§6 Forcing with classes of conditions

The more interesting applications of forcing to admissible sets M generally involve a class of conditions rather than an M-finite set. As an example:

Theorem 1 If $M = \langle |M| ; \in, A \rangle$ is a countable admissible structure and $A \subset On$, then there is an $F \subset On^3$ s.t. $N = \langle M[F] ; A, F \rangle$ is admissible and every ordinal $\nu \in On_M$ is countable by a map $f \in N$.

proof.

Let ρ_ν ($\nu \in On$) enumerate monotonically the p.r. closed ordinals.

The F which we shall adjoin will have the form:

$$F = \{ \langle f_\nu(m), \rho_\nu, m \rangle \mid \nu \in On_M, m < \omega \},$$

where f_ν maps ω onto ρ_ν.

Our conditions are pieces of information which fix the function F at finitely many places:

$\mathbb{P} =_{\mathrm{Df}}$ the set of finite maps p s.t. $\mathrm{dom}(p) \subset \{\rho_\nu \mid \nu \in On_M\} \times \omega$ and $p(\rho_\nu, n) < \rho_\nu$ whenever defined.

$p \leq q \longleftrightarrow_{\mathrm{Df}} p \supset q$.

We also define:

$$\mathbb{P}_\nu =_{\mathrm{Df}} \{p \in \mathbb{P} \mid \mathrm{dom}(p) \subset \rho_\nu \times \omega\}.$$

<u>Lemma 1</u> If $C \subset \mathbb{P}_\nu$ is compatible in $\mathbb{P}_\nu$, then C is compatible in $\mathbb{P}$.

proof.

Let $p \in \mathbb{P}$. Then $p' = p \upharpoonright (\rho_\nu \times \omega)$ is compatible with some $q \in C$. It follows that p is compatible with q. QED

For $G \subset \mathbb{P}$ set: $G_\nu =_{\mathrm{Df}} G \cap \mathbb{P}_\nu$.

<u>Lemma 2</u> If G is $\mathbb{P}$-generic over M, then G_ν is $\mathbb{P}_\nu$-generic over M. More generally, if M', M'' are transitive structures s.t. $\mathbb{P}_\nu$ is M''-definable and $\mathbb{P}$, M'' are M'-definable, then if G is $\mathbb{P}$-generic over M', $\mathbb{P}_\nu$ is G_ν-generic over M''.

proof.

Let $C \subset \mathbb{P}_\nu$ be compatible in $\mathbb{P}_\nu$ and M''-definable. Then C is compatible in $\mathbb{P}$ and M'-definable. Hence $G_\nu \cap C = G \cap C \neq \emptyset$

QED

Now set: $\mathfrak{M} =_{\mathrm{Df}} \langle M^{\mathbb{P}}; I, E, \check{A}, \dot{G} \rangle$,

where $\check{A}(x) = \bigcup_{\nu \in A} [\![x \equiv \check{\nu}]\!]$

$\dot{G}(x) = \bigcup_{p \in \mathbb{P}} [\![x \equiv \check{p}]\!] \cap [p]$.

Clearly, if G is $\mathbb{P}$-generic, then $N = \mathfrak{M}/G$ has the form:

$$N = \langle |N|; A, G \rangle .$$

Define models M_ν ($\nu \in \mathrm{On}_M$) by:

$$M_\nu = \langle |M_\nu|, A \cap \rho_\nu \rangle$$

where $|M_\nu| = \{x \in M \mid \mathrm{rn}(x) < \rho_\nu\}$.

$|M_\nu|$ is p.r. closed.

Define $BA(\mathbb{P}_\nu)$-valued structures

$$\mathfrak{M}_\nu = \langle M_\nu^{\mathbb{P}_\nu}; I_\nu, E_\nu, \check{A}_\nu, \dot{G}_\nu \rangle$$

in analogy with $\mathfrak{M}$, using $\mathbb{P}_\nu$, $A \cap \rho_\nu$ in place of $\mathbb{P}, A$.

Then $N_\nu = \mathfrak{M}_\nu / G_\nu$ has the form:

$$N_\nu = \langle |N_\nu|; A \cap \rho_\nu, G_\nu \rangle .$$

Clearly: $G_\nu^*(x) = G^*(x)$ for $x \in M_\nu^{\mathbb{P}_\nu}$.

Hence $N_\nu \subseteq N_\mu \subseteq N$ for $\nu \leq \mu < On_M$.

For $\nu = \tau + 1$, we have: $\mathbb{P}_\nu \in M_\nu$, hence: $|N_\nu| = M_\nu[G_\nu]$.

Thus:

<u>Lemma 3</u> $|N| = M[G]$; $N_\nu = M_\nu[G_\nu]$ for $\nu < On_M$.

proof.

For $\nu = \tau + 1$, the theorem is proven.

$|N| \subset M[G]$, since $|N| = G^* {}'' M^{\mathbb{P}}$ and $f(x) = G^*(x)$ is p.r. in G.

But $M[G] = \bigcup_{\tau \in On_M} M_{\tau+1}[G_{\tau+1}] \subset |N|$.

Similarly we prove: $|N_\lambda| = M_\lambda[G_\lambda]$ for limit λ. QED

Let $\Vdash_{\mathfrak{M}}$ be the forcing relation of $\mathfrak{M}$ + let $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ be its restriction to Σ_0 fmlae. Similarly for $\Vdash_{\mathfrak{M}_\nu}$, $\Vdash_{\mathfrak{M}_\nu}^{\Sigma_0}$. We do not yet know that $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ is M-definable, but we can show:

<u>Lemma 4</u> $\{\langle \nu, p, \varphi\rangle \mid p \Vdash_{\mathfrak{M}_\nu}^{\Sigma_0} \varphi\}$ is Δ_1

proof. Since $\Vdash_{\mathfrak{M}_\nu}^{\Sigma_0}$ is uniformly p.r. in $\check{A}$, $\mathring{C}_\nu$ and the parameter $\mathbb{P}_\nu$, it suffices to note that the functions $f_0(\nu) = \mathbb{P}_\nu$, $f_1(\nu, x) = \check{A}_\nu(x)$, $f_3(\nu, x) = \mathring{G}_\nu(x)$ are Δ_1. QED

The fact that $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ is Δ_1 now follows from:

<u>Lemma 5</u> If φ is a Σ_0 fmla of $\mathfrak{M}_\nu$ and if $p \in \mathbb{P}_\nu$, then

$$p \Vdash_{\mathfrak{M}_\nu} \varphi \longleftrightarrow p \Vdash_{\mathfrak{M}} \varphi .$$

proof.

