§1 Admissible sets

Let $H = H_\omega$ be the collection of hereditarily finite sets. We use the usual Levy hierarchy of set theoretic formulae:

$\Pi_0 = \Sigma_0$ = fmlae in which all quantifiers are bounded.

Σ_{n+1} = fmlae $\bigvee x\, \varphi$ where φ is Π_n

Π_{n+1} = " $\bigwedge x\, \varphi$ " φ is Σ_n.

The use of H offers an elegant way to develop ordinary recursion theory. Call a relation $R \subset H^n$ r.e. (or "H-r.e.") iff R is Σ_1-definable over H. We call R recursive (or H-recursive) iff it is Δ_1-definable (i.e. R and its complement $\neg R$ are Σ_1-definable). Then $R \subset \omega^n$ is rec (r.e.) in the usual sense iff it is the restriction of an H-rec. (H-r.e.) relation to ω. Moreover, there is an H-recursive function $\pi : \omega \leftrightarrow H$ s.t. $R \subset H^n$ is H-recursive iff $\{\langle x_1, \ldots, x_n\rangle \mid R(\pi(x_1), \ldots, \pi(x_n))\}$ is recursive.

(Hence $\{\langle x,y\rangle \mid \pi(x) \in \pi(y)\}$ is recursive.)

This suggests a way of relativizing the concepts of recursion theory to transfinite domains:

Let $N = \langle |N|, \in, A_1, A_2, \ldots \rangle$ be a transitive structure (with finitely or infinitely many predicates). We define:

$R \subset N^n$ is N-r.e. (N-rec.) iff

iff R is Σ_1 (Δ_1) definable over N.

Since N may contain infinite sets, we must also relativize the notion "finite":

u is N-<u>finite</u> iff $u \in N$.

There are, however, certain basic properties which we expect any recursion theory to possess. In particular:

- If A is recursive and u finite, then $A \cap u$ is finite
- If u is finite and $F: u \to N$ is recursive, then $F''u$ is finite.

The transitive structures $N = \langle |N|, \in, A_1, A_2, \ldots \rangle$ which yield a satisfactory recursion theory are called <u>admissible</u>. They were characterized by Kripke and Platek as those which satisfy the following axioms:

(1) $\emptyset, \{x, y\}, \bigcup x$ are sets

(2) The Σ_0-axiom of subsets (Aussonderung)

$$x \cap \{z \mid \varphi(z)\} \text{ is a set,}$$

where φ is any Σ_0 formula.

(3) The Σ_0-axiom of collection

$$\bigwedge x \bigvee y\, \varphi(x,y) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x,y)$$

where φ is any Σ_0 formula.

<u>Note</u> Applying (3) to: $x \in u \to \varphi(x,y)$, we get:

$$\bigwedge x \in u \bigvee y\, \varphi(x,y) \to \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x,y).$$

<u>Note</u> <u>Kripke-Platek set theory</u> (KP) consists of the above axioms together with the axiom of extensionality and the full axiom of foundation (i.e. for all formulae, not just Σ_0 ones). These latter axioms of course hold trivially in transitive domains.

KPC (KP with choice) is KP augmented by: Every set is enumerable by an ordinal.

We now show that admissible structures satisfy the criteria stated above.

<u>Lemma 1</u> Let $u \in M$. Let A be $\underline{\Delta}_1(M)$. Then $A \cap u \in M$.

prf.

Let $Ax \leftrightarrow \bigvee y\, A_0 y x$, $\neg Ax \leftrightarrow \bigvee y\, A_1 y x$,

where A_0, A_1 are Σ_0. Then

$\bigwedge x \bigvee y\, (A_0 y x \vee A_1 y x)$. Hence there is $v \in M$

s.t. $\bigwedge x \in u \bigvee y \in v\, (A_0 y x \vee A_1 y x)$. Hence

$u \cap A = u \cap \{x \mid \bigvee y \in v\, A_0 y x\} \in M$. QED

Before verifying the second criterion we prove:

<u>Lemma 2</u> M satisfies:

$$\bigwedge x \in u \bigvee y_1 \dots y_n\, \varphi(x, \vec{y}) \rightarrow$$
$$\rightarrow \bigvee v \bigwedge x \in u \bigvee y_1 \dots y_n \in v\, \varphi(x, \vec{y})$$

for Σ_0 φ.

proof.

