^{\bar{Q}_i} = \pi_{i\bar{\theta}}^Q \upharpoonright \bar{Q}_i$

for $i \leq \bar{\theta}$, since

$$\begin{aligned} & \langle \bar{Q}_{\bar{\theta}}, \langle \bar{\pi}_{i\bar{\theta}} \mid i \leq \bar{\theta} \rangle \rangle = \\ & = \lim \langle \langle Q_i \parallel \bar{\theta} \mid i \leq \bar{\theta} \rangle, \langle \pi_{i_j} \upharpoonright (Q_i \parallel \bar{\theta}) \mid i \leq \bar{\theta}, j \leq \bar{\theta} \rangle \rangle = \\ & = \langle Q_{\bar{\theta}} \parallel \tilde{\theta}, \langle \pi_{i\bar{\theta}} \upharpoonright (Q_i \parallel \bar{\theta}) \mid i \leq \bar{\theta} \rangle \rangle = \\ & = \langle Q_{\bar{\theta}} \parallel \tilde{\theta}, \langle \pi_{i\bar{\theta}} \upharpoonright \bar{Q}_i \mid i \leq \bar{\theta} \rangle \rangle \end{aligned}$$

Exactly as in (2)(d) we conclude:

$$(d) \sigma \upharpoonright \bar{Q}_{\bar{\theta}} = \bar{\pi}_{\bar{\theta}}^Q \upharpoonright \bar{Q}_{\bar{\theta}}.$$

Using this we prove:

(5) There is no truncation on the main branch $b^Q = \{i \mid i \leq_{T^Q} \theta\}$ of γ^Q ,

prf. Suppose not,

The last truncation occurs below $\bar{\theta}$. Hence by (4)(a)-(c) we know that for sufficiently large $i \leq_{T^Q} \bar{\theta}$ we have $\bar{Q}_i = Q_i$ and $\bar{\pi}_{i'}^Q = \pi_{i'}^Q$ for $i \leq_{T^Q} i' \leq_{T^Q} \bar{\theta}$. It then follows

$$\text{exactly as in (3) that } \bar{\theta} \in Q_{\bar{\theta}}. \text{ By (1)(g) } \#(\bar{\theta}) \cap P_{\bar{\theta}} = \#(\bar{\theta}) \cap Q_{\bar{\theta}} = \#(\bar{\theta}) \cap \bar{Q}_{\bar{\theta}}.$$

Using (2)(d) and (4)(d) we then imitate the classical proof that the coiteration of two mice terminates; let $j = j_p^i$ be least

$$\text{s.t. } \bar{\theta} \leq_{T^P} j+1 < \theta \text{ and } E_{j+1}^P \neq \emptyset. \text{ Then}$$

$$E_{j+1}^P(x) = \sigma(x) \upharpoonright \lambda_{j+1}^P \text{ by (2)(d). Similarly}$$

$$\text{if } j_Q \text{ has the same definition on the } Q\text{-side, then } E_{j+1}^Q(x) = \sigma(x) \upharpoonright \lambda_{j+1}^Q \text{ by}$$

(4)(d). The usual proof then shows $j_p = j_Q = j$; hence $E_{j+1}^Q = E_{j+1}^P$, which is impossible in a coiteration. QED(5)

Carrying the argumentation of (5) a bit further, we get:

$$(6) \bar{\theta} = \tilde{\theta}$$

prf. Suppose not.

Then $\bar{\theta} \in \bar{Q}_{\bar{\theta}}$ and $\bar{\theta} = \text{crit}(\pi_{\bar{\theta}\bar{\theta}}^{\alpha})$ by (4)(d).

Clearly $\tilde{\theta} = \bigcup_{i < \bar{\theta}} \pi_{i\bar{\theta}}^{-1}(\bar{\theta}) = \pi_{j\bar{\theta}}^{-1}(\bar{\theta})$ for $j < \bar{\theta}$.

