§2 primitive recursive functions

The p.r. functions $f: V^n \to V$ are generated by the following schemata:

(i) $f(\vec{x}) = x_i$

(ii) $= \{x_i, x_j\}$

(iii) $= x_i \setminus x_j$

(iv) $= \omega$

(v) $f(\vec{x}) = h(\vec{g}(\vec{x}))$

(vi) $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$

(vii) $f(y, \vec{x}) = g(y, \vec{x}, \langle f(z, \vec{x}) \mid z \in y \rangle)$ *

p.r. fcns have the following closure properties:

(1) a. $f(\vec{x}) = \bigcup x_i$ is p.r.

b. $= x_i \cup x_j = \bigcup \{x_i, x_j\}$ is p.r.

c. $= \{\vec{x}\}$ or $\langle \vec{x} \rangle$

 [redacted]

d. If $f(y, \vec{x})$ is p.r., so is

$$g(y \vec{x}) = \langle f(z, \vec{x}) \mid z \in y \rangle ,$$

$$\left(\begin{array}{l} \text{since } g(y \vec{x}) = \bigcup_{z \in y} h(y, \vec{x}) , \\ \text{where } h(y, \vec{x}) = \langle f(y, \vec{x}), y \rangle \end{array} \right)$$

<u>Def</u> $R \subset V^n$ is p.r. iff there is a p.r. fcn $r : V^n \to V$ s.t.

$$R = \{ \langle \vec{x} \rangle \mid r(\vec{x}) \neq \emptyset \} .$$

(2)a. $\notin$ is p.r., since $y \notin x \longleftrightarrow \{y\} \setminus x \neq \emptyset$

b. If f, R are p.r., then so is:

$$g(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } R\vec{x} \\ \emptyset & \text{if not} \end{cases}$$

prf. Let $R\vec{x} \longleftrightarrow r(\vec{x}) \neq 0$

Then $g(\vec{x}) = \bigcup_{y \in r(\vec{x})} f(\vec{x})$.

Let χ_R be the characteristic fcn of R.

c. R is p.r. $\longleftrightarrow$ χ_R is p.r. (prf. 2b)

d. " $\longleftrightarrow$ $\neg R$ "

e. Let $f_i : V^n \to V$, $R_i \subset V^n$ be p.r. $(i = 1, \ldots, m)$; let R_i be disjoint s.t. $\bigcup_i R_i = V^n$.

Then f is p.r., where

$$f(\vec{x}) = f_i(\vec{x}) \text{ if } R\vec{x}.$$

proof.

Set $\bar{f}_i(\vec{x}) = \begin{cases} f_i & \text{if } R_i\vec{x} \\ \emptyset & \text{if not} \end{cases}$

Then $f(\vec{x}) = \bigcup_{i=1}^{m} f_i(\vec{x})$. QED

~~f. If R is p.r., so~~ is

$$\cancel{f(y, \vec{x}) = y \cap \{z \mid Rz\vec{x}\}}.$$

~~proof. $f(y, \vec{x}) = \bigcup_{z \in y} \{h(y,$~~ $\bigcup_{z \in y} \{f$

f. If $R y \vec{x}$ is p.r., so is

$$f(y, \vec{x}) = y \cap \{z \mid R z \vec{x}\}$$

proof. $f(y, \vec{x}) = \bigcup_{z \in y} h(z, \vec{x})$,

where $h(y \vec{x}) = \begin{cases} \{y\} & \text{if } R y \vec{x} \\ \emptyset & \text{if not} \end{cases}$

g. If R is p.r. and $\bigwedge \vec{x} \overset{=1}{\bigvee} y\, R y \vec{x}$,

so is

$$f(y \vec{x}) = \begin{cases} \iota z \in y\, R z \vec{x} & \text{if } \bigvee z \in y\, R z \vec{x} \\ 0 & \text{if not} \end{cases}$$

$$= \bigcup (y \cap \{z \mid R z \vec{x}\})$$

h. If $R y \vec{x}$ is p.r., so is $\bigvee z \in y\, R z \vec{x}$

i. If $R_i \vec{x}$ is p.r. $(i = 1, \ldots, m)$, so are

$$\bigvee_{i=1}^{m} R_i \vec{x}, \quad \bigwedge_{i=1}^{m} R_i \vec{x}$$

Hence, R is p.r. if R is Σ_0-definable without parameters other than ω.

(3) The following fcns are p.r.