Assume $\bigwedge x \in u \bigvee y_1 \dots y_n\, \varphi(x, \vec{y})$. Then

$$\bigwedge x \bigvee w\, (\underbrace{x \in u \rightarrow \bigvee y_1 \dots y_n \in w\, \varphi(x, \vec{y})}_{\Sigma_0})$$

Hence there is $v' \in M$ s.t.

$\bigwedge x \in u \bigvee w \in v' \bigvee y_1 \dots y_n \in w\, \varphi(x, \vec{y})$.

Take $v = \bigcup v'$. QED (Lemma 2)

Finally we get:

<u>Lemma 3</u> Let $u \in M$, $u \subset \mathrm{dom}(F)$, where F is $\underline{\Sigma}_1(M)$. Then $F''u \in M$.

proof.
Let $y = F(x) \longleftrightarrow \bigvee z\, F'zyx$, where F' is $\underline{\Sigma}_0(M)$. Since $\bigwedge x \in u \bigvee y\; y = F(x)$, there is v s.t. $\bigwedge x \in u \bigvee y, z \in v\; F'zyx$.
Hence $F''u = v \cap \{y \mid \bigvee x \in u \bigvee z \in v\; F'zyx\}$.

QED (Lemma 3)

By similarly straightforward proofs we get;

<u>Lemma 4</u> If $Ry\vec{x}$ is Σ_1, so is $\bigvee y\, Ry\vec{x}$

<u>Lemma 5</u> If $Ry\vec{x}$ is Σ_1, so is $\bigwedge y \in u\, Ry\vec{x}$
(since $\bigwedge y \in u \bigvee z\, \varphi(y,z) \leftrightarrow \bigvee v \underbrace{\bigwedge y \in v \bigvee z \in v\; \varphi(y,z)}_{\Sigma_0}$)

<u>Lemma 6</u> If $R, Q \subset M^n$ are Σ_1, then so are $R \cup Q$, $R \cap Q$

<u>Lemma 7</u> If $R(y_1, \ldots, y_n)$ is Σ_1 and $f_i(x_1, \ldots, x_m)$ is a Σ_1 function for $i = 1, \ldots, n$ then $R(f_1(\vec{x}), \ldots, f_n(\vec{x}))$ is Σ_1.

proof
$R(\vec{f}(\vec{x})) \longleftrightarrow \bigvee y_1 \ldots y_n \left(\bigwedge_{i=1}^{n} y_i = f_i(\vec{x}) \wedge R(\vec{y})\right)$

Note The boldface versions of Lemmas 4 – 7 follow immediately.

Corollary 8 If the functions $f(z_1,\dots,z_n)$, $g_i(\vec{x})$ $(i=1,\dots,n)$ are Σ_1 in a parameter p, then so is $h(\vec{x}) \simeq f(g_1(\vec{x}),\dots,g_m(\vec{x}))$.

Lemma 9 The following functions are Δ_1:
$\bigcup x$, $x\cup y$, $x\cap y$, $x\setminus y$ (set difference), $\{x_1,\dots,x_n\}$, $\langle x_1,\dots,x_n\rangle$, $\mathrm{dom}(x)$, $\mathrm{rng}(x)$, $x''y$, $x\restriction y$, x^{-1}, $x\times y$, $(x)_i^n$, where:
$(\langle z_0,\dots,z_{n-1}\rangle)_i = z_i$; $(u)_i^n = \emptyset$ otherwise;

$$x[z] = \begin{cases} x(z) \text{ if } x \text{ is a function and } z\in \mathrm{dom}(x) \\ \emptyset \text{ if not} \end{cases}$$

Note As a corollary of Lemma 3 we have: If f is Σ_1, $u\in M$, $u\subset \mathrm{dom}(f)$, then $f\restriction u\in M$, since $f\restriction u = g''u$, where $g(x)\simeq \langle f(x), x\rangle$.