By Lemma 2 and the completeness theorem, it suffices to show that for every G which is $\mathbb{P}$-generic over M we have:

$$G \Vdash_{\mathfrak{M}} \varphi \longleftrightarrow G_\nu \Vdash_{\mathfrak{M}_\nu} \varphi ,$$

or equivalently:

$$\models_{\mathfrak{M}/G} \varphi^G \longleftrightarrow \models_{\mathfrak{M}_\nu/G_\nu} \varphi^{G_\nu} .$$

Since $G^*(x) = G^*_\nu(x)$ for $x \in M_\nu^{\mathbb{P}_\nu}$,

we may conclude: $\varphi^G = \varphi^{G_\nu}$.

By Lemma 3 we have:

$$\mathfrak{M}/G = \langle M[G] ; A , G \rangle$$

$$\mathfrak{M}_\nu/G_\nu = \langle M_\nu[G_\nu] ; A \cap \rho_\nu , G_\nu \rangle .$$

But $A \cap M_\nu[G_\nu] = A \cap \rho_\nu$; $G \cap M_\nu[G_\nu] = G_\nu$,

Since φ^G is Σ_0, the conclusion follows immediately. QED

An immediate corollary is:

<u>Lemma 6</u> $\Vdash^{\Sigma_0}_{\mathfrak{M}}$ is Δ_1.

proof.

$$p \Vdash_{\mathfrak{M}} \varphi \longleftrightarrow p \Vdash_{\mathfrak{M}_{\eta(p,\varphi)}} \varphi ,$$

where $\eta(p,\varphi)$ = the least η s.t. $p \in \mathbb{P}_\eta$, $\varphi \in \mathrm{Fml}_{\mathfrak{M}_\eta}$.

QED

We must now show that $N = \mathfrak{N}/G$ is admissible - i.e. that N satisfies the replacement axiom for Σ_0 - fmlae. In proving this, we shall make use of:

<u>Lemma ~~7~~</u> Let $D \subset \mathbb{P}$ be Σ_1 and compatible in $\mathbb{P}$. Let $\nu \in On_M$. Then for some $\beta \geq \nu$, $D_\beta = D \cap \mathbb{P}_\beta$ is compatible in $\mathbb{P}_\beta$ (hence in $\mathbb{P}$). (Note that D_β is not necessarily an element of M, since D is Σ_1 rather than Δ_1).

proof.

Set: $\sigma(p) = \mu\tau \; \bigvee q \in \mathbb{P}_\tau$ (p is compatible with q)

$$\eta(\tau) = \sup_{p \in \mathbb{P}_\tau} \sigma(p)$$

Then $\beta = \eta^\omega(\nu)$ satisfies the conclusion of the lemma. QED

<u>Lemma 8</u> If φ is a Σ_0 $\mathfrak{M}$-fmla, then:

$$p \Vdash \bigvee v\, \varphi \;\longrightarrow\; \bigvee x \in M^{\mathbb{P}} \; p \Vdash \bigvee v \in x\, \varphi .$$

proof.

~~The direction~~ Let $p \Vdash \bigvee v\, \varphi$. Then $D = \neg[p] \cup \{ q \leq p \mid \bigvee y \;\, q \Vdash \varphi(\underline{y}) \}$ is dense in $\mathbb{P}$. Let $p \in \mathbb{P}_\nu$. Then there is a $\beta \geq \nu$ s.t. $D \cap \mathbb{P}_\beta$ is dense in $\mathbb{P}_\beta$. Thus:

$$\bigwedge_{\substack{q \leq p \\ \in \mathbb{P}_\beta}} \bigvee_y \bigvee_{\substack{q' \leq q \\ \in \mathbb{P}_\beta}} q' \Vdash \varphi(y) .$$

By admissibility there is a $u \in M$ s.t.

$$\bigwedge_{\substack{q \leq p \\ \in \mathbb{P}_\beta}} \bigvee_{y \in u} \bigvee_{\substack{q' \leq q \\ \in \mathbb{P}_\beta}} q' \Vdash \varphi(y) .$$

~~But since every $p' \leq p$ is com-~~

~~We may conclude~~:

(*) ~~$\bigwedge q$~~

Thus $C = \{q \in \mathbb{P}_\beta \mid \bigvee y \in u \; q \Vdash \varphi(\underline{y})\}$ is dense in $[p] \cap \mathbb{P}_\beta$, hence in $\mathbb{P}_\beta$.

Thus:

(*) $\bigwedge q \leq p \bigvee q' \leq q \bigvee y \in u \; q' \Vdash \varphi(\underline{y})$.

Set: $x = \{\langle 1, y\rangle \mid y \in u \cap M^{\mathbb{P}}\}$. Then:

$$p \Vdash \bigvee v \, \varepsilon \, \underline{x} \, \varphi . \qquad \text{QED}$$

The fact that N is admissible follows by Lemma 8 and the fact that $\Vdash^{\Sigma_0}_{\mathfrak{M}}$ is Δ_1. We state this fact as a general lemma, for later reference:

<u>Lemma 9.</u> If M is an admissible structure, $\mathbb{P}$ is a Δ_1 system of conditions and $\mathfrak{M} = \langle M^{\mathbb{P}}; I, E, A_1, \ldots, A_n\rangle$ is a $BA(\mathbb{P})$-valued model s.t. $\Vdash^{\Sigma_0}_{\mathfrak{M}}$ is Δ_1 ~~that~~ and

if $p \Vdash \bigvee x\, \varphi \to \bigvee u\; p \Vdash \bigvee x \varepsilon \underline{u}\; \varphi$ for every $\varphi \in \mathrm{Fml}^{\Sigma_0}_{\mathfrak{M}}$, then $\mathfrak{M}/G$ is admissible for $\mathbb{P}$-generic G.

proof.