(a) $(x)_i^n = \iota y \in h(x) \bigvee \vec{z} \in h(x) \, (x = \langle \vec{z} \rangle \wedge y = z_i)$.
$(i < n < \omega)$, where
$$h(x) = \bigcup x \cup \bigcup^2 x \cup \ldots \cup \bigcup^n x$$

(b) $x(z) = \iota y \in \bigcup^2 x \, (\langle y, z \rangle \in x \wedge$
$\wedge \bigwedge y' \in \bigcup^2 x \, (y' \neq y \to \langle y', z \rangle \notin x))$

(c) $\mathrm{dom}(x) = \bigcup^2 x \cap \{ y \mid \bigvee z \in \bigcup^2 x \; \langle z, y \rangle \in x \}$

(d) $\mathrm{rng}(x) = $ " $\cap$ " " $\langle y, z \rangle$ "

(e) $x \times y = \bigcup_{z \in x} \bigcup_{w \in y} \{ \langle z, w \rangle \}$

(f) $x \upharpoonright y = x \cap (\mathrm{rng}(x) \times y)$

(g) $x '' y = \mathrm{rng}(x \upharpoonright y)$

(h) $x^{-1} = h''(x \upharpoonright \mathrm{dom}(x))$,
where $h(z) = \langle (z)_1^2, (z)_0^2 \rangle$.

<u>Note</u> Up to now, we have made no use of the recursion schema, which, however is needed for:

(4) a. $C(x) = x \cup \bigcup_{y \in x} C(y)$ is p.r.

b. $S(x) = x \cup \{x\}$ is p.r.

c. $rn(x) = \bigcup_{y \in x} S\,rn(y)$ is p.r.

d. If $f(y, \vec{x})$, then so is

$g(z, y, \vec{x}) = f^{rn(z)}(y, \vec{x})$, where

we define:

$$f^{0}(y, \vec{x}) = y$$

$$f^{s\nu}(y, \vec{x}) = f(f^{\nu}(y, \vec{x}), \vec{x})$$

$$f^{\lambda}(y, \vec{x}) = \bigcup_{\nu < \lambda} f^{\nu}(y, \vec{x})$$

for $Lim(\lambda)$.

<u>Def</u> $f : On^{n} \longrightarrow On$ is p.r. iff f is the restriction of a p.r. function to On.

(4) e. $\alpha + \beta$, [redacted] $\alpha \cdot \beta$, α^{β}, ... etc. are p.r.

(<u>Note</u> We have as yet made no use of the constant fcn ω)

(4) f. There is a p.r. pairing function $\langle\ \rangle : On \longleftrightarrow On$ s.t. $\alpha, \beta \leq \langle \alpha, \beta \rangle$ for all α, β.

proof.

$\langle\ \rangle$ is Gödel's pairing function:

$$\langle\ \rangle : \langle On^2, <^* \rangle \xrightarrow{\sim} \langle On, < \rangle,$$

where $<^*$ is defined by:

$$\langle \alpha, \beta \rangle <^* \langle \gamma, \delta \rangle \underset{Df}{\longleftrightarrow} \max(\alpha, \beta) < \max(\gamma, \delta) \vee.$$

$$\vee. \max(\alpha, \beta) = \max(\gamma, \delta) \wedge \alpha < \gamma. \vee.$$

$$.\vee. \max(\alpha, \beta) = \max(\gamma, \delta) \wedge \alpha = \gamma \wedge \beta < \delta.$$

To show that $\langle\ \rangle$ is p.r., note that:

$$\langle 0, \beta \rangle = \sup_{\nu < \beta} \langle \nu, \nu \rangle$$

$$\langle \alpha, \beta \rangle = \langle 0, \beta \rangle + \alpha \quad \text{if } \alpha < \beta$$

$$\langle \beta, \alpha \rangle = \langle 0, \beta \rangle + \beta + \alpha \quad \text{if } \alpha \leq \beta.$$

Thus, we can define the function $\langle 0, \beta \rangle$ by the recursion

$$\langle 0, \beta \rangle = \sup_{\nu < \beta} (\langle 0, \nu \rangle + \nu \cdot 2),$$

and define the other cases as above.

QED

$l(\alpha)$, $r(\alpha)$ are the inverses of $\langle\ \rangle$ (i.e. $\langle l(\alpha), r(\alpha)\rangle = \alpha$). These are obviously p.r. by the fact that $l(\alpha), r(\alpha) \leq \alpha$.

<u>Ordered</u> n-<u>tuples</u> of ordinals are defined by iteration:

$$\langle \alpha_1, \dots, \alpha_n \rangle =_{Df} \langle \alpha_1, \langle \alpha_2, \dots, \alpha_n \rangle \rangle.$$

We also set: $\langle \alpha \rangle =_{Df} \alpha$.

The inverse functions $(\alpha)^n_i$ $(i < n < \omega)$ s.t. $(\langle \alpha_0, \dots, \alpha_{n-1}\rangle)^n_i = \alpha_i$ are defined by:

$$(\alpha)^n_i = \begin{cases} l r^i(\alpha) & \text{if } i < n-1 \\ r^{n-1}(\alpha) & \text{if not.} \end{cases}$$

Note that $\langle (\alpha)^n_i \mid \alpha \in On,\ i < n < \omega \rangle$ is the restriction of a p.r. function to a p.r. domain.

Def $\mathcal{U}$ is p.r. closed iff
$f''\mathcal{U}^n \subset \mathcal{U}$ for each p.r.f.

