§2 The Model K^c

In this section we provide more justification for the claim, made in §11, that weak iterability holds for the stages N_ξ, M_ξ in the construction of K^c. We also discuss weakenings of the background condition used in that construction. Steel can weaken it considerably while still proving that each N_ξ is a weak MS-mouse. Our form of weak iterability seems to require a bit more.

Def Let F be an extender. By a *background certificate* for F we mean a pair $\langle N, F^* \rangle$ s.t.

(a) N is a transitive ZFC^- model with $V_\kappa \in N$, ${}^\omega N \subset N$ (where $\kappa = crit(F)$).

(b) F^* is an extender at κ on N s.t. $V_{\lambda+1} \subset Ult(N, F^*)$ (where $\lambda = lh(F)$).

(c) $F \upharpoonright N = (F^* \upharpoonright dom(F)) | \lambda$.

(Note Then $\kappa = \overline{\overline{V_\kappa}}$ and $cf(\kappa) > \omega$).

<u>Def</u> Let F be an extender at κ, λ. F <u>fixes</u> a set U iff there is a map $k : U \cap \lambda \longrightarrow \kappa$ s.t. for all $X \in U \cap \mathcal{P}(\kappa) \cap N$ and all $\alpha_1, \ldots, \alpha_n \in U \cap \lambda$ we have:
$\langle k(\vec{\alpha}) \rangle \in X \longleftrightarrow \langle \vec{\alpha} \rangle \in F(X)$.

F is ω-<u>complete</u> iff F fixes every countable U.

<u>Lemma 2</u> Let $\langle N, F^* \rangle$ be a background certificate. Then F^* is ω-complete.

pf.
Set $U_1 = U \cap \mathcal{P}(\kappa) \cap N$. Then $U_1 \in N$ by ω-closure. Let $\langle \alpha_i \mid i < \omega \rangle$ enumerate $U \cap \lambda$. Set $A =$ the set of $\langle \delta_{i_0}, \ldots, \delta_{i_{n-1}} \rangle \in A^{<\omega}$ s.t. for all $h \leq n$ and $X \in U_1$:

$$\langle \delta_{i_0}, \ldots, \delta_{i_{h-1}} \rangle \in X \longleftrightarrow \langle \alpha_{i_0}, \ldots, \alpha_{i_{h-1}} \rangle \in F^*(X).$$

(1) $A \in N$

proof.
For $u = \langle i_0, \ldots, i_{n-1} \rangle \in \omega^{<\omega}$ set:
$F_u = \{ X \in U_1 \mid \langle \alpha_{i_0}, \ldots, \alpha_{i_{n-1}} \rangle \in F^*(X) \}$.
Then $F_u \in N$ and $\langle F_u \mid u \in \omega^{<\omega} \rangle \in N$ by ω-closure. Hence $A \in N$.

QED (1)

Let $R = \{\langle u, v\rangle \mid u, v \in A \wedge u \supsetneqq v\}$. The claim reduces to:

(2) R is not well founded.

proof. Let $\pi : N \to_{F^*} N'$.
It suffices to show: $N' \models \pi(R)$ is not well founded. By absoluteness of well foundedness it is enough to show $\pi(R)$ is not well founded. But this is trivial, since $\langle \alpha_i \mid i < \omega\rangle$ is a branch through $\pi(R)$

QED (Lemma 2)

<u>Corollary 2.1</u> Let $\langle N, F^*\rangle$ be as above. Let $U_1 = U \cap \mathbb{P}(\kappa) \cap N \in N$ s.t. $\overline{\overline{U}}_1 < \kappa$ and let $U_0 = U \cap \lambda$ be countable. Then F^* fixes U.

prf. It suffices to show: $A \in N$, since the rest of the proof is as before. $F_u \in N$ for $u \in \omega^{<\omega}$, since $F_u \subset U_1$ where $U_1 \in N$, $\overline{\overline{U}}_1 < \kappa$ and $V_\kappa \in N$. Hence $\langle F_u \mid u \in \omega^{<\omega}\rangle \in N$ as before. Hence $A \in N$. QED (2.1)

In place of §9 Lemma 4 we shall use:

<u>Lemma 3</u> Let $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M \text{ mod}(\vec{p})$. Let F be an extender on M at κ, λ, where κ is regular. Let:

$\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{**} \langle M|\rho_0, F \rangle$,

where $\bar{F}$ is weakly amenable (hence close to $\bar{M}$). Let $U_1 \subset \mathbb{P}(\kappa) \cap M$ s.t. $\text{rng}(f) \subset U_1$ whenever $f : \xi \to \mathbb{P}(\kappa)$ is M-definable in parameters from $\text{rng}(\sigma) \cup \{p_i \mid i < \omega\}$ and $\xi < \kappa$. Let $U_0 \subset \lambda$ s.t. $\text{rng}(g) \subset U_0$. Suppose, moreover that $\bar{M}$ has cardinality $< \kappa$. Let F fix $U = U_0 \cup U_1$ as witnessed by $k : U_0 \longrightarrow \kappa$. Then:

(a) There is $\bar{\pi} : \bar{M} \longrightarrow_{\bar{F}} \bar{M}'$.