Lemma 10 If $f: M^{n+1}\to M$ is Σ_1 in the parameter p, then so are:
$F(u,\vec{x}) = \{f(z,\vec{x}) \mid z\in u\}$, $F'(u,\vec{x}) = \langle f(z,\vec{x}) \mid z\in u\rangle$.

proof.
$$y = F(u,\vec{x}) \longleftrightarrow \bigwedge z\in y \bigvee v\in u \; z = f(y,\vec{z}) \wedge \bigwedge v\in u \bigvee z\in y \; z = f(y,\vec{x}).$$

But $F'(u,\vec{x}) = \{f'(z,\vec{x}) \mid z\in u\}$, where $f'(y,\vec{x}) = \langle f(y,\vec{x}), \vec{x}\rangle$. QED

(Note If f is as in Lemma 10, then $F'(u,\vec{x}) = \langle f(y,\vec{x}) \mid y \in u \rangle$ is Σ_1 in p, since F' is obtained by applying Lemma 10 to $f'(y,\vec{x}) = \langle f(y,\vec{x}), y \rangle$.)

(Note The proof of Lemma 10 shows that, even if f is not defined everywhere, F is Σ_1 in p, where:

$$F(u,\vec{x}) \simeq \{ f(y,\vec{x}) \mid y \in u \},$$

where this equation means that $F(u,\vec{x})$ is defined and has the displayed value iff $f(y,\vec{x})$ is defined for all $y \in u$. Similarly for F'.)

Lemma 12 (Set Recursion Axiom)

Let G be an $n+2$-ary Σ_1 function in the parameter p. Then there is F which is also Σ_1 in p s.t.

$$F(y,\vec{x}) \simeq G(y,\vec{x},\langle F(z,\vec{x}) \mid z \in y \rangle)$$

(where this equation means that F is defined with the displayed value iff $F(z,\vec{x})$ is defined for all $z \in y$ and G is defined at $\langle y, \vec{x}, \langle F(z,\vec{x}) \mid z \in y \rangle \rangle$.)

prf.

Set $u = F(y,\vec{x}) \longleftrightarrow \bigvee f\,(\varphi(f,\vec{x}) \wedge \langle u, y \rangle \in f)$

where

$$\varphi(f,\vec{x}) \longleftrightarrow (f \text{ is a function} \wedge \bigcup \mathrm{dom}(f) \subset \mathrm{dom}(f) \wedge \bigwedge y \in \mathrm{dom}(f)\; f(y) = G(y,\vec{x}, f \upharpoonright y))$$

The equation is verified by $\in$-induction on y. QED (Lemma 12)

Corollary 13 TC, rn are Δ_1 functions, where
$TC(x)$ = the transitive closure of $x = x \cup \bigcup_{z \in x} TC(z)$
$rn(x)$ = the rank of $x = lub\{rn(z) \mid z \in x\}$

Lemma 14 ω, $On \cap M$ are Σ_0 classes

prf.
$x \in On \leftrightarrow (\bigcup x \subset x \wedge \bigwedge z, w \in x\, (z \in w \vee w \in z))$
$x \in \omega \leftrightarrow (x \in On \wedge \neg\, Lim(x))$
where $Lim(x) \leftrightarrow (x \neq 0 \wedge x \in On \wedge x = \bigcup x)$.

Corollary 15 The ordinal functions $\alpha+1$, $\alpha+\beta$, $\alpha \cdot \beta$, α^β, ... are Δ_1

An even more useful version of Lemma 12 is

Lemma 16 Let G be as in Lemma 12. Let $h: M \to M$ be Σ_1 in p s.t. $\{\langle x, z\rangle \mid x \in h(z)\}$ is well founded. There is F which is Σ_1 in p s.t.
$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) \mid z \in h(y)\rangle)$$
The proof is just as before.