(5)a. If $\mathcal{U}$ is p.r. closed and transitive and $f: V^m \to V$ is defined by the sequence of schemata $S_1, \ldots, S_p$, then the same sequence relativized to $\mathcal{U}$ defines $f \upharpoonright \mathcal{U}^m$.
(We prove this by induction on p)

(5)a continues to hold if we replace "p.r. closed" by "closed under the functions p.r. in $f_1, \ldots, f_n$" for arbitrary $f_1, \ldots, f_n$.

. . . . .

If ~~U is~~ $\langle \mathcal{U}, \in \rangle$ satisfies the extensionality axiom (i.e. if $\mathcal{U} \cap x = \mathcal{U} \cap y \to x = y$ for $x, y \in \mathcal{U}$), then, by a lemma of Mostowski, there is exactly one transitive set $\mathfrak{v}$ such that $\langle \mathfrak{v}, \in \rangle$ is isomorphic to $\langle \mathfrak{U}, \in \rangle$. The isomorphism, which is also unique, is defined by:

$$\pi(x) = \pi''(x \cap \mathfrak{U}) \quad \text{for } x \in \mathfrak{U}.$$

If $\mathfrak{U}$ is p.r. closed, we may apply the following <u>condensation lemma</u>:

(5)b. If $\mathfrak{U}$ is p.r. closed and $\pi : \langle \mathfrak{U}, \in \rangle \xleftrightarrow{\sim} \langle \mathfrak{v}, \in \rangle$, where $\bigcup \mathfrak{v} \subset \mathfrak{v}$, then

$$\pi f(\vec{x}) = f(\pi(\vec{x}))$$

for all $\vec{x} \in \mathfrak{U}$ and all p.r. f.
(Corollary: $\mathfrak{v}$ is p.r. closed).

We prove (5)b by induction on the defining schemata of f. We display a sample case of this induction:

Let $f(y,x) = g(y,x,\langle f(z,x) \mid z \in y \rangle)$. Then
$\pi(f(y,x)) = g(\pi(y), \pi(x), \pi(\langle f(z,x) \mid z \in y \rangle))$, where
$\pi(\langle f(z,x) \mid z \in y \rangle) = \pi''\langle f(z,x) \mid z \in U \cap y \rangle =$
$= \langle f(\pi(z), \pi(x)) \mid z \in U \cap y \rangle = \langle f(z, \pi(x)) \mid z \in \pi(y) \rangle$.
Hence $\pi(f(y,x)) = f(\pi(y), \pi(x))$.

. . . . .

A trivial consequence of the condensation lemma is:

(5)c. If U is the p.r. closure of X and
$\overline{U}, \pi$ are as above, and $\pi \upharpoonright X = id \upharpoonright X$,
Then $\pi = id \upharpoonright U$ (hence $U = \overline{U}$).

proof. $\pi f(\vec{x}) = f(\vec{x})$ for $x_1, \ldots, x_n \in X$

Syntax and p.r. functions:

As before, we consider the 1-st order language with the predicates $\varepsilon, \equiv, \dot{A}_1, \dots, \dot{A}_n$, and the constants $\underline{x}$ $(x \in \mathbb{V})$. $\dot{A}_i$ is an m_i-place predicate. As before, the language includes bounded quantifiers $\bigwedge x{:}\varepsilon t$, $\bigvee x{:}\varepsilon t$.

$v_x = \langle x, 0\rangle$ is p.r.; hence:

$Vbl = \{v_i \mid i \in \omega\}$ is the value of a constant p.r. function

$\underline{x} = \langle x, 1\rangle$ is p.r., hence so is the function:

$Const_u = \{\underline{x} \mid x \in u\}$

The class $Const = \{\underline{x} \mid x \in \mathbb{V}\}$ is p.r., since:

$$x \in Const \longleftrightarrow x \in Const_{C(x)}.$$

$(x \varepsilon y)$, $(x \equiv y)$, $A_i \vec{x}$ $(i = 1, \dots, n)$ are p.r. functions; hence so is:

$$
\begin{aligned}
PFml_u = {} & \{ t \varepsilon t' \mid t, t' \in Vbl \cup Const_u \} \cup \\
& \cup \{ t \equiv t' \mid \quad \text{''} \quad \} \cup \\
& \cup \bigcup_{i=1}^{n} \{ A_i \vec{t} \mid t \in Vbl \cup Const_u \}.
\end{aligned}
$$

$PFml = \bigcup_u PFml_u$ is a p.r. class, since: $x \in PFml \leftrightarrow x \in PFml_{C(x)}$.