It suffices to show that the Σ_0 replacement axiom is forced; i.e. if φ is a Σ_0 fmla of $\mathfrak{M}$, then

$$\Vdash (\bigwedge x \bigvee y\, \varphi \to \bigwedge u \bigvee v \bigwedge x \varepsilon u \bigvee y \varepsilon v\; \varphi),$$

or equivalently:

$$p \Vdash \bigwedge x \bigvee y\, \varphi \longrightarrow p \Vdash \bigwedge u \bigvee v \bigwedge x \varepsilon u \bigvee y \varepsilon v\, \varphi$$

for $p \in \mathbb{P}$. Let $p \Vdash \bigwedge x \bigvee y\, \varphi$. Then:

$$\bigwedge x \in M^{\mathbb{P}} \bigvee v \in M^{\mathbb{P}} \quad p \Vdash \bigvee y \varepsilon \underline{v}\; \varphi(\underline{x}, y).$$

Let $u \in$ ~~[illegible]~~ $M^{\mathbb{P}}$. Then there is a $w \in M^{\mathbb{P}}$ s.t.

$$\bigwedge x \in \mathrm{dom}(u) \bigvee v \in w \quad p \Vdash \bigvee y \varepsilon \underline{v}\; \varphi(\underline{x}, y),$$

Set: $v' = \bigcup w$. Then:

$$p \Vdash \bigwedge x \varepsilon \underline{u} \bigvee y \varepsilon \underline{v}'\, \varphi. \qquad \text{QED}$$

Thus $N = \langle M[G], A, G\rangle$ is admissible.

The only remaining step ~~[illegible]~~ in the proof of Thm 1 is to show that N is equivalent to a structure $N' = \langle M[F]; A, F\rangle$, where $F \subset On_M^3$ is as above.

(Equivalence ~~[illegible]~~, of course, means: $|N| = |N'|$ and the Δ_1 relations of N are the Δ_1 relations of N').

$F = \bigcup G$ obviously has the required properties. F (as a relation) is p.r. in G since:

$$\langle \kappa, \nu, n\rangle \in F \longleftrightarrow \{\langle \kappa, \nu, n\rangle\} \in G.$$

Conversely, G is p.r. in F since:

$$p \in G \longleftrightarrow p \subset F.$$

This completes the proof of Theorem 1.

Def Call α <u>admissible</u> <u>in</u> $A_1, \ldots, A_m \subset On$ iff

$$\langle L_\alpha[A_1, \ldots, A_m] ; A_1 \cap \alpha, \ldots, A_m \cap \alpha \rangle$$

is an admissible structure.

As a corollary to Thm 1, we obtain:

<u>Theorem 2</u> Let λ be a (countable) limit of admissible ordinals. Let $A \subset \lambda$. Then there is a $B \subset \lambda$ s.t.

(a) If $\alpha < \lambda$ is admissible in $C \subset \alpha$ and C is $\langle L_\lambda[A], A\rangle$ - definable then α is admissible in B, C.

(Thus, in particular, if $\alpha < \lambda$ is admissible in A, α is admissible in A, B).

(b) ρ_ν is countable by a map $f \in L_{\rho_{\nu+1}}[B]$ for $\nu < \lambda$.

proof of Thm 2.

As before, force with conditions:

$\mathbb{P}$ = the set of finite maps p s.t. $\mathrm{dom}(p) \subset \{\rho_\nu \mid \nu < \lambda\} \times \omega$ and $p(\rho_\nu, n) < \rho_\nu$ if defined.

Let G be $\mathbb{P}$-generic over $\langle L_\lambda[A], A\rangle$. Let $F = \bigcup G$. Then

$$F = \{\langle f_\nu(n), \rho_\nu, n\rangle \mid \nu < \lambda, n < \omega\},$$

where f_ν maps ω onto ρ_ν.

Set: $B = \{\langle \alpha, \beta, \gamma\rangle \mid \langle \alpha, \beta, \gamma\rangle \in F\}$.

(b) follows immediately.

To prove (a), we note that G_α is $\mathbb{P}_\alpha$-generic over $\langle L_\alpha[C], C\rangle$. Hence, by the proof of Thm 1, $\langle L_\alpha[C, B], C, B \cap \alpha\rangle$ is admissible.

QED

We now give a second application of forcing with a class of conditions to add, as before, a class of ordinals to an admissible structure M. This application will differ from the previous one, however, in that no new <u>sets</u> will be added to the structure.

<u>Def</u> α is <u>Mahlo</u> (or <u>recursively Mahlo</u>) in $A \subset \alpha$ iff α is admissible in A and every normal function which is Δ_1 in $\langle L_\alpha[A], A \rangle$ takes a value which is admissible in A.

Theorem 3 Let α be Mahlo in A. Assume that for all $\beta < \alpha$, β is countable in $L_{\beta'}[A]$, where β' = the least p.r. closed $\beta' > \beta$. Then there is a $B \subset \alpha$ s.t.

(i) $\langle L_\alpha[A], A, B \rangle$ is admissible

(ii) If $\gamma < \alpha$ is admissible in A, then γ is admissible in A, B

(iii) α is non Mahlo in A, B.

proof.

It suffices that B satisfy (i) and:

(iv) B is unbounded in α, but if $\gamma < \alpha$ is admissible in A, then $B \cap \gamma \in L_\gamma[A]$.

(ii) obviously follows from (iv).

To see that (iii) follows,

note that the monotone enumeration of the limit pts. of B is a Δ_1 normal function (in $\langle L_\alpha[A], A, B\rangle$) ~~containing no~~ none of whose values are admissible in A.

Our conditions will describe initial segments of the set B which we wish to add:

$\mathbb{P} =_{df}$ the collection of bounded subsets p of α s.t.

(i) For all $\beta < \alpha$, $p \cap \beta \in L_{\beta'}[A]$, where $\beta' =$ the least p.r. closed $\beta' > \beta$.