We also note:

Lemma 17.1 Let $u \in H_\omega$. Then the class u and the constant function $f(x) = u$ are Σ_0

prf. $\in$-induction on u:
$$x \in u \leftrightarrow \bigvee_{z \in u} x = z$$
$$x = u \leftrightarrow (\bigwedge z \in x\; z \in u \wedge \bigwedge_{z \in u} z \in x). \text{ QED}$$

Lemma 17.2. If $\omega \in M$, then the constant function $x = \omega$ is Σ_0.

prf.

$$x = \omega \leftrightarrow (\bigwedge z \in x \; z \in \omega \wedge \emptyset \in x \wedge \bigwedge z \in x \; z \cup \{z\} \in x)$$

Lemma 17.3 If $\omega \in M$, the constant fcn $x = H_\omega$ is Σ_1 (hence Δ_1).

prf.

$$x = H_\omega \leftrightarrow (\bigwedge z \in x \bigvee u \bigvee f \bigvee m \in \omega (\bigcup u \subset u \wedge x \subset u \wedge \wedge f : m \leftrightarrow x)) \wedge$$
$$\wedge \emptyset \in x \wedge \bigwedge z, w \in x (\{z, w\}, z \cup w \in x)$$

Lemma 18 Fin, $\mathbb{P}_\omega(x)$ are Δ_1, where $\mathrm{Fin} = \{x \in M \mid \bar{\bar{x}} < \omega\}$, $\mathbb{P}_\omega(x) = \mathrm{Fin} \cap \mathbb{P}(x)$

prf.

$$x \in \mathrm{Fin} \leftrightarrow \bigvee n \in \omega \bigvee f \; f : n \leftrightarrow x$$
$$x \notin \mathrm{Fin} \leftrightarrow \bigvee y (y = \omega \wedge \bigwedge n \in y \bigvee f \bigvee u \subset x \; f : n \leftrightarrow u)$$
$$y = \mathbb{P}_\omega(x) \leftrightarrow \bigwedge u \in y (u \in \mathrm{Fin} \wedge u \subset x) \wedge$$
$$\wedge \bigwedge z \in x (\{z\} \in y \wedge \bigwedge u, v \in y \; u \cup v \in y)$$

QED

The constructible hierarchy relative to a class A is defined by:

$L_0[A] = \emptyset$; $L_{\nu+1}[A] = \mathrm{Def}(\langle L_\nu[A] \quad L_\nu[A]\rangle)$

$L_\lambda[A] = \bigcup_{\nu<\lambda} L_\nu[A]$ for limit λ,

where $\mathrm{Def}(\mathfrak{A})$ is the set of $B \subset \mathfrak{A}$ which are $\mathfrak{A}$-definable in parameters from $\mathfrak{A}$.

We also define $L_\nu = L_\nu[\emptyset]$.

The constructible hierarchy <u>over a set</u> u is defined by:

$L_0(u) = TC(\{u\})$, $L_{\nu+1}(u) = \mathrm{Def}(L_\nu(u))$,

$L_\lambda(u) = \bigcup_{\nu<\lambda} L_\nu(u)$ for limit λ.

It is easily seen that:

<u>Lemma 19</u>. If $A \subset M$ is $\Delta_1(M)$ in p, then $\langle L_\nu[A] \mid \nu \in M\rangle$ is $\Delta_1(M)$ in p.

• If $u \in M$, then $\langle L_\nu(u) \mid \nu \in M\rangle$ is $\Delta_1(M)$ in u.

By set recursion we can also define a sequence $\langle <_\nu^A \mid \nu < \infty\rangle$ s.t.

• $<_\nu^A$ well orders $L_\nu[A]$

• $<_\mu^A$ end extends $<_\nu^A$ for $\nu \leq \mu$.

Then:

<u>Lemma 20</u> ~~If~~ $A \in M$ is $\Delta_1(M)$ in p, then $\langle <_\nu^A \mid \nu \in M\rangle$ is $\Delta_1(M)$ in p.

<u>Def</u> $L_\nu^A = \langle L_\nu[A], A \cap L_\nu[A] \rangle$.