Recall that $\tau = \bar{\theta} + P_{\bar{\theta}} = \bar{\theta} + Q_{\bar{\theta}}$ and that $\int_{\tau}^{P_{\bar{\theta}}} = \int_{\tau}^{Q_{\bar{\theta}}}$. Hence it suffices to show:

Claim $\tau \leq \tilde{\theta}$,

since then $\#(\bar{\theta}) \cap P_{\bar{\theta}} = \#(\bar{\theta}) \cap Q_{\bar{\theta}} = \#(\bar{\theta}) \cap \bar{Q}_{\bar{\theta}}$

and we can use (2)(d), (4)(d) to get a contradiction just as before.

Clearly the claim holds if $\tilde{\theta}$ is a cardinal in $Q_{\bar{\theta}}$, so suppose not. Then $\bar{\theta} = \pi_{0\bar{\theta}}^{-1}(\tilde{\theta})$ is not a cardinal in $K \parallel \bar{\theta} = Q_0$. Let α be the largest $\alpha < \bar{\theta}$ which is a cardinal in $K \parallel \bar{\theta}$. Then $\pi_{0\bar{\theta}}(\alpha)$ is the largest cardinal in $Q_{\bar{\theta}}$. Hence $\tau \leq \pi_{0\bar{\theta}}(\alpha) < \tilde{\theta}$, since τ is a cardinal in $Q_{\bar{\theta}}$.

Q.E.D. (6)

The following key observation is due to Mitchell and Schindler ([MSch]):

(7) $\mathcal{P}(\bar{\theta}) \cap \kappa \subset \bar{H}$

prf.

Let $X \in \mathcal{P}(\bar{\theta}) \cap \kappa$. Then $X \in \kappa \parallel \bar{\theta} = \mathcal{Q}_0$. Let $Y = \pi_{\bar{\theta}}(X)$. Then $Y \in J_{\bar{\theta}}^{E^{\mathcal{Q}_0}} = J_{\bar{\theta}}^{E^{\mathcal{P}_0}} \subset \bar{H}$

and $X = \pi_{\bar{\theta}}^{-1} \circ Y = \pi_{\bar{\theta}}^{-1} \circ Y$,

since $\pi_{\bar{\theta}} \upharpoonright \bar{\theta} = \pi_{\bar{\theta}} \upharpoonright \bar{\theta}$. QED (7)

It follows of course that $\kappa \parallel \alpha \in \bar{H}$ for $\alpha < \bar{\theta}$, since $\kappa \parallel \alpha$ is coded by an $X \in \mathcal{P}(\bar{\theta}) \cap \kappa$. Pursuing this line of reasoning we in fact get:

(8) Let $\pi_{\bar{\theta}}(\bar{\sigma}) = \bar{\sigma}$. Then $\kappa \parallel \bar{\sigma} \in \bar{H}$ and is \bar{H} -definable in the parameter $\kappa \parallel \bar{\theta}$.

proof.

Set $Z =$ the set of $\alpha > \bar{\theta}$ s.t. $\rho_{\kappa \parallel \alpha}^{\omega} = \bar{\theta}$.

Then Z is unbounded in $\bar{\sigma}$. Set;

$U = \langle \kappa \parallel \alpha \mid \alpha \in Z \rangle$. Since $\kappa \parallel \bar{\sigma} = \bigcup \text{rng}(U)$ and U is a function on a subset of $\bar{\sigma} \in \bar{H}$, it suffices to show:

Claim $U =$ the set of $\langle N, \alpha \rangle \in \bar{H}$

s.t. $\alpha > \bar{\theta}$, N is a core premouse,

$\alpha = \text{ht}(N)$, $N \parallel \bar{\theta} = \kappa \parallel \bar{\theta}$ and N

obeys the following condensation principle wrt $\kappa \parallel \bar{\theta}$:

□

(*) Let $\pi: \bar{N} \prec N$, $\text{crit}(\pi) \in \bar{\theta}$ and $\bar{\alpha} = \text{ht}(\bar{N})$. Then $\bar{N} = \text{Kl} \bar{\alpha}$.
 (Note such π , if it exists, lies in \bar{H} , since it is obtained by collapsing $H^N(\delta) =$ the set of $x \in N$ which are N -definable in parameters $< \delta$.)