(b) There is $\sigma' : \bar{M}' \longrightarrow_{\Sigma^*} M \text{ mod}(\vec{p})$ defined by: $\sigma'(\bar{\pi}(f)(\alpha)) = \sigma(f)(kg(\alpha))$ for $f \in \Gamma^*(\bar{\kappa}, \bar{M})$, $\alpha < \bar{\lambda}$ (where $\bar{F}$ is at $\bar{\kappa}, \bar{\lambda}$). (Hence $\sigma' \bar{\pi} = \sigma$ and $\sigma \restriction \bar{\lambda} = kg$.)

Proof of Lemma 3:

Note first that each $f \in \Gamma^*(\bar{\kappa}, \bar{m})$ has a uniquely defined $\sigma(f)$ mod $(\vec{p})$.

Let $\langle \alpha, f \rangle \in \mathbb{D}^* = \mathbb{D}^*(\bar{M}, F)$. Let φ be $\Sigma_0^{(n)}$ where $\omega\rho^n > \bar{\kappa}$. Then

$$\{\xi \mid \bar{M} \models \varphi(f(\xi))\} \in \bar{F}_\alpha \longleftrightarrow$$

$$\longleftrightarrow \{\xi \mid M \models \varphi(\sigma(f)(\xi)) \bmod (\vec{p})\} \in F_{g(\alpha)}$$

$$\longleftrightarrow kg(\alpha) \in \{\xi \mid M \models \varphi(f)(\xi) \bmod (\vec{p})\}$$

$$\longleftrightarrow M \models \varphi(\sigma(f)(kg(\alpha))) \bmod (\vec{p})$$

This verifies the existence of $\bar{\pi} : \bar{M} \longrightarrow^{*}_{\bar{F}} \bar{M}'$ and of $\sigma' : \bar{M}' \longrightarrow M$ defined as above. As shown, moreover, that $\sigma' : \bar{M}' \longrightarrow_{\Sigma_0^{(n)}} M$ mod $(\vec{p})$ for $\omega\rho^n_{\bar{M}} > \bar{\kappa}$. We also note that if $\sigma' : \bar{M}' \longrightarrow_{\Sigma^*} M$ mod $(\vec{p})$, then $\vec{p} = \min(\bar{M}', \sigma', \vec{p})$, since $\vec{p} = \min(\bar{M}, \sigma, \vec{p})$ and $\mathrm{rng}(\sigma) \subset \mathrm{rng}(\sigma')$.

Hence the theorem is proven if $\omega\rho^\omega_{\bar{M}} > \bar{\kappa}$. Now let:

(1) $\omega\rho^{n+1} \le \bar{\kappa} < \omega\rho^n$ in $\bar{M}$.

By the regularity of κ, the minimality of $\vec{p}$ and the fact that $\mathrm{card}(\bar{M}) < \kappa$, we easily get:

(2) $\omega\rho_{n+1} < \kappa < \omega\rho_n$.

<u>Claim 1</u> Let $\bar{R}(z,y)$ be $\Sigma_1^{(n)}(\bar{M}')$ and $R(z,y)$ be $\Sigma_1^{(n)}(M,\vec{p})$ by the same definition. Let $\bar{x} \in \bar{M}'$, $x = \sigma'(x)$. Set:
$\bar{P} = \{\xi^{n+1} \mid \bar{R}(\xi^{n+1}, \bar{x})\}$, $P = \{\xi^{n+1} \mid R(\xi^{n+1}, x)\}$
Then $\bar{P}$ is $\Sigma_1^{(n)}(\bar{M})$ in a parameter $\bar{r}$ and P is $\Sigma_1^{(n)}(M,\vec{p})$ in $r = \sigma(\bar{r})$ by the same definition.

prf.
Let $\bar{x} = \pi(f)(\alpha)$. Then $x = \sigma(f)(kg(\alpha))$.

Suppose either $f = \bar{p}$ or f is a good $\Sigma_1^{(n-1)}(\bar{M})$ function in the parameter $\bar{p}$ by a functionally absolute definition. Then $\sigma(f)$ has the same definition over M mod $(\vec{p})$ in $p = \sigma(\bar{p})$
We know:
$\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{**} \langle M|\rho_0, F \rangle$.

Hence there are G, H s.t.

(i) F_α, ${}^{\bar\kappa}\mathbb{P}(\bar u) \cap \bar M$ are $\Sigma_1(\bar M)$ in a parameter $\bar q$ and G, H are $\Sigma_1(M|\rho_0)$ in $q = \sigma(\bar q)$ by the same definition.