$\langle L_\nu^A, B_1, B_2, \ldots \rangle = \langle L_\nu[A], A \cap L_\nu[A], B_1, B_2, \ldots \rangle$

It follows easily that:

<u>Lemma 21</u> Let $M = \langle L_\alpha^A, B_1, \ldots \rangle$ be admissible. Then $<_M =_{Df} \bigcup_{\nu<\alpha} <^A$ is a $\Delta_1(M)$ well ordering of M. Moreover, there is a $\Delta_1(M)$ map $h: M \to M$ s.t. $h(x) = \{z \mid z <_M x\}$.

Using this, it follows easily that every $\Sigma_1(M)$ relation is uniformizable by a $\Sigma_1(M)$ function.

Thus the KP axioms give us a "reasonable" recursion theory. They do not suffice, however, to get Σ_1-uniformization. In fact, since we have not posited the axiom of choice, we do not even have N-finite uniformization. However, the admissible structures dealt with in these notes will almost always satisfy Σ_1-uniformization. This can happen in different ways. If $N = L_\tau^A =$
$=_{Df} \langle L_\tau[A], A\rangle$, there is a well ordering $<$ of N s.t. the function $h(x) = \{z \mid z < x\}$ is Σ_1. We can then uniformize $R(y, \vec{x})$ as follows: Let $R(y, \vec{x}) \longleftrightarrow \bigvee z\, R'(y, z, \vec{x})$, where R' is Σ_0. R is then uniformized by:

$$\bigvee z \big(R'(y, z, \vec{x}) \wedge$$
$$\wedge \bigwedge \langle y', z'\rangle \in h(\langle y, z\rangle)\ \neg R(y', z', \vec{x})\big)$$

The same holds for $N = L_\tau(a)$ where a is a transitive set with a well ordering constructible from a below τ.

If N is a ZFC$^-$ model with a definable well ordering $<$, then every definable relation has a definable uniformization. If $N^* = \langle N, A_1, A_2, \ldots \rangle$ is the result of adding all N-definable predicates to N, then the $\Sigma_1(N^*)$ relations are exactly the N-definable relations, so uniformization holds trivially.

<u>Ill founded ZF⁻ models</u>

We now prove a lemma about arbitrary (possibly ill founded) models of ZF⁻ (where the language of ZF⁻ may contain predicates other than '$\in$'.) Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \ldots \rangle$ be such a model. For $X \subset A$ we of course write $\mathfrak{A} | X = \langle X, \in_{\mathfrak{A}} \cap X^2, \ldots \rangle$. By the <u>well founded core</u> of $\mathfrak{A}$ we mean the set of all $x \in A$ s.t. $\in_{\mathfrak{A}} \cap C(x)^2$ is well founded, where $C(x)^2$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$. Let $wfc(\mathfrak{A})$ denote the restriction of $\mathfrak{A}$ to its well founded core. Then $wfc(\mathfrak{A})$ is a well founded structure satisfying the axiom of extensionality, and is, therefore, isomorphic to a transitive structure. Hence there is $\mathfrak{A}'$ s.t. $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ and $wfc(\mathfrak{A}')$ is transitive. We say that a model $\mathfrak{A}$ of ZF⁻ is <u>solid</u> iff $wfc(\mathfrak{A})$ is transitive.

Thus every consistent set of sentences in ZF^- has a solid model. Note that if $\mathfrak{A}$ is solid, then $\omega \subset wfc(\mathfrak{A})$.

By Σ_0-absoluteness we of course have:

(1) $wfc(\mathfrak{A}) \models \varphi(\vec{x}) \longleftrightarrow \mathfrak{A} \models \varphi(\vec{x})$

if $x_1, \ldots, x_n \in wfc(\mathfrak{A})$ and φ is a Σ_0-formula.

By $\in$-induction on $x \in wfc(\mathfrak{A})$ it follows that the rank function is absolute:

(2) $rn(x) = rn^{\mathfrak{A}}(x)$ for $x \in wfc(\mathfrak{A})$.