(C) If $\alpha \in \bar{Z}$, then $\langle \text{Kl} \alpha, \alpha \rangle \in U$, since $\text{Kl} \alpha$ is a mouse & hence satisfies (*) by the condensation lemma for mice (noting that $\bar{\theta}$ is a limit cardinal in $\text{Kl} \alpha$), and, as remarked, $\text{Kl} \alpha \in \bar{H}$.

(D) Let $\langle N, \alpha \rangle \in U$. Set $\langle N', \alpha' \rangle = \sigma \langle N, \alpha \rangle$. Then $N' \parallel \theta = \text{Kl} \theta$ and N' has the condensation property (*) wrt. $\text{Kl} \theta$. But $\sigma \upharpoonright N: N \prec N'$ and $\bar{\theta} = \text{crit}(\sigma \upharpoonright N)$. Hence $N = \text{Kl} \alpha$ and $\langle N, \alpha \rangle \in \bar{Z}$. QED (8)

(9) $\pi_{\bar{\theta}}^{\bar{\alpha}} \upharpoonright (\text{Kl} \bar{\alpha}) \in \bar{H}$ is \bar{H} -definable in the parameters $\text{Kl} \bar{\theta}, \bar{\pi}_{\bar{\theta}}^{\bar{\alpha}}$.
 proof.

For $\alpha \in \bar{Z}$ set: $\pi_{\alpha} = \pi_{\bar{\theta}}^{\bar{\alpha}} \upharpoonright (\text{Kl} \alpha)$,

$N_{\alpha} = \pi_{\bar{\theta}}^{\bar{\alpha}} (\text{Kl} \alpha) = Q_{\bar{\theta}} \parallel \alpha' = P_{\bar{\theta}} \parallel \alpha'$,

where $\pi_{\bar{\theta}}^{\bar{\alpha}}(\alpha) = \alpha'$. Then $N_{\alpha} \in \bar{H}$

and $\pi_{\alpha} \in \bar{H}$, since π_{α} is

obtained by collapsing the N_α -definable hull of $\text{rng}(\pi_{\alpha, \bar{\theta}})$.

It suffices to show:

$\langle \langle N_\alpha, \pi_\alpha \rangle \mid \alpha \in \mathbb{Z} \rangle =$ the set of tuples $\langle N, \pi, \alpha \rangle$

and $\pi : \text{Kl}(\alpha) \rightarrow N$ and $\pi \upharpoonright (\text{Kl}(\bar{\theta})) = \overline{\pi_{\alpha, \bar{\theta}}}$ and $\pi(\bar{\theta}) = \bar{\theta}$.

proof.

(C) follows by the above remarks.

(D) Let $\langle N_\alpha, \pi_\alpha \rangle$ be as above.

Claim $N_\alpha = \overline{\pi_{\alpha, \bar{\theta}}(\text{Kl}(\alpha))}$, $\pi_\alpha = \overline{\pi_{\alpha, \bar{\theta}} \upharpoonright (\text{Kl}(\alpha))}$.

Let f be a $\text{Kl}(\alpha)$ -definable map of $\text{Kl}(\alpha)$ onto $\text{Kl}(\alpha)$. Let f' have the same def. over N_α and let \tilde{f} have the same def. over $\tilde{N} = \overline{\pi_{\alpha, \bar{\theta}}(\text{Kl}(\alpha))}$.