(ii) $G \subset F_{g(\alpha)}$

(iii) $H \subset \{X \in {}^{\kappa}\mathbb{P}(u) \mid \bigwedge i < u\ (X_i \cap \kappa \setminus X_i \in G)\}$

We prove the claim for $\bar r = \langle \bar u, \bar p, \bar q \rangle$, $r = \langle u, p, q \rangle$. Let $\bar R(\varsigma, x) \leftrightarrow \bigvee z^n \bar Q'(z^n, \varsigma, x)$, where $\bar Q'$ is $\Sigma_0^{(n)}(\bar M')$. Let $\bar Q$ have the same $\Sigma_0^{(n)}$ definition over $\bar M$. Then for $\varsigma \in H^n_{\bar M} = H^n_{\bar M'}$, we have:

$$(1)\ \bar R(\varsigma, \bar x) \longleftrightarrow \bigvee u \in H^n_{\bar M}\ \underbrace{\bigvee z \in \Pi(u)\ \bar Q(z, \varsigma, \bar x)}_{\Sigma_0^{(n)}}$$

$$\longleftrightarrow \bigvee u \in H^n_{\bar M}\ \{\delta < \bar\kappa \mid \bigvee z \in u\ \bar Q(z, \varsigma, f(\delta))\} \in \bar F_\alpha$$

$$\longleftrightarrow \bigvee u^n \bigvee v^n (v^n = \{\delta < \bar\kappa \mid \bigvee z \in u^n\ Q(z, \varsigma, f(\delta))\} \wedge v^n \in \bar F_\alpha).$$

Clearly it suffices to show:

<u>Claim</u> $R(\varsigma, x) \longleftrightarrow$

$$\longleftrightarrow \bigvee u^n \bigvee v^n (v^n = \{\delta < \kappa \mid \bigvee z \in u^n Q(z, \varsigma, \sigma(f)(\delta))\} \wedge v^n \in G)$$

for $\varsigma \in H_n(M, \vec p)$, where Q has

The same $\Sigma_0^{(n)}(M, \vec{p})$ def. as $\bar{Q}$ over $\bar{M}$. We first prove the easy direction:

($\leftarrow$) Let $v = \{\delta < \kappa \mid \bigvee z \in u\, Q(z, \xi, \sigma(f)(\delta))\} \in G$. Then $G \subset F_{g(\alpha)}$ + hence $g(\alpha) \in F(v)$. Hence $\delta g(\alpha) \in v$. Hence $Q(z, \xi, \sigma(f)(\delta g(\alpha)))$ for a $z \in H_n$, where $\sigma(f)(\delta g(\alpha)) = x$. Hence $R(\xi, x)$.

QED($\rightarrow$)

($\rightarrow$) Assume $R(\xi, x)$. Then $\bigvee z \in u\, Q(z, \xi, x)$ for some $u \in H_n$.

By the minimality of $\vec{p}$ we may assume w.l.o.g. that $u = h(\zeta)$ for some $\zeta < \omega\rho_{n+1} < \kappa$, where h is a $\Sigma_1^{(n)}(M, \vec{p})$ map in a parameter $s = \sigma(\bar{s})$, (since, by minimality, the set of such u is cofinal in H_n).

Set:

$$X(\zeta, \xi) \simeq \{\delta < \kappa \mid \bigvee z \in h(\zeta)\, Q(z, \xi, \sigma(f)(\delta))\}$$

for $\zeta < \omega\rho_{n+1}$, $\xi < \kappa$. Set:

$X(\zeta) \simeq \langle X(\zeta, \xi) \mid \xi < \kappa \rangle$. It is easily seen that X is a partial

$\Sigma_1^{(n)}(M,\vec{p})$ map from $\omega\rho_{n+1}$ to H_n in the parameters r, s. Moreover, $X(\xi)$ is defined iff $h(\xi)$ is defined. Let $\bar{X}$ have the same functionally absolute $\Sigma_1^{(n)}(\bar{M})$ definition in $\bar{r}, \bar{s}$. Then in $\bar{M}$ we have:

(3) $\bigwedge \xi^{n+1} \in dom(\bar{X})\ \bar{X}(\xi) \in {}^{\bar{\kappa}}\#(\bar{\kappa})$,

This statement is $\Pi_1^{n+1}(\bar{M})$ in $\bar{r}, \bar{s}$ (using our $\Sigma_1^{(n)}(\bar{M})$ def of $\bar{H} = {}^{\bar{\kappa}}\#(\bar{\kappa})$ in $\bar{r}$). Hence the corresponding $\Pi_1^{n+1}(M,\vec{p})$ statement holds in r, s:

(4) $\bigwedge \xi^{n+1} \in dom(X)\ X(\xi) \in H$.

Hence:

(5) $X(\xi, \varsigma) \in G \vee (\kappa \setminus X(\xi, \varsigma)) \in G$.