$(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$, $(x \leftrightarrow y)$, $\neg x$, $\bigwedge x y$, $\bigvee x y$, $\bigwedge x{:}\varepsilon y z$, $\bigvee x{:}\varepsilon y z$ are p.r.; hence so is:

$$
\begin{aligned}
F(v, u) = {} & \{ (x \wedge y) \mid x, y \in v \} \cup \\
& \cup \{ (x \vee y) \mid x, y \in v \} \cup \dots \cup \\
& \cup \{ \bigwedge x y \mid x \in Vbl \wedge y \in v \} \cup \\
& \cup \{ \bigwedge x{:}\varepsilon y z \mid x \in Vbl \wedge y \in Const_u \wedge z \in v \} \cup \\
& \cup \dots .
\end{aligned}
$$

Thus, $\mathrm{Fml}_u^\nu = F^\nu(\mathrm{PFml}_u, u)$ is p.r.

In particular, $\mathrm{Fml}_u = \mathrm{Fml}_u^\omega$ is p.r.

The class $\mathrm{Fml} = \bigcup_u \mathrm{Fml}_u$ is p.r.,

since: $x \in \mathrm{Fml} \leftrightarrow x \in \mathrm{Fml}_{C(x)}$.

Similarly, the function $\mathrm{Fml}_u^{\Sigma_0}$ and the class Fml^{Σ_0} are p.r.

The functions:

$$\mathrm{Fr}(\varphi) = \begin{cases} \text{the set of vbls. free in } \varphi \\ \text{if } \varphi \in \mathrm{Fml} \\ \emptyset \text{ if not} \end{cases}$$

$$s(\varphi, x, t) = \begin{cases} \varphi(x/t) \text{ if } x \in \mathrm{Vbl},\ t \in \mathrm{Const}, \\ \qquad\qquad \varphi \in \mathrm{Fml} \\ \emptyset \text{ if not} \end{cases}$$

are p.r., since both are obtainable by a recursion of the form:

$$f(\varphi, \vec{x}) = g(\varphi, \vec{x}, \langle f(z, \vec{x}) \mid z \in C(\varphi) \rangle).$$

Semantics and p.r. functions

Let Mod(m) mean that m is a model of [redacted] our language:

$$\mathrm{Mod}(m) \longleftrightarrow \bigvee u \bigvee a_1 \ldots a_n \left(\bigwedge_{i=1}^{n} a_i \subset u^{m_i} \wedge \right.$$

$$\left. \wedge\ m = \langle u, \in \cap u^2, a_1, \ldots, a_n \rangle \right).$$

Mod is a p.r. predicate, since the quantifiers can be bounded by $C(m)$

Set:

$$|m| = \begin{cases} (m)_0^{n+2} & \text{if } \mathrm{Mod}(m) \\ \emptyset & \text{if not} \end{cases}$$

(i.e. $|\langle u, \in \cap u^2, \ldots \rangle| = u$).

Let $\models_m \varphi$ mean that m is a model and φ is a true statement of the m-language.

Theorem $f(m) = \{\varphi \mid \models_m \varphi\}$ is p.r.

proof. We show that:

$$f(\nu, m) = \{\varphi \in Fml^\nu \mid \models_m \varphi\} \text{ is p.r.}$$

Then $f(m) = f(\omega, m)$.

Clearly, $g(m) = \{\varphi \in Fml^0 \mid \models_m \varphi\}$ is p.r.

Set:

$$h(v, u) = v \cup \{(\varphi \wedge \psi) \mid \varphi, \psi \in v\} \cup$$
$$\cup \{(\varphi \wedge \psi) \mid \varphi \in v \vee \psi \in v\} \cup \dots \cup$$
$$\cup \{\bigwedge x \varphi \mid \bigwedge y \in u \ \ \varphi(x/\underline{y}) \in v\} \cup$$
$$\cup \{\bigwedge x{:}\varepsilon \underline{z}\, \varphi \mid \bigwedge y \in u \cap z \ \ \varphi(x/\underline{y}) \in v\} \cup$$
$$\cup \dots$$

Then $f(\nu, m) = h^\nu(g(m), |m|)$.

QED

Similarly, $f(m) = \{\varphi \mid \models^{\Sigma_0}_{m} \varphi\}$ is p.r.

Let $A_i \subset V^{m_i}$ $(i = 1, \dots, n)$.

Let $\models^{\Sigma_0} \varphi$ mean that φ is a Σ_0-statement which is true in $\langle V, \in, A_1, \dots, A_n \rangle$.

<u>Corollary</u> The predicate $\models^{\Sigma_0}$ is p.r. in $A_1, \dots, A_n$

proof. Set:

$h(u) = \langle u, \in \cap u^2, A_1 \cap u^{m_1}, \dots, A_n \cap u^{m_n} \rangle$.

h is p.r. in $A_1, \dots, A_n$. But

$$\models^{\Sigma_0} \varphi \longleftrightarrow \models^{\Sigma_0}_{h(C(\varphi))} \varphi \qquad \text{QED}$$

The constructible hierarchy

Let $Def(\mathfrak{M})$ be the set of $x \subset |\mathfrak{M}|$ definable by an $\mathfrak{M}$-formula with one free vbl.