Let $\xi_1, \dots, \xi_m < \bar{\theta}$. Pick $\delta < \bar{\theta}$ s.t. $\xi_1, \dots, \xi_m < \delta$. Let φ be a 1-st order formula in the language of $\text{Kl}(\alpha)$. Set:

$$a = \{ \langle \gamma_1, \dots, \gamma_m \rangle \in \delta^m \mid \text{Kl}(\alpha) \models \varphi[\vec{\gamma}] \}$$

Then $a \in \text{Kl}(\bar{\theta})$ and:

$$N_\alpha \models \varphi[f'(\xi_1), \dots, f'(\xi_m)] \iff$$

$$\iff \langle \vec{\xi} \rangle \in \overline{\pi_{\alpha, \bar{\theta}}(a)} \iff \tilde{N} \models \varphi[\tilde{f}(\vec{\xi})]$$

Hence there is an isomorphism $k: N_\alpha \xrightarrow{\sim} \tilde{N}$ defined by $k(f'(\xi)) = \tilde{f}(\xi)$. Hence $N_\alpha = \tilde{N}$. But $\pi(f(\xi)) = \tilde{f}(\pi_{\bar{\theta}}(\xi)) = \pi_{\bar{\theta}}(f(\xi))$. Hence $\pi = \pi_{\bar{\theta}} \upharpoonright (K \cap \alpha)$.

QED (9)

(10) $\bar{H} \models cf(\bar{\tau}) < \bar{\theta}$.

proof.

$\pi_{\bar{\theta}}^{\bar{\theta}}$ takes $\bar{\tau}$ cofinally to τ since the iteration is by Σ_0 ultraproducts on the main branch and $\bar{\tau}$ is a successor cardinal in K . Since $\pi_{\bar{\theta}}^{\bar{\theta}} \upharpoonright \bar{H} \in \bar{H}$ it suffices to show that $cf(\tau) < \bar{\theta}$ in \bar{H} . Since $\sigma: \bar{H} \prec H_\Omega$, it suffices to show that $cf(\tau^*) < \theta$, where $\tau^* = \sigma(\tau)$.

For this it suffices to show:

Claim 1 σ takes τ cofinally to τ^* ,

since then $cf(\tau^*) \leq \tau < \theta$.

We use the fact that $\pi_{\bar{\theta}}^{\bar{\theta}} \upharpoonright P_{\bar{\theta}} = \sigma \upharpoonright P_{\bar{\theta}}$.

There is a $i_0 \leq \bar{\tau}$ s.t. no truncation occurs on the main branch above i_0 .

Hence π_{i_j} is total on P_i for $i_0 \leq i \leq \bar{\tau}$.

Let $\gamma =$ the maximal $\gamma \leq \omega$ s.t.

$\pi_i < P_{P_i}^\gamma$ for all $i \in [i_0, \bar{\tau})$.

Subclaim 2 ~~At~~ $m > \gamma$, then $\kappa_i > \rho_{P_i}^m$
 for all $i \in [\bar{\theta}, \theta)_T$.

proof.

Let i_1 be the least $i \leq_T \theta$ s.t. $\kappa_i \geq \gamma + 1$.

Then $i_1 < \bar{\theta}$, since $\sigma: H \prec H_\Omega$.

It follows easily that $\rho_{P_i}^{\gamma+1} = \rho_{P_{i_1}}^{\gamma+1}$
 and hence $\kappa_i > \rho_{P_i}^{\gamma+1}$ for $i_1 \leq_T i \leq_T \theta$,
 since the κ_i 's are increasing. QED

But then, if $\bar{\theta} \leq_T i = T(i+1)$, $i+1 \leq_T \theta$,

$\prod_{P_{i+1}}^{E_{i+1}}$ is a Σ_0 ultraproduct of $\prod_{P_i}^{E_i}$
 by E_{V_i} . This means, in particular,

that successor cardinals in $\prod_{P_i}^{E_i}$ are
 taken cofinally to their images. (This
 holds for $\rho_{P_i}^\gamma$ as well if it is a successor,
 since $\rho_{P_{i+1}}^\gamma$ is a cardinal in P_{i+1} .)

(Clearly $\tau \leq \rho_{P_{\bar{\theta}}}^\gamma$, since $\rho_{P_{\bar{\theta}}}^\gamma$ is a

cardinal in $P_{\bar{\theta}}$. By induction
 on i it follows easily that $\pi_{\bar{\theta}, i}^P$

takes τ cofinally to its image

for $\bar{\theta} \leq_T i \leq_T \theta$. QED (10)

Now let $X' \prec H_\Omega$ be defined like X , but with $X \cup \{\theta\} \subset X'$. Let $\sigma': H' \xrightarrow{\sim} X'$, where H' is transitive, let $\theta' = \theta \cap X'$. Then, as before, $\theta' = \text{crit}(\sigma')$, $\sigma'(\theta') = \theta$, where $\sigma': H' \prec H_\Omega$. Set $\tilde{\sigma} = \sigma'^{-1} \circ \sigma$.