Using this we prove:

<u>Lemma 22</u> Let $\mathfrak{A}$ be a solid model of ZF^-. Then $wfc(\mathfrak{A})$ is admissible.

proof.

Let φ be Σ_0 and let

(3) $wfc(\mathfrak{A}) \models \bigwedge x \bigvee y\, \varphi(x, y, \vec{z})$

where $x, y, z_1, \ldots, z_n \in wfc(\mathfrak{A})$.

Let $u \in wfc(\mathfrak{A})$. By (3) + Σ_0 absoluteness:

(4) $\mathfrak{A} \models \bigwedge x \bigvee y\, \varphi(x, y, \vec{z})$,

Since $\mathfrak{A}$ is a ZFC^- model, there must then be $v \in \mathfrak{A}$ of minimal

$\mathfrak{A}$ - rank $rn^{\mathfrak{A}}(v)$ s.t.

(5) $\mathfrak{A} \models \bigwedge x \in u \bigvee y \in v \; \varphi(x, y, \vec{z})$.

It suffices to note that $rn^{\mathfrak{A}}(v) \in$
$\in wfc(\mathfrak{A})$, hence $rn^{\mathfrak{A}}(v) = rn(v)$ and
$v \in wfc(\mathfrak{A})$. (Otherwise there is $r \in \mathfrak{A}$
s.t. $\mathfrak{A} \models r < rn(v)$ and there is $v' \in \mathfrak{A}$
s.t. $\mathfrak{A} \models v' = \{x \in v \mid rn(x) < r\}$. Hence
v' satisfies (5) and $rn^{\mathfrak{A}}(v') < rn^{\mathfrak{A}}(v)$.
Contr!) By Σ_0 absoluteness, then:

(6) $wfc(\mathfrak{A}) \models \bigwedge x \in u \bigvee y \in v \; \varphi(x, y, \vec{z})$

QED (Lemma)

As immediate corollaries we have:

<u>Cor 22.1</u> Let $\delta = On \cap wfc(\mathfrak{A})$. Then
$L_\delta(a)$ is admissible for $a \in wfc(\mathfrak{A})$

<u>Cor 22.2</u> $L_\delta^A = \langle L_\delta[A], A \cap L_\delta[A] \rangle$
admissible whenever A is $\mathfrak{A}$-
-definable.
(prf. We may suppose w.l.o.g. that
A is one of the predicates of $\mathfrak{A}$.)

Note In Lemma 22 we can replace ZF⁻ by KP. In this form it is known as Villes Lemma. However, a form of Lemma 22 was first employed in our paper [NA] with Harvey Friedman. If memory serves us, the idea was due to Friedman.

[NA] Jensen, H. Friedman A Note on Admissible Sets Springer Lecture Notes on Mathematics Vol. 72 (1968)

Kleene's T-predicate

If an admissible structure $M = \langle |M|, A_1, \ldots, A_n \rangle$ has only finitely many predicates and constants, then there is a universal Σ_1 relation – i.e. a Σ_1 relation $T(u, x)$ s.t. every Σ_1 class $A \subset M$ has the form $Ax \leftrightarrow T(u, x)$, for some $u \in M$. We call such T a Kleene T-predicate. If M also satisfy Σ_1 uniformization, then there is a universal Σ_1 function $F(u, x)$ s.t. whenever f is a Σ_1 function, then $f(x) \simeq F(u, x)$, for some $u \in M$.