Def is a p.r. function, since

$$Def(\mathfrak{M}) = h''Fml_{\mathfrak{M}}, \text{ where}$$

$$h(\varphi) = |\mathfrak{M}| \cap \{x \mid \models_{\mathfrak{M}} \varphi(v_0/\underline{x})\}.$$

Let $A_i \subset V^{m_i}$ $(i = 1, \ldots, m)$.

Set: $\widetilde{Def}(u) = Def(\langle u, \in \cap u^2, A_1 \cap u^{m_1}, \ldots, A_m \cap u^{m_m} \rangle)$

The constructible hierarchy over u relative to $A_1, \ldots, A_m$ is defined by:

$$L_\nu[u; A_1, \ldots, A_m] = \widetilde{Def}^\nu(C(u)).$$

Clearly, $\langle L_\nu[u; \vec{A}] \mid u \in V, \nu \in On \rangle$ is a p.r. function.

We set: $L[u; \vec{A}] = \bigcup_{\nu \in On} L_\nu[u, \vec{A}]$

Write '$L_\alpha[u]$' for '$L_\alpha[u, \vec{A}]$'.

Let $\bigcup u \subset u$. Let 'p.r.' stand for 'p.r. in $\vec{A}$' (i.e. in the characteristic functions of $\vec{A}$).

By induction on β, we get:

<u>Thm</u> If $y \in L_{\beta+1}[u]$, then y is definable in $\langle L_\beta[u], \in, \vec{A} \rangle$ by a formula containing only the constants $\underline{x}$ $(x \in u)$, L_ν $(\nu < \beta)$.

<u>Corollary</u> For all α, if $y \in L_\alpha[u]$, then y ~~is definable in~~ has the form:

$$y = \{ z \in L_\beta \mid \models \varphi(\underline{z}) \},$$

where $\beta < \alpha$ and φ is a Σ_0 formula containing only constants $\underline{x}$ $(x \in u)$ and L_ν $(\nu < \alpha)$.

 [redacted]

Since $\models^{\Sigma_0}$ is p.r. and $\langle L_\nu[u] \mid \nu \in On \rangle$ is p.r. in the parameter u, it follows that, whenever $L_\alpha[u]$ is p.r. closed, then $L_\alpha[u]$ is the p.r. closure of $u \cup \{u\} \cup \alpha$. If we suppose that u is closed under the formation of finite sets (i.e. $\vec{x} \in u \to \{\vec{x}\} \in u$), then we may sharpen this to:

<u>Thm</u> Let u be closed under finite subsets. Then there is a function g which is p.r. in the parameter u s.t. $L[u] = g''(u \times On)$. (Hence, if $L_\alpha[u]$ is p.r. closed, then $L_\alpha[u] = g''(u \times \alpha)$).

proof.
Using the [redacted] maps $\prec \succ : On^2 \longleftrightarrow On$ ~~[illegible]~~ and the inverse functions $(\alpha)_i^n$ introduced in [redacted] (4)f., we define:

If $x = \langle \varphi, n \rangle$, where $\varphi \in \mathrm{Fml}_{U}^{\Sigma_0}$ and $n < \omega$, and if $\alpha = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$, we set:

$$g(x, \alpha) = \left\{ z \in L_{\alpha_0} \;\middle|\; \models \varphi \left({}^{v_0 \cdots v_{n-1}} \big/ {}_{z \, \underline{L_{\alpha_1}} \cdots \underline{L_{\alpha_{n-1}}}} \right) \right\}.$$

(Note that $\langle n, \varphi \rangle \in U$ by closure of U under finite subsets). In all other cases we set: $g(x, \alpha) = \emptyset$. QED

Characterizing p.r. closed ordinals

Call an ordinal α p.r. closed iff $f``\alpha^n \subset \alpha$ for every p.r. $f: On^n \to On$. In the following, we attempt to characterize the class of p.r. closed ordinals. We shall also show that, if α is p.r. closed, then so are V_α and L_α.

Define a sequence of functions $a_m : On \to On$ $(m < \omega)$ as follows:

$$a_0(\alpha) =_{Df} \alpha + 1 \quad ; \quad a_{m+1}(\alpha) =_{Df} a_m^{\alpha+2}(\alpha).$$

(In ordinary recursion theory, $a_m \upharpoonright \omega$ is known as the m-th Ackermann branch. $a = \langle a_n(m) \mid n, m < \omega \rangle$ is known as the Ackermann function. Ackermann's discovery that a, though effective, is not p.r. led to the theory of the general recursive functions.) The maps a_m are obviously p.r. and are defined without the use of schema (iv) (the constant fcn ω).

We can easily show that each a_n is strictly monotone and that

$d < a_0(d) < a_1(d) < \cdots < a_n(d) < \cdots$.