Then $\tilde{\sigma}: H \prec H'$. Let $\tilde{\sigma}^{-1}(y^p, y^q) = \langle y^p, y^q, y^i \rangle = \langle \langle P_i^p \rangle, \langle v_i^p \rangle, \langle \pi_{i_i}^p \rangle, T^p \rangle$
 $y^q = \langle \langle Q_i^q \rangle, m, T^q \rangle$. As before:

$$T^p = T \cap (\theta + 1)^2, \quad \pi_{i_i}^p = \pi_{i_i}^p \text{ for } i, i \leq \bar{\theta}'$$

$$i, i \leq \bar{\theta}', \quad \theta' \leq_T \theta, \quad \pi_{i_i}^q = \pi_{i_i}^q \parallel (Q_i \parallel \theta')$$

for $i, i \leq \bar{\theta}'$. Since $\bar{\theta}, \theta' \in \{i \mid i \leq_T \theta\}$ and $\bar{\theta} < \theta'$, we have $\bar{\theta} \leq_T \theta'$.

(Clearly $\tilde{\sigma}(\bar{\theta}) = \theta'$ and $\bar{\theta} = \text{crit}(\tilde{\sigma})$.)

Moreover $\tilde{\sigma}(\bar{\pi}_{i_{\bar{\theta}}}^q) = \pi_{i_{\theta'}}^q$. Set:

$\tau' = \theta' + \kappa$. As before, $\kappa \parallel \tau' \in H'$ and is H' -definable from $\pi_{i_{\theta'}}^q$

by the same def. as $\kappa \parallel \bar{\tau}$ from $\bar{\pi}_{i_{\bar{\theta}}}^q$ in \bar{H} . Hence

$\tilde{\sigma}(\kappa \parallel \bar{\tau}) = \kappa \parallel \tau'$. By (10) there

is $f \in \bar{H}$ mapping $a \mu < \bar{\theta}$ cofinally to $\bar{\tau}$.

But then $\tilde{\sigma}(f)$ maps μ cofinally to τ' . It follows that $\tilde{\sigma} \upharpoonright \bar{\tau}$ maps $\bar{\tau}$ cofinally to τ' . Set $\pi = \tilde{\sigma} \upharpoonright (K \parallel \bar{\tau})$, $F = \tilde{\sigma} \upharpoonright \mathcal{F}(\mu)$.

Then $\pi : K \parallel \bar{\tau} \xrightarrow{F} K \parallel \tau'$. The structure $N = \langle K \parallel \tau', \mathbb{R} \rangle$ has all the properties of a premouse except, perhaps, the initial segment condition. Hence it makes sense to ask whether F is robust in N . We prove:

(11) F is robust in $N = \langle K \parallel \tau', \mathbb{R} \rangle$.
 proof.

Let $u \subset \theta'$, $w \subset \mathcal{F}(\bar{\theta}) \cap K$ be countable.
 Let $g : \omega \leftrightarrow u$, $X : \omega \xrightarrow{\text{onto}} w$, where
 $X = \langle X_i : i < \omega \rangle$. Set:

$$D = \{ \langle i_1, \dots, i_m, i \rangle \mid \langle g(i_1), \dots, g(i_m) \rangle \in F(K, i) \}$$

$$A = \{ \langle a_1, \dots, a_m, \varphi \rangle \mid \varphi \text{ is a } \Sigma_1 \text{ formula} \\ \wedge a_1, \dots, a_m \subset \omega \wedge \bigcup_{c, \infty}^{E^K} \models \varphi(g'' a_1, \dots, g'' a_m) \}$$

where $c = \sup u$