In order to obtain a Kleene T-predicate, we must arithmetize the language. The details follow:

<u>Arithmetizing the M-language</u>:

Let $M = \langle |M|, A_1, \ldots, A_m \rangle$. We arithmetize the M-language as follows:

Vbls $\quad v_i = \langle 0, i \rangle \quad (i < \omega)$

Constants $\quad \underline{x} = \langle 1, x \rangle \quad (x \in M)$

$x \varepsilon y = \langle 2, \langle x, y \rangle \rangle$; $x \equiv y = \langle 3, \langle x, y \rangle \rangle$;

$\dot{A}_j \vec{x} = \langle 4, \langle j, \vec{x} \rangle \rangle \quad (j = 1, \ldots, m)$;

$(x \dot{\vee} y) = \langle 5, \langle x, y \rangle \rangle$

$(x \wedge y) = \langle 6, \quad " \quad \rangle$

$\rightarrow \quad = 7$

$\longleftrightarrow \quad = 8$

$\neg x \quad = \langle 9, x \rangle$

$\bigwedge x\, y \quad = \langle 10, \langle x, y \rangle \rangle$

$\bigvee x\, y \quad = \langle 11, \quad " \quad \rangle$

$\bigwedge x{:}\varepsilon z\, y \quad = \langle 13, \langle x, z, y \rangle \rangle$

$\bigvee \quad = \langle 14, \quad " \quad \rangle$.

~~φ is a primitive fmla (PFml) iff φ has the form $t \equiv t'$, $t \varepsilon t'$, $\dot{A}_j \vec{t}$ where $t, t', \vec{t}$ are Vbls or Constants.~~

Clearly, $\langle v_i \mid i<\omega \rangle$, $\langle \underset{\sim}{x} \mid x \in M \rangle$ are Δ_1, as are

$Vbl = \{ v_i \mid i < \omega \}$; $Const = \{ \underset{\sim}{x} \mid x \in M \}$.

A <u>term</u> is a constant or vbl.

The set PFml of <u>primitive fmlae</u> consists of all objects of the form $t \equiv t'$, $t \varepsilon t'$, $\dot{A}_i \vec{t}$, where $t', t, \vec{t}$ are terms. It is easily seen that PFml is Δ_1.

The set Fml of <u>fmlae</u> is the closure of PFml under $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$, $\bigwedge x \varphi$, $\bigvee x \varphi$, $\bigwedge x \varepsilon t \varphi$, $\bigvee x \varepsilon t \varphi$ where x is a vbl, t is a term & $x \neq t$. To show that Fml is Δ_1, we make use of the <u>component</u> <u>fcn</u>, Cm, defined by:

$Cm(x \vee y) = \{x, y\}$

(similarly for $\wedge, \rightarrow, \leftrightarrow, \neg$)

$Cm(\bigwedge x z) = \{z\}$

(similarly for $\bigvee x z$, $\bigwedge x \varepsilon y z$, $\bigvee x \varepsilon y z$)

~~Clearly Cm~~

$Cm(x) = \emptyset$ in all other cases.

Then Cm is Δ_1. ~~and~~ The relation $\{\langle x, y\rangle \mid x \in Cm(y)\}$ is well founded, since $Cm(y) \subset C(y)$. Now let σ be the characteristic fcn of Fml. It is easily seen that σ satisfies an equation of the form:

$$\sigma(\varphi) = G(\varphi, \sigma \restriction Cm(\varphi))$$

where G is Δ_1. Hence σ is Δ_1 + so is $Fml = \{x \mid \sigma(x) = 1\}$.

Similarly, the function

$Fr(\varphi)$ = the set of vbls occurrin free in φ

is Δ_1.

If φ is a fmla, x a vbl *, t a term + if t can be substituted for all free occurrences of x in φ without confusion, we denote the result of this substitution by $\varphi(x/t)$. In all other cases $\varphi(x/t)$ is undefined. Then $\varphi(x/t)$ is a Σ_1 fcn (of φ, x, t) + ~~is defined~~

~~$S(\varphi, x, t)$ is Δ_1, where~~;

has a Δ_1 domain. To see this, we show that S is Δ_1, where

$$S(\varphi, x, t) = \begin{cases} \varphi(x/t) & \text{if defined} \\ 0 & \text{otherwise.} \end{cases}$$

The restriction of $S(\varphi, x, t)$ to primitive φ is obviously Δ_1. But then S has the form