Lemma 1. Set : $|x_1, \dots, x_m| =_{Df} \max(\mathrm{rn}(x_1), \dots, \mathrm{rn}(x_m))$

If f is p.r. without using ω (i.e. obtainable from schemata (i) – (iii), (v) – (viii) alone), then there is an n s.t.

$$|f(\vec{x})| < a_n(|\vec{x}|) \quad \text{for all } \vec{x}.$$

Lemma 1 is proved by induction on the defining schemata of f. For this purpose it is convenient to replace schema (vii) by:

(vii)′ $f(y, \vec{x}) = g(\vec{x}, \bigcup_{z \in y} f(z, \vec{x}))$.

That (vii)′ is sufficient may be shown as follows:

Let $f(y, x) = h(y, x, \langle f(z, x) \mid z \in y \rangle)$.

Set $\bar{f}(y, x) =_{Df} \{\langle f(y, x), y \rangle\}$. Then $\bar{f}$ is definable by the recursion:

$$\bar{f}(y,x) = \{\langle h(y,x,\bigcup_{z\in y}\bar{f}(z,x)),\ y\rangle\}$$
$$= g'(y,x,\bigcup_{z\in y}\bar{f}(z,x)).$$

Replacing the first occurrence of 'y' on the right side by '$\mathrm{dom}(\bigcup_{z\in y}\bar{f}(z,x))$', we arrive at an equation of the form:

$$\bar{f}(y,x) = g(x,\bigcup_{z\in y}\bar{f}(z,x)).$$

We display a typical case of the induction. Let $|g(x,v)| < a_m(|x,v|)$ and let $f(y,x) = g(x,\bigcup_{z\in y}f(z,x))$.

Then by induction on y we get:

$$|f(y,x)| < a_m^{|y|+1}(|x|),$$

since, assuming it to hold for $z\in y$, we have:

$$|f(y,x)| < a_m\Big(|x, \bigcup_{z\in y} f(z,x)|\Big)$$

$$\leq a_m\Big(\big||x|, \sup_{\nu<|y|} a_m^{\nu+1}(|x|)\big|\Big)$$

$$= a_m\Big(\big||x|, a_m^{|y|}(|x|)\big|\Big)$$

$$= a_m^{|y|+1}(|x|).$$

Hence, $|f(y,x)| < a_m^{|y,x|+2}(|y,x|) = a_{m+1}(|y,x|)$.

QED

Corollary 2

(a) α is p.r. closed iff α is closed under each of the Ackermann branches a_m $(m<\omega)$.

(b) V_α is p.r. closed iff α is p.r. closed.

Note that if $f_1,\ldots,f_m$ are arbitrary functions s.t. $\bigvee m \bigwedge \vec{x}\ |f_i(\vec{x})| < a_m(|\vec{x}|)$, $(i=1,\ldots,m)$, then the proof of Lemma 1 goes through for the class of functions p.r. in $f_1,\ldots,f_m$. Hence:

Corollary 3 Let $f_1,\ldots,f_m$ satisfy the conclusion of Lemma 1. Then Lemma 1 + Corollary 2 continue to hold if we substitute 'p.r. in $f_1,\ldots,f_m$' for 'p.r.'.

Corollary 3 applies, in particular, when $f_1, \ldots, f_n$ are the characteristic functions of classes $A_1, \ldots, A_n$.

. . . .

Now let $A_1, \ldots, A_n$ be fixed and write 'p.r.' for 'p.r. in $A_1, \ldots, A_n$'. Let u be a transitive set. For which ordinals $\alpha > 0$ is $L_\alpha[u] = L_\alpha[u; A_1, \ldots, A_n]$ p.r. closed? Obviously, if α is such an ordinal, then α is p.r. closed and $\alpha > rn(u)$. These conditions are, in fact, sufficient. To show this, we employ the following <u>stability</u> <u>lemma</u> due to Carol Karp:

Lemma 4 (Stability Lemma) If f is p.r., then there is a Σ_0-formula $\varphi_f(z, y, \vec{x})$ (containing no constants other than $\underline{\omega}$) and a p.r. normal function $\bar{f} : On \to On$ s.t.

(a) $y = f(\vec{x}) \longleftrightarrow V_z \models \varphi_f(\underline{z}, \underline{y}, \underline{\vec{x}})$

(b) If u is transitive, $\vec{x} \in L_{\bar{f}(\alpha)}[u]$, and $|\vec{x}| < \bar{f}(\alpha)$, then:

(i) $f(\vec{x}) \in L_{\bar{f}(\alpha)}[u]$

(ii) $y = f(\vec{x}) \longleftrightarrow V_z \in L_{\bar{f}(\alpha)}[u] \models \varphi_f(\underline{z}, \underline{y}, \underline{\vec{x}})$

Lemma 4 is proved by induction on the defining schemata of f. We display a sample case of the induction:

Let $g(x, v)$ satisfy the conclusion of Lemma 4 and let

$$f(y, x) = g(x, \langle f(z, x) \mid z \in y \rangle).$$

Let $\varphi_q(z, w, x, v)$, $\bar{q}$ be as in Lemma 4. Without loss of generality we may make the further assumption:

$$(*)\quad |x|, |y| < q(\alpha) \to |\langle f(z,x) \mid z \in y\rangle| < q(\alpha)$$

This is possible by Lemma 1, since $f'(y,x) =_{\mathrm{df}} \langle f(z,x) \mid z \in y\rangle$ is a p. r. function.