(ii) If $\beta \leq \alpha$ is admissible in A, then $p \cap \beta \in L_\beta[A]$.

$$p \leq q \underset{df}{\longleftrightarrow} q = p \cap \sup\{\nu+1 \mid \nu \in q\}.$$

Set: $\mathfrak{M} = \langle L_\alpha[A]^{\mathbb{P}}; I, E, \check{A}, \mathring{B} \rangle$,

where: $\check{A}(x) = \bigcup_{\nu \in A} [\![x \equiv \check{\nu}]\!]$

$$\mathring{B}(x) = \bigcup_{\nu < \alpha} \left([\![x \equiv \check{\nu}]\!] \wedge \{ p \in \mathbb{P} \mid \nu \in p \} \right).$$

Then $N = \mathfrak{M}/G = \langle |N|; A, B \rangle$, for $\mathbb{P}$-generic G, where B is an unbounded subset of α satisfying (iv).
We must show that N is admissible and that $|N| = L_\alpha[A]$.

<u>Lemma 1</u>. If G is $\mathbb{P}$-generic, then $|\mathfrak{M}/G| = L_\alpha[A]$.

proof.
Let $\beta < \alpha$ be p.r. closed.
Let $p = \bigcup G \cap \beta$. Then $p \in G$.
For $x \in L_\beta[A]^{\mathbb{P}}$ set:

$$p^*(x) = \{ p^*(y) \mid \bigvee q \geq p \ \langle q, y \rangle \in x \}.$$

Since $G \cap L_\beta[A] = \{ q \in L_\beta[A] \mid q \geq p \}$,
we have:

$p^*(x) = G^*(x)$ for $x \in L_\beta[A]^{\mathbb{P}}$.

But $p^* \upharpoonright L_\beta[A]^{\mathbb{P}} \in L_\alpha[A]$. QED

<u>Lemma 2</u> $\Vdash^{\Sigma_0}_{\mathfrak{M}}$ is Δ_1 in $\langle L_\alpha[A], A\rangle$.

proof.

Let $\beta < \alpha$ be ~~a limit of~~ p.r. closed, let β' be the next largest p.r. closed ordinal. Let $p \in L_{\beta'}[A]$, $p \in \mathbb{P}$ be s.t. $\sup(p) \geq \beta$. Then for any $\mathbb{P}$-generic G s.t. $p \in G$ we have:

(1) $p^*(x) = G^*(x)$ for $x \in L_\beta[A]^{\mathbb{P}}$

(p^* being defined as in the proof of lemma 1)

(2) $\mathbf{B} \cap \beta = p \cap \beta$ (where $\mathbf{B} = G^*(\mathring{B})$).

In particular, if φ is a Σ_0 fmla of $\mathfrak{M}$ containing only constants $\underset{\sim}{x}$ s.t. $x \in L_\beta[A]^{\mathbb{P}}$, then

$$G \Vdash \varphi \iff \models_{\langle L_\beta[A, p\cap\beta],\, A\cap\beta,\, p\cap\beta\rangle} \varphi^{(p)}$$

(where $\varphi^{(p)}$ is the result of replacing $\underset{\sim}{x}$ by $\underline{p^*(x)}$).

Since this holds for <u>every</u> $\mathbb{P}$-generic $G \ni p$, we conclude:

$$p \Vdash \varphi \iff \models_{\langle L_\beta[A, p],\, A\cap\beta,\, p\cap\beta\rangle} \varphi^{(p)} .$$

Now let $p' \in \mathbb{P} \cap L_\beta[A]$. Every $\mathbb{P}$-generic $G \ni p'$ contains a $p \in L_{\beta'}[A]$ s.t. $p' \geq p$ and $\sup(p) \geq \beta$. Hence:

$$p' \Vdash \varphi \iff \bigwedge p \in L_{\beta'}[A]\ (p \leq p' \wedge$$
$$\wedge \sup(p) \geq \beta \rightarrow \models_{\langle L_\beta[A,p],\, A\cap\beta,\, p\cap\beta\rangle} \varphi^{(p)}$$

$$\iff S(\varphi, p', \beta)$$

where S is Δ_1.

Hence:

$$p \Vdash \varphi \leftrightarrow S(\varphi, p, \eta(\varphi, p)),$$

where $\eta(p, \varphi)$ = the least p.r. closed β s.t. $p, \varphi \in L_\beta[A]$. QED

Our main tool in showing that $N = \mathfrak{N}/G$ is admissible will be:

<u>Lemma 3</u> Let D_i $(i < \omega)$ be a sequence of sets which are dense in $\mathbb{P}$ and closed under extensions. Let $\{\langle p, i\rangle \mid p \in D_i\}$ be Σ_1 in $\langle L_\alpha[A], A\rangle$. Then $\bigcap_i D_i$ is dense in $\mathbb{P}$.

proof.

Let $p \in D_i \leftrightarrow \bigvee x\, Rxip$,

where R is Σ_0.

For $\delta < \alpha$ set:

$$D_i^\delta =_{\text{df}} \{p \in \mathbb{P} \cap L_\delta[A] \mid \bigvee x \in L_\delta[A]\ Rxip\}$$

Claim There are arbitrarily large p.r. closed $\delta < \alpha$ s.t.

(i) D_i^δ is dense in $\mathbb{P} \cap L_\delta[A]$ for $i < \omega$

(ii) δ is not admissible in A.

proof.

Let $\gamma < \alpha$ be s.t. $\vec{x} \in L_\gamma[A]$, where $\vec{x}$ are the constants occurring in the Σ_0 definition of R.

Define $\eta : \mathbb{P} \to \alpha$ by:

$\eta(p) =$ the least β s.t.

$\bigwedge i<\omega \; \bigvee q \leq p \; \bigvee x \; (q, x \in L_\beta[A] \wedge Rxiq)$.

Then η is Δ_1. Set:

$\bar{\eta}(\beta) =$ the least p.r. closed $\beta' > \beta$ s.t. $\bigwedge p \in \mathbb{P} \cap L_\beta[A] \;\; \eta(p) < \beta'$.

Then $\bar{\eta}$ is Δ_1.

Let $\delta = \bar{\eta}^{\omega}(\gamma)$.

Then $\delta > \gamma$ is p.r. closed and satisfies (i).