But $\tilde{X} = \langle X(\xi, \varsigma) \mid \xi, \varsigma < \omega\rho_{n+1} \rangle$ is M-definable in parameters from $rng(\sigma) \cup \{p_i \mid i < \omega\}$. Hence:

(6) $g(\alpha) \in X(\xi, \varsigma) \longleftrightarrow X(\xi, \varsigma) \in F_{g(\alpha)}$

for $\xi, \varsigma < \omega\rho_{n+1}$

For our specific ξ, ς we have $\delta g(\alpha) \in X(\xi, \varsigma)$, hence $X(\xi, \varsigma) \in F_{g(\alpha)}$. Hence $u \setminus X(\xi, \varsigma) \notin G \subset F_{g(\alpha)}$. Hence $X(\xi, \varsigma) \in G$. Hence for $u = k(\xi)$ we have: $u \in H_n$ and

$\{\delta < \alpha \mid \bigvee z \in u\ Q(z, \varsigma, \sigma(f)(\delta))\} \in G$.

QED ($\rightarrow$). QED (Claim 1).

By this we easily get:

<u>Claim 2</u> $\{z^{n+1} \mid \bar{R}(z^{n+1}, \bar{x})\}$ is $\Sigma_1^{(n)}(\bar{M})$ in some $\bar{r}$ and $\{z^{n+1} \mid R(z^{n+1}, x\}$ is $\Sigma_1^{(n)}(M, \vec{p})$ in $r = \sigma(\bar{r})$ by the same definition.

But $\rho^i_{\bar{M}} = \rho^i_{\bar{M}}$ for $i > n$. It follows easily that:

<u>Claim 3</u> Let $i > n$. Let $\bar{R}$ be $\Sigma_1^{(i)}(\bar{M}')$ and R be $\Sigma_1^{(i)}(M)$ by the same definition. Then $\{z^i \mid \bar{R}(z^i, \bar{x})\}$ is $\Sigma_1^{(i)}(\bar{M})$ in some $\bar{r}$ and $\{z^i \mid R(z^i, x)\}$ is $\Sigma_1^{(i)}(M, \vec{p})$ in $r = \sigma(\bar{r})$ by the same definition.

Now let φ be a $\Sigma_1^{(i)}$ formula ($i \geq n$). Let $\bar{x} \in \bar{M}'$, $x = \sigma'(\bar{x})$. The statement $\bar{M}' \models \varphi(\bar{x})$ is $\Sigma_1^{(i)}(\bar{M})$ in a parameter $\bar{\pi}$ and $M \models \varphi(x) \bmod(\vec{p})$ is $\Sigma_1^{(i)}(M, \vec{p})$ in $q = \sigma(\bar{q})$ by the same definition. Hence: $\bar{M}' \models \varphi(\bar{x}) \longleftrightarrow M \models \varphi(x) \bmod(\vec{p})$, since $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M \bmod(\vec{p})$.

QED (Lemma 3)

. . . . . .

We assume Θ to be inaccessible and define N_ξ, M_ξ ($\xi < \bar{\theta} \leq \theta$) as in §10 except that in Case 2.1 we relax the condition that F be the restriction of an extender F^* on V. We instead require:

(*) Let $\kappa = \mathrm{crit}(F)$. Whenever $\xi < \kappa$ and $f : \xi \to \mathrm{dom}(F)$, then F has a background certificate $\langle N, F^* \rangle$ s.t. $f \in N$.

(<u>Note</u> (*) implies that κ is regular, hence inaccessible.)

In order to define $K^c = N = N_\Theta$ we show inductively that N_ξ is defined and satisfies (a), (b) of §10 (p. 8). As before, (b) follows from (a). We must show:

<u>Lemma 4</u> N_η is a weak mouse (i.e. if $\sigma : P \xrightarrow{\Sigma^*} N_\eta$ and P is countable, then P is countably iterable).

(<u>Note</u> We of course assume A0 of the previous section.)

As before this reduces to:

<u>Lemma 4.1</u> Let $\sigma : P \xrightarrow{\Sigma^*} N_\eta \min(\vec{\rho})$, where P is countable. Then P has a countable normal iteration strategy S. Moreover if $\mathcal{I} = \langle\langle P_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is a countable normal S-iteration of length $\theta + 1$, then:

(i) There is $\sigma' : P_\theta \xrightarrow{\Sigma^*} N_\gamma \min(\vec{\rho}\,')$ for a $\gamma \leq \xi$ s.t.

(ii) If $\pi_{0\theta}$ is not total, then $\gamma < \xi$

(iii) If $\pi_{0\theta}$ is total, then $\gamma = \xi$, $\vec{\rho}\,' = \vec{\rho}$, and $\sigma' \pi_{0\theta} = \sigma$.

(Lemma 4 follows from Lemma 4.1 as Corollary 2 follows from Lemma 1 in §10.)