Let $\psi(h, v, x)$ be the formula:

$$\mathrm{Fcn}(h) \wedge \bigcup \mathrm{dom}(h) \subset \mathrm{dom}(h) \wedge$$
$$\wedge \bigwedge y \in \mathrm{dom}(h) \bigvee z \in v\ \varphi_q(z, h(y), x, h\restriction y$$

ψ is a Σ_0 formula. Moreover, if $\models \psi(\underline{h}, \underline{v}, \underline{x})$, then $h(y) = f(y,x)$ for all $y \in \mathrm{dom}(h)$.

Define a p.r. function ~~by~~ $\bar{f}(\alpha,\beta)$ by:

$$\bar{f}(0,\beta) = \beta$$

$$\bar{f}(\alpha+1,\beta) = \bar{g}\,(\bar{f}(\alpha,\beta)+1)$$

$$\bar{f}(\lambda,\beta) = \sup_{\nu<\lambda} \bar{f}(\nu,\beta) \quad \text{for } Lim(\lambda).$$

<u>Claim</u> If $|y| < \alpha$; $|x| < \beta$; $y, x \in L_\beta$, ~~then there~~ and $\delta = \bar{f}(\alpha,\beta)$, then there is an $h \in L_\delta$ s.t.

$$y \in dom(h) \wedge \models \Psi(\underline{h}, \underline{L_\delta}, \underline{x}).$$

(For the sake of brevity, we are writing 'L_β' in place of '$L_\beta[u]$').

proof. by induction on α.

If $\alpha = 0$ or α is a limit ordinal, the proof is trivial. Now let the claim hold for α and let $|y| = \alpha+1$. ~~[illegible]~~; Let $\delta = \bar{f}(\alpha,\beta)$ and

~~$h = $ [illegible]~~ set:

$$h^* =_{nf} \bigcup \{h \in L_\delta \mid \models \Psi(\underline{h}, \underline{L_\delta}, \underline{x})\}.$$

Then $h^* \in L_{\delta+1}$ and $y \subset \mathrm{dom}(h^*)$;

hence $h^* \upharpoonright y = \langle f(z,x) \mid z \in y \rangle$.

By (*), we conclude: $|h^* \upharpoonright y| < \bar{g}(\delta+1)$.

Since $h^* \upharpoonright y \in L_{\bar{g}(\delta+1)}$, we have:

$$f(y,x) = g(x, h^* \upharpoonright y) \in L_{\bar{g}(\delta+1)}$$

$$\forall z \in L_{\bar{g}(\delta+1)} \models \varphi_g(\underline{z}, \underline{f(y,x)}, \underline{h^* \upharpoonright y}).$$

Thus, setting $\bar{h} = h^* \cup \{\langle f(y,x), y\rangle\}$, we get:

$$y = \mathrm{dom}(\bar{h}) \wedge \models \Psi(\underline{h^*}, \underline{L_{\bar{g}(\delta+1)}}, \underline{x}).$$

But $\bar{g}(\delta+1) = \bar{f}(\alpha+1, \beta)$. This proves the claim.

Now let $\bar{f}(\alpha)$ enumerate the ordinals closed under $\bar{f}(\alpha,\beta)$. We obtain this by setting $\tilde{f}(\alpha) =_{nf} \bar{f}(\alpha,\alpha)$ and defining:

$$\bar{f}(0) = 0;\quad \bar{f}(\alpha+1) = \tilde{f}^{\omega}(\bar{f}(\alpha)+1);$$

$$\bar{f}(\lambda) = \sup_{\nu<\lambda} \bar{f}(\nu) \text{ for limit } \nu.$$

Let $\varphi_f(v,z,y,x)$ be the formula:

$$\bigvee h \in v\,(\psi(h,v,x) \wedge \langle z,y\rangle \in h).$$

Then φ_f, $\bar{f}$ have the required properties. QED

Corollary 5 Let $\alpha > 0$. $L_\alpha[u; A_1, \ldots, A_n]$ is closed under functions p.r. in $A_1, \ldots, A_n$ iff α is p.r. closed and $rn(u) < \alpha$.

Corollary 6 If f is p.r. in $A_1, \ldots, A_n$, then there is a Σ_0 formula φ_f and a p.r. function $\tilde{f}$ s.t.