To show that δ satisfies (ii), we note that $\eta \upharpoonright (\mathbb{P} \cap L_\delta[A])$ is Δ_1 in $\langle L_\delta[A], A \cap \delta \rangle$. Thus, if δ were admissible in A, $\langle \bar{\eta}^i(\gamma) \mid i < \omega \rangle$ would be Δ_1 in $\langle L_\delta[A], A \cap \delta \rangle$. Contradiction!

QED (Claim)

Now let $p \in \mathbb{P}$. We must show that there is $q \leq p$ with $q \in \bigcap_i D_i$. Pick δ s.t. $p \in L_\delta[A]$, D_i^δ is dense in $\mathbb{P} \cap L_\delta[A]$ and δ is not admissible in A.

~~Define~~ $\cancel{q_i\ (i<\omega)}$ ~~by~~:

$\cancel{q_0 = p}$

$\cancel{q_{i+1}}$ ~~= the least~~ $\cancel{q \leq q_i}$ ~~(in the canonical well ordering of~~ $\cancel{L[A]}$~~) s.t. sup~~(q)

Let $f : \omega \leftrightarrow \delta$, $f \in L_{\delta'}[A]$, where δ' is the next largest p.r. closed ordinal. Define $q_i \in \mathbb{P} \cap L_\delta[A]$ $(i < \omega)$ by:

$q_0 = p$

$q_{i+1} =$ the least $q \leq q_i$ (in the canonical well ordering of $L[A]$) s.t. $q \in D_i^\delta$ and $\sup(q) \geq f(i)$.

Set $q = \bigcup_i q_i$. Then $q \in \mathbb{P}$, since $\sup(q) = \delta$ and $q \in L_{\delta'}[A]$.

Hence $q \leq p \wedge q \in \bigcap_i D_i$.

QED

We are now ready to show:

<u>Lemma 4</u> If G is $\mathbb{P}$-generic, then $\mathfrak{M}/G$ is admissible.

proof.

Since all sets are countable in $\mathfrak{M}/G$, it suffices to show that $\mathfrak{M}/G$ satisfies:

$$\bigwedge i<\omega \bigvee y\, R i y \rightarrow \bigvee u \bigwedge i<\omega \bigvee y \in u\, R i y$$

for Σ_0 R. That is, we must prove:

$$\Vdash \bigwedge x \in \check{\omega} \bigvee y\, \varphi \rightarrow \bigvee u \bigwedge x \in \check{\omega} \bigvee y \in u\, \varphi$$

for Σ_0 $\mathfrak{M}$-fmlae φ.

Let $p \Vdash \bigwedge x \in \check{\omega} \bigvee y\, \varphi$.

Set: $D_i = \{q \leq p \mid \bigvee y \;\; q \Vdash \varphi(\underline{i}, \underline{y})\} \cup$

$$\cup \neg[p]$$

for $i<\omega$. Then D_i is dense in $\mathbb{P}$ and $\{\langle i,x\rangle \mid x \in D_i\}$ is Σ_1.

Hence $\bigcap_i D_i$ is dense in $\mathbb{P}$.

This means that for every $p' \leq p$ there is a $p'' \leq p'$ s.t.

$$\bigwedge i<\omega \ \bigvee y \quad p'' \Vdash \varphi(\underset{\sim}{i}, \underset{\sim}{y}).$$

By the replacement axiom there is a v s.t.

$$\bigwedge i<\omega \ \bigvee y \in v \quad p'' \Vdash \varphi(\underset{\sim}{i}, \underset{\sim}{y}).$$

Set : $\tilde{v} = \{ \langle 1, y \rangle \mid y \in v \cap \mathbb{P} \}$.

Then :

$$p'' \Vdash \bigwedge x \in \check{\omega} \ \bigvee y \in \underset{\sim}{\tilde{v}} \ \varphi ,$$

Hence $\quad p \Vdash \bigvee v \bigwedge x \in \check{\omega} \ \bigvee y \in v \ \varphi .$

QED

We now turn to the major theorem of this section:

<u>Theorem 4</u> Let $B \subset On$. Let α_ν $(\nu < \delta)$ be a countable sequence of countable ordinals s.t. $\alpha_\nu > \omega$ and α_ν is admissible in $B \cap \alpha_\nu$, $\{\alpha_\iota \mid \iota < \nu\}$ for $\nu < \delta$. Then there is a $b \subset \omega$ s.t.

(i) α_ν is the ν-th $\alpha > \omega$ s.t. α is admissible in b

(ii) $B \cap \alpha_\nu$ is Δ_1 in $L_{\alpha_\nu}[b]$

for $\nu < \delta$.

Thm 4 follows immediately from the conjunction of the following two theorems:

Theorem 4.1 Let λ be a countable limit of admissible ordinals. Let $B \subset \lambda$. Let $\mathfrak{a} \subset \lambda$ be s.t. $\omega \notin \mathfrak{a}$ and each $\alpha \in \mathfrak{a}$ is admissible in $B \cap \alpha$, $\mathfrak{a} \cap \alpha$. Then there is a $C \subset \lambda$ s.t for all $\alpha < \lambda$:

(i) α is admissible in C iff $\alpha \in \mathfrak{a}$

(ii) If α is admissible in C, then $B \cap \alpha$, $\mathfrak{a} \cap \alpha$ are Δ_1 in $\langle L_\alpha[C], C \cap \alpha \rangle$.

(iii) If α is p.r. closed and α' is the next largest p.r. closed ordinal, then α is countable in $L_{\alpha'}[C \cap \alpha]$.

Theorem 4.2 Let λ be a countable limit of admissible ordinals and let $C \subset \lambda$ satisfy (iii) of Thm 4.1. Then there is a $b \subset \omega$ s.t. for all $\alpha < \lambda$ s.t $\alpha > \omega$:

(i) If α is admissible in C, then α is admissible in b

(ii) If α is admissible in b, then $C \cap \alpha$ is Δ_1 in $L_\alpha[b]$

(Hence α is admissible in b iff α is admissible in C).

. . . . .

We begin with the proof of Thm 4.1.