If we only need to prove that N_γ is a weak MS-mouse, then a weaker version of Lemma 4.1 will do. Steel sketches a proof of this in [S] for the case $lh(\mathcal{T}) = \omega$, making several other simplifying assumptions. We now show how Steel's sketch can be modified to give our result. Our simplifying assumptions are:

(*) $lh(\mathcal{T}) = \omega$ and $\mathcal{T}$ has no truncations.

Since no truncations occur, we are constrained to find a cofinal branch b and a map σ' s.t. $\sigma' : P_b \xrightarrow[\Sigma^*]{} N_\gamma \bmod (\vec{p})$. Following Steel we devise, for any map $\tau : P \xrightarrow[\Sigma^*]{} Q \bmod (\vec{p})$ a tree $\mathcal{U} = \mathcal{U}(\tau, Q, \vec{p})$ s.t. any branch through $\mathcal{U}$ gives a cofinal b in $\mathcal{T}$ and a $\sigma : P_b \xrightarrow[\Sigma^*]{} Q \bmod (\vec{p})$ s.t. $\sigma \pi_b = \tau$. The definition is the same as in [S] (p. 10) except that in (b) the condition: $(P_e, x_0, \ldots, x_k) \equiv (Q, y_0, \ldots, y_k)$ is interpreted in the Σ^*-language with reference to Σ^* formulae only, which in the case of Q we interpret mod$(\vec{p})$. If $U = U(\tau, Q, \vec{p})$ and $\sigma : P_i \underset{\Sigma^*}{\longrightarrow} Q$ min$(\vec{p})$ s.t. $\tau = \sigma \pi_{0 i}$, we define $p(i, \sigma, \tau, Q, \vec{p})$ exactly as Steel does.

We assume that no such branch b and map σ' exist. This means that $U = U(\sigma, N_\eta, \vec{p})$ is well founded. We derive a contradiction. Following Steel we define $\langle \sigma_i, Q_i, P_i, \vec{p}^{\,i} \rangle$ s.t.

(1) $P_i = \langle P_i, \in, \delta_i \rangle$ is a coarse premouse in the sense of [S].

(2) $\sigma_i : P_i \underset{\Sigma^*}{\longrightarrow} Q_i$ min$(\vec{p}^{\,i})$ and Q_i is a model on the sequence $\vec{N}^{P_i}$ (defined in $V_{\delta_i}^{P_i}$ like $\vec{N}$ in V_Θ).

We recall from §10 that if $\nu \leq ht(N_\eta)$ and $E_\nu^{N_\eta} \neq \emptyset$, then there exist $\xi \leq \eta$ and a canonical $k_\nu : N_\eta \| \nu \underset{\Sigma^*}{\longrightarrow} N_\xi$. If $\nu = ht(N_\eta)$, then $\xi = \eta$ and $k_\nu = id$. Otherwise we still have: $k_\nu \upharpoonright wp_\beta^\omega = id$ for all $\beta < ht(N_\eta)$.

(We also write:

$k_\nu = k_{\nu,\eta}(\vec{N})$.) Now set:

$k^*_i = (k_{\sigma_i(\nu_i),\eta_i}(\vec{N}))^{R_i}$ where $Q_i = N^{R_i}_{\eta_i}$.

Set $\lambda^*_i = k^*_i \sigma_i(\lambda_i)$. We also require:

(3) $V^{R_h}_{\lambda^*_h+1} = V^{R_i}_{\lambda^*_h+1}$ and $\lambda^*_h < \lambda^*_i$ for $h < i$.

(4) $\sigma_i \upharpoonright \lambda_h = k^*_h \sigma_h \upharpoonright \lambda_h$ for $h < i$.

(Hence $j < h < i \rightarrow \sigma_h \upharpoonright \lambda_j = \sigma_i \upharpoonright \lambda_j$, since $\sigma_h(\lambda_j)$ is a cardinal in Q_h, hence $\omega\rho^\omega_{Q_h \| \beta} \geq \sigma_h(\lambda_j)$ for $\beta < ht(Q_h)$ and $k^*_h \upharpoonright \sigma_h(\lambda_j) = id$.)

(5) Let $U = U(\sigma_i \pi_{0i}, Q_i, \vec{\rho}^{\,i})$ and $p = p(i, \sigma_i, \sigma_i \pi_{0i}, Q_i, \vec{\rho}^{\,i})$. Then U is well founded and the order type of the set of cutoff points of R_i is at least $|p|_U$.

(6) $R_i \in R_{i-1}$ for $i > 0$.

As in [S] we set:

$R_0 = \langle V_{\theta+\xi}, \in, \theta \rangle$ where $\xi \geq |U|$, $U = U(\sigma, N_\eta, \vec{\rho})$,

$\sigma_0 = \sigma$, $Q_0 = N_\eta$, $\vec{\rho}^{\,0} = \vec{\rho}$.