(i) $\bigwedge \vec{x}\;\; f(\vec{x}) \in \tilde{f}(\vec{x})$

(ii) $\bigwedge \vec{x}\, y\;\; (y = f(\vec{x}) \longleftrightarrow \bigvee z \models \varphi_f(z,y,\vec{x}))$

$$\longleftrightarrow \bigvee z \in \tilde{f}(\vec{x}) \models \varphi_f(z,y,\vec{x}))$$

Thus f is Δ_1 in every domain which is closed under relations p.r. in $A_1, \ldots, A_n$

proof. Set $\tilde{f}(\vec{x}) = L_{\bar{f}(|\vec{x}|)}[C(\{\vec{x}\})]$, where $\bar{f}$ is as in Lemma 4.

Admissible sets and p.r. functions

If $M = \langle |M|; \in, A_1, \ldots, A_m \rangle$ is admissible, then the set of its Δ_1 maps $f : M^n \to M$ is obviously closed under the schemata (i)–(viii) (assuming $\omega \in M$). Thus, in particular, $|M|$ ~~[illegible]~~ p.r. closed. Admissibility is, however, a stronger condition than p.r. closure, since M will contain numerous p.r. closed elements. To see this, note that the function $\langle a_m(\nu) \mid m < \omega, \nu \in On_M \rangle$ is Δ_1 (the proof is left to the reader). Hence, since $\omega \in M$, the function

$$p(\alpha) = \sup_{m<\omega} a_m(\alpha)$$

is Δ_1. But $p(\alpha)$ is p.r. closed since, if $\beta < p(\alpha)$, then $\beta < a_m(\alpha)$ for some n, which means that $a_m(\beta) < a_{m+n}(\alpha) < p(\alpha)$ for $m < \omega$. The monotone enumeration of the p.r. closed ordinals is defined by: $q(0) = p(\omega)$; $q(\alpha+1) = p\,q(\alpha)$;

$$q(\lambda) = \sup_{\nu<\lambda} q(\nu) \quad \text{for } \nu < \lambda.$$

q is clearly Δ_1. If $u \in M$, then $\langle L_{q(\alpha)}[u] \mid \alpha > rn(u) \rangle$ is a Δ_1 heirarchy of p.r. closed sets.

<u>Def</u> An ordinal α is called <u>admissible</u> iff L_α is admissible.

α is called admissible <u>in</u> $\vec{A}$ iff $L_\alpha[\emptyset; \vec{A}]$ is admissible.

α is called admissible <u>over</u> u (in $\vec{A}$) iff $L_\alpha[u]$ $(L_\alpha[u; \vec{A}])$ is admissible.

It is apparent from the foregoing that the admissible ordinals are considerably sparser than the p.r. closed ordinals. Moreover, the set of admissible ordinals is not closed. To see this, note that the predicate 'α is admissible' is Δ_1 in any admissible set M. Let γ be the limit of the first ω admissibles. If L_γ were admissible, there would be a Δ_1 enumeration of the set of smaller ~~a adm~~

admissibles which, since it is unbounded in γ, would have to be of order type γ. Contradiction!

A recursion Theorem for p.r. functions

Def. Let h be a p.r. function and let the relation $x \in h(y)$ be well founded. h is <u>manageable</u> iff there is a p.r. $\sigma : V \to On$ s.t.

$$x \in h(y) \longrightarrow \sigma(x) < \sigma(y).$$

(<u>Note</u> Many unmanageable h exist. Fr. ins. let R be a recursive well ordering of ω having order type $>$ the first p.r. closed ordinal. Set $h(x) = \{n \mid n R x\}$).

<u>Theorem</u> If g is p.r. and h is manageable, then f is p.r., where f is defined by:

$$f(y, \vec{x}) = g(y, \langle f(z, \vec{x}) \mid z \in h(y) \rangle).$$

(<u>Note</u> Virtually the same proof yields the stronger theorem:
If g, h are p.r. and if σ is p.r. s.t. $z \in h(y \vec{x}) \to \sigma(z) < \sigma(y)$ for all $z, y, \vec{x}$, then f is p.r. where:

$$f(y, \vec{x}) = g(y, \langle f(z, \vec{x}) \mid_z z \in h(y, \vec{x}) \rangle)).$$

proof.

Define: $|x| = \sup_{y \in h(x)} (|y|+1)$.

~~Set~~ Then $|x| \leq \sigma(x)$.

Set:

$$\Theta(z, \vec{x}; u) = \bigcup_{\substack{y \in u \\ h(y) \subset \mathrm{dom}(z)}} \{\langle g(y, \vec{x}, z \upharpoonright h(y)), y\rangle\}$$

By induction on α, if u is h-closed (i.e. $x \in u \to h(x) \subset u$), then

$$\Theta^{\alpha}(\emptyset, \vec{x}, u) = \langle f(y, \vec{x}) \mid y \in u \wedge |y| < \alpha \rangle.$$

Set:

$h^*(v) =$ the h-closure of v

$= \tilde{h}^{\omega}(v)$, where

$$\tilde{h}(v) = v \cup \bigcup_{y \in v} h(y).$$

<u>Then</u> $f(y, \vec{x}) = \Theta^{\sigma(y)+1}(\emptyset, \vec{x}, h^*(\{y\}))\ (y)$.

QED