<u>Lemma 1</u> There is an $A \subset \lambda$ s.t. for $\alpha < \lambda$

(a) α is admissible in $B \cap \alpha$, and iff α is admissible in A

(b) If α is admissible in A, then $B \cap \alpha$, $C \cap \alpha$ are Δ_1 in $\langle L_\alpha[A], A\rangle$

(c) If α is p.r. closed, then α is countable in $L_{\alpha'}[A]$, where α' is the next largest p.r. closed ordinal.

proof of Lemma 1.

By Thm 2 there is an $A' \subset \lambda$ satisfying (iii) s.t., whenever $\alpha < \lambda$ is admissible in $\mathcal{A} \cap \alpha$, $B \cap \alpha$, then α is admissible in $\mathcal{A} \cap \alpha$, $B \cap \alpha$, $A' \cap \alpha$. Set:

$$A = \{3 \cdot \nu \mid \nu \in \mathcal{A}\} \cup \{3 \cdot \nu + 1 \mid \nu \in B\} \cup \{3 \cdot \nu + 2 \mid \nu \in A'\}$$

Then A satisfies (i) – (iii). QED

Now ~~at~~ let $\mathcal{A}^*$ be the set of $\alpha < \lambda$ s.t. α is admissible in A. (Hence $\mathcal{A} \subset \mathcal{A}^*$).

<u>Lemma 2</u> There is an $f : \mathcal{A}^* \to \lambda$ s.t.

(i) $f(\alpha) < \alpha$ for $\alpha \in \mathcal{A}^*$

(ii) f is 1-1

(iii) $\{\alpha \in \mathcal{A}^* \setminus \varkappa \mid f(\alpha) < \varkappa\}$ is finite for $\varkappa < \lambda$

(iv) If $\alpha \in \mathcal{A}^*$, then α is admissible in A, $f \restriction \alpha$, $\mathrm{rng}(f \restriction \alpha)$. (i.e. $\langle L_\alpha[A, f\restriction\alpha, \mathrm{rng}(f\restriction\alpha)]; A\cap\alpha, f\restriction\alpha, \mathrm{rng}(f\restriction\alpha)\rangle$ is admissible)

proof of Lemma 2.

Let ρ_ν $(\nu \leq \lambda)$ be the monotone enumeration of the p.r. closed ordinals $\leq \lambda$. Let Θ_ν $(\nu \leq \lambda)$ be the set of maps f f defined on $\mathfrak{a}^* \cap \rho_\nu$ s.t. f satisfies (i)–(iv) and $f \in L_{\rho_{\nu+1}}[A]$. Set:

$$f \leq f' \underset{Df}{\longleftrightarrow} \bigvee \nu \tau \, (\nu \leq \tau \wedge f \in \Theta_\nu \wedge f' \in \Theta_\tau \wedge$$
$$\wedge f \subset f' \wedge \operatorname{rng}(f' \setminus f) \subset \rho_\tau \setminus \rho_\nu).$$

By induction on $\kappa < \lambda$ we prove:

<u>Claim</u> $\Theta_\kappa \neq \emptyset$; moreover, if $\nu < \kappa$, $\nu \notin \mathfrak{a}^*$ and $f \in \Theta_\nu$, then there is an $f' \in \Theta_\kappa$ s.t. $f \leq f'$.

proof.

<u>Case 1</u> $\kappa = 0$: $\emptyset \in \Theta_0$.

<u>Case 2</u> $\kappa = \tau + 1$; $\tau \notin \mathfrak{a}^*$. Then $\Theta_\kappa = \Theta_\tau$

<u>Case 3</u> $\kappa = \alpha + 1$; $\alpha \in \mathfrak{a}^*$.

If $\nu < \alpha$, $\notin \mathfrak{a}^*$, $f \in \Theta_\nu$, then pick $\bar{f} \in \Theta_\alpha$ s.t. $\bar{f} \geq f$.

Let $\iota \in \alpha \setminus \rho_\nu$ s.t. $\iota \notin \mathrm{rng}(\bar{f})$.

Set: $f' = f \cup \{\langle \iota, \alpha \rangle\}$.

<u>Case 4</u> $\mathrm{Lim}(\kappa)$; $\kappa \notin a^*$.

Since ρ_κ is countable in $L_{\rho_{\kappa+1}}[A]$, there is a $g \in L_{\rho_{\kappa+1}}[A]$ s.t. $g : \omega \to \kappa$ is monotone + cofinal in κ. We may assume: $g(0) = \nu$ and $g(i) \notin a^*$ for $i < \omega$. Define f_i $(i < \omega)$ by:

$f_0 = f$

$f_{i+1} =$ the least $f' \in \Theta_{g(i+1)}$ (in the canonical well ordering of $L[A]$) s.t. $f' \geq f_i$.

Set: $f' = \bigcup_i f_i$. Then $f' \in \Theta_\kappa$ and $f' \geq f$.

<u>Case 5</u> $\kappa = \alpha \in \mathfrak{a}^*$.

By Thm 3 there is a $B \subset \alpha$ s.t. α is admissible in A, B, B is unbounded in α, $B \cap \alpha'$ is bounded in α' for every $\alpha' < \alpha$ s.t. $\alpha' \in \mathfrak{a}^*$ and $B \cap \rho_\nu \in L_{\rho_{\nu+1}}[A]$ for $\nu < \alpha$. From the proof of Thm 3 it is clear that we may assume: $B \in L_{\rho_{\alpha+1}}[A]$, for $L_\alpha[A]$ is countable in $L_{\rho_{\alpha+1}}[A]$; hence we can find $G \in L_{\rho_{\alpha+1}}[A]$ s.t. G is $\mathbb{P}$-generic over $\langle L_\alpha[A], A\rangle$, $\mathbb{P}$ being the system of conditions used in the proof of Thm 3.

Assume: $B \in L_{\rho_{\alpha+1}}[A]$ and let $g : \alpha \to \alpha$ be the monotone enumeration of the limit points of B.