Following [S] closely, we now define $R_{i+1}, \sigma_{i+1}, Q_{i+1}, \vec{\rho}^{\,i+1}$. We know that $\tau_i = \kappa_i^{+P_i}$ is a cardinal in P_i. (τ_i is cardinal in P_j, where $j = T(i+1)$, since there is no truncation. If $i = j$ we are done. If not, τ_i is a cardinal in $J_{\lambda_i}^{EP_j} = J_{\lambda_j}^{EP_i}$ and λ_j is a cardinal in P_i.) Hence $\sigma_i(\tau_i)$ is a cardinal in Q_i. Hence $k_i^* \upharpoonright \sigma_i(\tau_i) = id$, since $\omega\rho^\omega_{Q_i \| \beta} \geq \sigma_i(\tau_i)$ for $\beta < ht(Q_i)$. It follows that $k_i^* \sigma_i(\tau_i) = \sigma_i(\tau_i)$. (To see this let ρ = the minimal $\rho = \omega\rho^\omega_{Q_i \| \beta}$ for a $\beta < ht(Q_i)$. By the construction of k_i^* in §10 (as a composition of core maps) and by §8 Lemma 5 it follows that ρ is a cardinal in N_{ξ_i}. But then $k_i^*(\rho) = \rho$ if $\rho = \sigma_i(\tau_i)$, since $\sigma_i(\tau_i)$ is a successor cardinal in Q_i.)

Since $\omega\rho^{\omega}_{Q_i\|\beta} \geq \sigma_i(\tau_i)$ for $\beta < ht(Q_i)$, the same argument shows: $k^*_i \sigma_i \restriction (\tau_i+1) = \sigma_i \restriction (\tau_i+1)$, Hence $\sigma_i \restriction (\tau_i+1) = k^*_i \sigma_i \restriction (\tau_i+1) = \sigma_i \restriction (\tau_i+1)$.

Let $k^*_i : Q_i \| \sigma_i(\nu_i) \longrightarrow_{\Sigma^*} N^{R_i}_{\xi_i}$, where $\xi_i \leq \gamma_i$. (Recall $k^*_i = id$ if $\xi_i = \gamma_i$)

. . . . . .

Let $\langle N, F \rangle$ be a background certificate for $N_{\xi_i} = \langle J^{E^{N_{\xi_i}}}_{\gamma}, E^*_i \rangle$ in R_i, chosen s.t. $f \in N$ for every $f : \varsigma \to Q_i \cap \mathbb{P}(\sigma_i(\kappa_i))$ s.t. $\varsigma < \sigma_i(\kappa_i)$ and f is Q_i-definable in parameters from $rng(\sigma_i) \cup \{\rho^i_n \mid n<\omega\}$ (There are only countably many.) Set: $\sigma^*_i = k^*_i \sigma_i$. Let $\pi : N \to_F N'$. Then $\lambda^*_i, \sigma^*_i \in N'$. Pick $\beta < lh(F)$ s.t. $\sigma^*_i = \pi(\tilde{\sigma})(\beta)$, $\lambda^*_i = \pi(\tilde{\lambda})(\beta)$, where $\tilde{\sigma}, \tilde{\lambda}$ map $\sigma_i(\kappa_i)$ into $V^{R_i}_{\sigma_i(\kappa_i)}$. Following [S] we assert:

<u>Claim</u> For F_β many ξ, there are in $V^N_{\sigma_i(\kappa_i)}$: a coarse premouse R (in the sense of [S]), a model Q in the sequence $\vec{N}^R$, $\sigma : P_{i+1} \xrightarrow{\Sigma^*} Q$ min($\vec{\rho}$) s.t.

(i) $V^R_{\tilde{\lambda}(\xi)+1} = V^N_{\tilde{\lambda}(\xi)+1}$

(ii) $\sigma \upharpoonright \lambda_i = \tilde{\sigma}(\xi)$

(iii) Let $U = U(\sigma\pi_{0,i+1}, Q, \vec{\rho})$, $p = p(i+1, \sigma, \sigma\pi_{0,i+1}, Q, \vec{\rho})$. Then U is well founded and there are in order type at least $|p|_U$ many cutoff points of R.

. . . .

Before proving the Claim, we reveal that in addition to (1)–(6), we need another property:

(7) $\langle \sigma_i, \sigma_i \upharpoonright \lambda_i \rangle : \langle P_i, E^{P_i}_{\gamma_i} \rangle \xrightarrow{**} \langle Q_i, E^{Q_i}_{\sigma_i(\gamma_i)} \rangle$

Hence:

(7') $\langle \sigma_i, \sigma^*_i \upharpoonright \lambda_i \rangle : \langle P_i, E^{P_i}_{\gamma_i} \rangle \xrightarrow{**} \langle Q_i, E^*_i \rangle$.

We pass over the verification

of (7) which must be handled inductively in the manner of §10 Lemma 4.