~~$S(\varphi, x, t) = G(\varphi, x, t, \langle S(\psi, x, t) | \psi$~~

$$S(\varphi, x, t) = G(\varphi, x, t, \langle S(\psi, x, t) \mid \psi \in Cm(\varphi) \rangle)$$

since:

$$S(\varphi \vee \psi, x, t) = (S(\varphi, x, t) \vee S(\psi, x, t))$$

if $S(\varphi, x, t), S(\psi, x, t) \neq 0$;

$= 0$ otherwise

(similarly for $\wedge, \rightarrow, \leftrightarrow, \neg$)

$$S(\Lambda z \varphi, x, t) = \begin{cases} \Lambda z \varphi & \text{if } \Lambda z \varphi \in Fml \text{ and } z = x \\ \Lambda z\, S(\varphi, x, t) & \text{if } S(\varphi, x, t) \neq 0,\ z \neq t \ \&\ z \in Vbl \\ 0 & \text{otherwise} \end{cases}$$

(similarly for $V z \varphi$, $\Lambda z \in s\, \varphi$, $V z \in s\, \varphi$).

We note finally that the relation $\{\langle \varphi, \psi, x\rangle \mid V t\ \ \psi = \varphi(x/t)\}$ in Δ_1. To see this, we first note that $Tm(\psi)$ = the set of all terms occurring in ψ

is a Δ_1 fcn. But then

$\bigvee t \; \varphi = \psi(x/t) \longleftrightarrow$

$\longleftrightarrow \varphi, \psi \in Fml \wedge \bigvee t \in Tm(\varphi) \; \varphi = \psi(x/t)$.

. . . .

The collection $St = \{\varphi \in Fml \mid Fr(\varphi) = \emptyset\}$ of <u>statements</u> is Δ_1.

The collection Fml^{Σ_0} (St^{Σ_0}) of Σ_0 fmlae (statements) is Δ_1.

Let $\models u$ mean: u is a true statement

" $\overset{\Sigma_0}{\models} u$ " : u is a true Σ_0 statement.

<u>Theorem</u> $\models^{\Sigma_0}$ is Δ_1

proof. $\models^{\Sigma_0}$ satisfies the recursion:

$\models \underline{z} \,\varepsilon\, \underline{x} \quad \longleftrightarrow \quad z \in x$

$\models \underline{z} \equiv \underline{x} \quad \longleftrightarrow \quad z = x$

$\models \dot{A}_i \, \vec{\underline{x}} \quad \longleftrightarrow \quad A_i \, \vec{x}$

$\models (\varphi \wedge \psi) \longleftrightarrow \; \models \varphi \wedge \models \psi$

(similarly for $\vee, \rightarrow, \longleftrightarrow, \neg$)

$\models \bigwedge v_i \,\varepsilon\, \underline{x} \; \varphi \longleftrightarrow \bigwedge z \in x \models \varphi(^{v_i}/_{\underline{z}})$

(similarly for $\bigvee$).

Let σ be the characteristic function of $\models^{\Sigma_0}$. Then σ satisfies a recursion of the form:

$$\sigma(u) = G(u, \sigma \upharpoonright \Theta(u)),$$

where Θ is defined by:

$\Theta(\varphi) = \emptyset$ if φ is a primitive statement

$\Theta(\varphi \wedge \psi) = \{\varphi, \psi\}$

(similarly for $\vee, \rightarrow, \leftrightarrow, \neg$)

$\Theta(\bigwedge v_i \in \underline{z}\, \varphi) = \{\varphi(v_i/\underline{x}) \mid x \in z\}$

(similarly for $\bigvee$)

$\Theta(u) = \emptyset$ in all other cases.

Hence, σ is Δ_1. QED

<u>The Kleene T-predicate</u>:

Set:

$$Tuy\vec{x} \underset{\mathrm{Df}}{\longleftrightarrow} \models^{\Sigma_0} u(v_0 \ldots v_n / \underline{y}\, \underline{\vec{x}})$$

Then $R \subset M^n$ is Σ_1 iff

$$\bigvee u \bigwedge \vec{x}\, (R\vec{x} \longleftrightarrow \bigvee y\, Tuy\vec{x}).$$