~~Then α is admissible~~

Then g is a normal function

taking no ~~admissible~~ values in a^*, $g \in L_{\rho_{\alpha+1}}[A]$ and α is admissible in A, g. Moreover, $g \upharpoonright \rho_\nu \in L_{\rho_{\nu+1}}[A]$ for $\nu < \alpha$. ~~We may~~

Let $\nu < \alpha$; ~~We~~ $f \in \Theta_\nu$. We may assume without loss of generality that $g(0) = \nu$. Define a sequence f_ν $(\nu < \alpha)$ by:

$f_0 = f$

$f_{\nu+1} =$ the least $f' \in \Theta_{\nu+1}$ (in the canonical well ordering of $L[A]$ s.t. $f' \geq f_\nu$

$f_\delta = \bigcup_{\nu < \delta} f_\nu$ for limit δ

(hence $f_\delta \in \Theta_\delta$; $f_\delta \geq f_\nu$ for $\nu < \delta$).

~~So~~ The sequence f_ν $(\nu < \alpha)$ is Δ_1 in $\langle L_\alpha[A, g], A, g \rangle$.

Set: $f' = \bigcup_\nu f_\nu$.

Then f' is Δ_1 in $\langle L_\alpha[A,g], A, g\rangle$

since:

$$\nu = f'(\tau) \longleftrightarrow \nu = f_{\tau+1}(\tau).$$

$\mathrm{rng}(f')$ is Δ_1 in $\langle L_\alpha[A,g], A, g\rangle$

since:

$$\iota \in \mathrm{rng}(f') \longleftrightarrow \iota \in \mathrm{rng}(f_{\iota+1}).$$

~~But~~ Hence $f' \geq f$ and $f' \in \Theta_\alpha$.

QED (Claim).

Lemma 2 follows almost immediately from the claim: Let λ_i $(i<\omega)$ be monotone + cofinal in λ. Assume: $\lambda_i \notin a^*$. Select f_i $(i<\omega)$ s.t. $f_i \in \Theta_{\lambda_i}$ and $f_{i+1} \geq f_i$. Set: $f = \bigcup_i f_i$.

Then f satisfies (i) – (iv) of Lemma 2.

QED

Now let f be as in Lemma 2 and define a map $\tilde{f} : (\mathfrak{a}^* \setminus \mathfrak{a}) \times \omega \to \lambda$ by: $\tilde{f}(\alpha, i) = g_\alpha(i)$, where:

g_α = the least g (in the canonical well ordering of $L[A]$) s.t. g is a monotone map from ω to α and $\sup_{i<\omega} g(i) = \alpha$.

Set:

$$D = \{\langle f(\beta), \tilde{f}(\beta, i)\rangle \mid \beta \in \mathfrak{a}^* \setminus \mathfrak{a} \wedge i < \omega\},$$

and

$$D_\alpha = \{\langle f(\beta), \tilde{f}(\beta, i)\rangle \mid \beta \in (\mathfrak{a}^* \setminus \mathfrak{a}) \cap \alpha \wedge i < \omega\}$$

for $\alpha < \lambda$. Then if $\alpha \in \mathfrak{a}^*$, D_α is Δ_1 in $\langle L_\alpha[A], A \cap \alpha, f \upharpoonright \alpha, \mathrm{rng}(f \upharpoonright \alpha)\rangle$. It follows that, if $\alpha \in \mathfrak{a}$, $D \cap \alpha$ is also Δ_1, since $D \cap \alpha$ differs from D_α by at most finitely many points. On the other hand, if $\alpha \in \mathfrak{a}^* \setminus \mathfrak{a}$, α is not admissible in $D \cap \alpha$, since, if it were, we should have: $\mathrm{rng}(g_\alpha) = \{\nu \mid \langle f(\alpha), \nu\rangle \in D \cap \alpha\}$ is Δ_1 in $\langle L_\alpha[D], D\cap\alpha\rangle$; hence g_α, being the monotone enumeration of $\mathrm{rng}(g_\alpha)$, would be Δ_1. But this violates the replacement axiom.

We now define the set C, whose existence is asserted in Theorem 4.1.

$$C = \{4\cdot\alpha \mid \alpha\in A\} \cup \{4\cdot\langle\alpha,\beta\rangle + 1 \mid \alpha = f(\beta)\} \cup$$
$$\cup \{4\cdot\alpha + 2 \mid \alpha \in \mathrm{rng}(f)\} \cup \{4\alpha+3 \mid \alpha\in D\}.$$

For $\alpha < \lambda$ we define C_α similarly, using $A\cap\alpha$, $f\restriction\alpha$, $\mathrm{rng}(f\restriction\alpha)$, D_α in place of $A, f, \mathrm{rng}(f), D$. Clearly, if $\alpha\in\mathfrak{a}^*$, then C_α is Δ_1 in $\langle L_\alpha[A]; A\cap\alpha, f\restriction\alpha, \mathrm{rng}(f\restriction\alpha)\rangle$. Hence $C\cap\alpha$ is Δ_1 for $\alpha\in\mathfrak{a}$, since C_α differs from C_α by at most finitely many elements. Thus α is admissible in C if $\alpha \in \mathfrak{A}$. On the other hand, if $\alpha \notin \mathfrak{A}$, then α is not admissible in C, since, ~~if it were, we should have: $D \cap \alpha$ is Δ_1 in L_α~~, if it were, we should have: $\alpha \in \mathfrak{A}^* \setminus \mathfrak{A}$, since $A \cap \alpha$ is Δ_1 in $\langle L_\alpha[C], C \cap \alpha \rangle$. But then ~~$D \cap \alpha$ is~~ α is not admissible in C, since $D \cap \alpha$ is Δ_1 in $\langle L_\alpha[C], C \cap \alpha \rangle$ and α is not admissible in $D \cap \alpha$. This establishes (i) of Theorem 4.1. (ii) follows from Lemma 1 and the fact that $A \cap \alpha$ is Δ_1 in $\langle L_\alpha[C], C \cap \alpha \rangle$. (iii) is established by the following lemma:

<u>Lemma 3</u> ρ_α is countable in $L_{\rho_{\alpha+1}}[C \cap \alpha]$ for $\alpha < \lambda$.

proof of Lemma 3.

We assume the following facts:

(1) ρ_0 is countable in L_{ρ_1}

(2) ρ_ν is mappable onto $\rho_{\nu+1}$ by a function $g \in L_{\rho_{\nu+2}}$.