. , , , ,

We now prove the Claim. Suppose not. Let X = the set of ξ for which the claim holds. Then $X \notin F_\beta$. We derive a contradiction. The main step is devising an appropriate map:

$\sigma' : P_{i+1} \xrightarrow{\Sigma^*} Q_i \ \min(\vec{\rho}\,')$.

Set: $W_0 = \{\beta\} \cup \sigma_i{}^{*}{}''\lambda_i$;

W_1 = The union of all $\mathrm{rng}(f)$ s.t. $f : \zeta \to Q_i \cap \mathcal{P}(\sigma_i(\kappa_i))$ for a $\zeta < \sigma_i(\kappa_i)$ and f is Q_i-definable from $\mathrm{rng}(\sigma_i) \cup \{\rho'_n \mid n < \omega\}$

Then W_0 is countable, $W_1 \subset Q_i \cap \mathcal{P}(\sigma_i(\kappa_i))$ and $W_1 \in N$. Moreover $\overline{\overline{W}}_1 < \sigma_i(\kappa_i)$ in N. Let W_2 = the set of $Y \subset \sigma_i(\kappa_i)$ which are N-definable in $\tilde{\sigma}, \tilde{\lambda}$ and parameters from $TC(\mathcal{Y})$. Then $W_2 \subset N \cap \mathcal{P}(\sigma_i(\kappa_i))$ and

$X \in W_2$ and W_2 is countable. Set:
$W = \{\beta\} \cup W_0 \cup W_1 \cup W_2$. Let
$\pi : W_0 \cup \{\beta\} \longrightarrow \sigma_i(\kappa_i)$ fix W wrt F in V_i
the sense of our earlier definition.
Then:
$\langle \pi(\vec{\alpha}) \rangle \in X \longleftrightarrow \langle \vec{\alpha} \rangle \in F(X)$
for $\alpha_1, \ldots, \alpha_n \in \{\beta\} \cup W_0$ and
$X \in W_1 \cup W_2$.
Clearly $\pi \restriction W_0$ fixes $W_0 \cup W_1$ wrt.
E_i^*. Hence, by (7′) we may
define:
$\sigma' : P_{i+1} \xrightarrow[\Sigma^*]{} Q_i \ \min(\vec{p}^{\,i})$
by: $\sigma'(\pi_{i,i+1}(f)(\alpha)) = \sigma_i(f)(\pi\sigma_i^*(\alpha))$.

Now let $\bar{\beta} = \pi(\beta)$. Let
$U = U(\sigma_i \pi_{0i}, Q_i, \vec{p}^{\,i})$ and $p =$
$= p(i, \sigma_i, \sigma_i \pi_{0i}, Q_i, \vec{p}^{\,i})$. Then R_i
has at least $|p|_U$ many cutoff
points. Set $q = p(i+1, \sigma', \sigma_i \pi_{0i}, Q_i, \vec{p}^{\,i})$.
Then $\sigma_i = \sigma' \pi_{i,i+1}$ + it follows

that q extends p in U. Hence $|q|_U < |p|_U$ and there is $\gamma \in R_1$ s.t. $\gamma =$ the $|q|_U$-th cutoff point of R_1. Pick $Y \prec V_\gamma^{R_1}$ s.t. $V_{\tilde{\lambda}(\bar{\beta})+1}^{R_1} \cup \{Q_1\} \subset Y$ and $\overline{\overline{Y}} < \sigma_i(\kappa_i)$ and Y is ω-closed in R_1.

Let $\delta : R \longleftrightarrow^{\sim} Y$ where R is transitive. Set: $Q = \delta^{-1}(Q_1)$, $\sigma = \delta^{-1}(\sigma')$, $\vec{\rho} = \delta^{-1}(\vec{\rho}\,')$. Then $\langle Q, R, \sigma, \vec{\rho} \rangle \in V_{\sigma_i(\kappa_i)}^{R_1} =$ $= V_{\sigma_i(\kappa_i)}^{N}$. Moreover $V_{\tilde{\lambda}(\bar{\beta})+1}^{N} = V_{\tilde{\lambda}(\beta)}^{R}$ $= V_{\tilde{\lambda}(\beta)+1}^{R_1}$. For $\alpha < \lambda_i$ we have $\sigma'(\alpha) = \pi\sigma_i^*(\alpha)$. But then $\sigma_i^*(\alpha) < \lambda_i^*$ $\pi(\tilde{\lambda})(\beta)$ + hence $\langle \sigma_i^*(\alpha), \beta \rangle \in$ $\in F(Z)$ where $Z = \{\langle \eta, \tau \rangle \mid \eta < \tilde{\lambda}(\tau)\}$. Hence $\sigma'(\alpha) = \pi\sigma_i^*(\alpha) < \tilde{\lambda}(\bar{\beta})$.
Clearly $\delta \upharpoonright \tilde{\lambda}(\bar{\beta}) = \mathrm{id}$. Hence