By (2) we get:

(3) If $Lim(\alpha)$ and $\rho_\alpha > \alpha$, then α is mappable onto ρ_α by a function $g \in L_{\rho_\alpha}$.

proof of (3).

Suppose not. Let κ be the least $\kappa \leq \alpha$ s.t. α is not mappable onto ρ_κ by a map $g \in L_{\rho_{\kappa+1}}$. For $\nu < \kappa$ let $g_\nu : \alpha \to \rho_\nu$ be the least $g \in L_{\rho_{\nu+1}}$ (in the canonical well ordering of L) s.t. g maps α onto ρ_ν. κ is a limit ordinal by (2); hence the function

$$g(\langle \nu, \iota \rangle) = g_\nu(\iota)$$

maps $\kappa \times \alpha$ onto ρ_κ. It is easily seen that $g \in L_{\rho_{\kappa+1}}$. But since $\kappa \leq \alpha$ there is an $h \in L_{\rho_{\kappa+1}}$ which maps ~~$\kappa \times \alpha$ onto α~~. α onto $\kappa \times \alpha$. Hence: $g \circ h \in L_{\rho_{\kappa+1}}$ and $g \circ h$ maps α onto ρ_κ. Contradiction! QED (3).

Now suppose Lemma 3 to be false. Let α be the least ordinal for which Lemma 3 fails. Then $\alpha = \rho_\alpha$ by (1)-(3), (hence $L_{\rho_\alpha}[C \cap \alpha] = L_\alpha[C]$).

(4) α is admissible in C. Moreover, if $R \subset L_\alpha[C]$ and $R \in L_{\rho_{\alpha+1}}[C \cap \alpha]$, then:

(*) $\bigwedge x \bigvee y\, Rxy \rightarrow \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, Rxy$

holds in $L_\alpha[C]$.

proof of (4).

Since each $x \in L_\alpha[C]$ is countable in $L_\alpha[C]$, it suffices to show:

$(**)$ $\bigwedge n < \omega \bigvee y\, R x y \rightarrow \bigvee v \bigwedge n < \omega \bigvee y \in v\, R n y$.

Suppose not. Set:

$$h(n) = \mu\gamma \bigvee y \in L_\gamma[C]\; R n y \qquad (n < \omega).$$

Then $\sup_{n<\omega} h(n) = \alpha$. Set:

$$q_n =_{\mathrm{df}} \text{the least } y \in L_\alpha[C] \text{ (in the canonical well ordering of } L[C]) \text{ s.t. } q \text{ maps } \omega \text{ onto } h(n).$$

Then $q(\langle n, m\rangle) = q_n(m)$ maps ω onto α and $q \in L_{\rho_{\alpha+1}}[C \cap \alpha]$. Contradiction!

QED

Using (4), we get:

(5) α is Mahlo in C.

proof of (5)

Let D be a closed, unbounded subset of α which is Δ_1 in $\langle L_\alpha[C], C\cap\alpha\rangle$. We must show: $\beta \in D$ for some $\beta < \alpha$ which is admissible in C.

Let $\vec{x}$ be all constants occurring in the Δ_1 definition of D. Let u be the set of $x \in L_\alpha[C]$ which are $\langle L_\alpha[C], C\cap\alpha\rangle$ - definable in the parameters $\vec{x}$. Since there is an $\langle L_\alpha[C], C\cap\alpha\rangle$ - definable well ordering of $L_\alpha[C]$, we have: ~~$\langle u, C\cap u\rangle$ is an elementary submodel of~~

(*) $\langle u, C\cap u\rangle$ is an elementary submodel of $\langle L_\alpha[C], C\cap\alpha\rangle$.

It follows that u is transitive, for by (*), if $x \in u$ there is a $g \in u$ s.t. g maps ω onto x. Hence: $x = g``\omega \subset u$.

Let $\beta = \alpha \cap u$. By the condensation lemma we have: $u = L_\beta[C]$.
By (*) $\langle L_\beta[C], C \cap \beta \rangle$ satisfies the admissibility axioms, since $\langle L_\alpha[C], C \cap \alpha \rangle$ does. Hence β is admissible in C. We now claim: $\beta < \alpha$. To ~~see~~ see this, we let $\tilde{u}$ be the set of fmlae (with $\varepsilon, \equiv, \mathring{C}$) containing at most the constants $\vec{x}$ and the free vbl v_0. Set:

$$h(\varphi) =_{Df} \text{the least } y \in L_\alpha[C] \text{ (in the canonical well ordering of } L[C]) \text{ s.t. } \models_{\langle L_\alpha[C], C \cap \alpha \rangle} \varphi(v_0/\underline{y}) \text{ if such } y \text{ exists}$$

$$=_{Df} \emptyset \text{ if not.}$$

Then $\tilde{u} \in L_\alpha[C]$ and $h \in L_{\alpha+1}[C \cap \alpha]$.
By (4) there is a $v \in L_\alpha[C]$ s.t. $u = h''\tilde{u} \subset v$. Hence $\beta = \alpha \cap u < \alpha$.

By (*), $D \cap \beta$ is closed and unbounded in β. Hence β is a limit point of D. Hence $\beta \in D$ QED (5).

(5) leads to a contradiction, however, since:

~~(6) If $\alpha < \lambda$ is acl~~

(6) $\alpha < \lambda$ is not Mahlo in C.

proof.

We may assume: $\alpha \in \mathcal{C}$. Since $f \restriction \alpha$, $\mathrm{rng}(f \restriction \alpha)$ are Δ_1 in $\langle L_\alpha[C], C \cap \alpha \rangle$, we can define a Δ_1 normal function g by:

$g(0) = 0$

$g(\nu+1) =$ the least $\beta > g(\nu)$ s.t. $f^{-1}(\nu) < \beta$ for all $\nu \in \mathrm{rng}(f \restriction \alpha) \cap g(\nu)$

$g(\delta) = \sup_{\nu<\delta} g(\nu)$ for limit δ.

Clearly, g takes no values in $\mathcal{C}^*$.

QED

This completes the proof of Thm 4.1.