$\sigma'(\alpha) = \delta\sigma(\alpha) = \sigma(\alpha)$. Hence $\sigma \upharpoonright \lambda_i =$
$= \sigma' \upharpoonright \lambda_i$. But for $\alpha < \lambda_i$, $\gamma =$
$= \sigma'(\alpha) = r\sigma_i^*(\alpha)$ we have:

$$\langle \alpha, \gamma \rangle \in \tilde{\sigma}(\bar{\beta}) \longleftrightarrow \langle \alpha, \sigma_i^*(\alpha) \rangle \in \pi(\tilde{\sigma})(\beta)$$
$$\longleftrightarrow \quad " \quad \in \sigma_i^* \upharpoonright \lambda_i$$
$$\longleftrightarrow \sigma_i^*(\alpha) = \sigma_i^*(\alpha),$$

Hence $\tilde{\sigma}(\bar{\beta}) = \sigma' \upharpoonright \lambda_i = \sigma \upharpoonright \lambda_i$.

Finally we note that since $V_\gamma^{R_i}$ has $q = p(i+1, \sigma', \sigma' \circ \pi_{0,i+1}, Q_i)$ many cutoff pts, then the same statement holds in R – i.e. R has $p(i+1, \sigma, \sigma \circ \pi_{0,i+1}, Q)$ many cutoff points and $U = U(\sigma, \sigma \circ \pi_{0,i+1}, Q)$ is well founded. Thus we have shown: $\bar{\beta} \in X$. Hence $\beta \in F(X)$, since $\bar{\beta} = r(\beta)$. Contr!

This proves the Claim. An N choose $\xi \to \langle R(\xi), Q(\xi), \sigma(\xi), \vec{p}(\xi) \rangle$ with the above properties for $\xi \in X$. Set: $R_{i+1} = \pi(R)(\beta)$, $Q_{i+1} = \pi(Q)(\beta)$ etc.

The verifications are straight-forward. This completes the construction of $\langle R_i, Q_i, \sigma_i, \vec{\rho}^i \rangle$ $(i<\omega)$ giving us our contradiction.
Thus the Claim is proven. We believe - but haven't checked - that the rest of Steel's proof can be modified in the same way. In this sense Lemma 4 is now "proven".

. . . . .

We now consider the problem of constructing a version of K^c which segments need only be weak MS-mice. This is a somewhat easier problem and we can get by with a weaker background condition for the N_ξ. In place of Lemma 4.1 it suffices to prove:

<u>Lemma 4.1'</u> Let $\sigma : P \to_{\Sigma_0^{(k)}} N_\xi$ where P is countable. Then P has a countable normal k-iteration strategy S. Moreover, if $\mathcal{I} = \langle\langle P_i\rangle, \ldots, T\rangle$ is a countable normal S-iteration of length $\theta+1$, then there is $\sigma' : P_\theta \to N_\gamma$ s.t.

(i) If $\pi_{0\theta}$ is total, then $\gamma = \xi$, $\sigma'\pi_{0\theta} = \sigma$, and σ' is $\Sigma_0^{(n)}$-preserving whenever $n \le k$ and $\omega\rho^n_{P_\theta} > \lambda_i$ for all $i < \theta$.

(ii) If $\pi_{0\theta}$ is not total, then $\gamma < \xi$ and σ' is $\Sigma_0^{(n)}$-preserving whenever $\omega\rho^n_{P_\theta} > \lambda_i$ for all $i < \theta$.

From this follows:

<u>Lemma 4'</u> Let $\sigma : P \to_{\Sigma^*} N_\xi$ where P is countable. Then P is countably MS-iterable.

To obtain this it suffices to use a weaker background condition in Case 2.1 of the def. of N_ξ: We require only that, whenever $Z \subset \mathrm{dom}(F)$ is countable, there be a background certificate $\langle N, F^*\rangle$ s.t. $Z \subset N$. (Hence

$\kappa = \text{crit}(F)$ need no longer be regular, although we still have $\kappa = \bar{\bar{\nu}}_\kappa$ and $\text{cf}(\kappa) > \omega$.) The proof of Lemma 4.1' for the special case that $\mathcal{I}$ satisfies (*) is much as before. In the decisive step we construct $\sigma' : P_{i+1} \to Q_i$ as follows. We choose our background certificate for $E_i = E^{N_{\xi_i}}_{ht(N_{\xi_i})}$ as $\langle N, F \rangle$ s.t. $\text{rng}(\sigma_i^*) \subset N$. Define W_0, W_2 as before and set $W_1 = \text{rng}(\sigma_i^*) \cap \mathcal{F}(\sigma_i$ Set $W = W_0 \cup W_1 \cup W_2$ and proceed as before: Let s fix W and define σ' by: $\sigma'(\pi_{i,i+1}(f)(\alpha)) = \sigma_i(f)(s\sigma_i^*(\alpha))$. Then σ' is defined and is $\Sigma_0^{(n)}$ - preserving whenever $n \leq k$ and $\omega\rho^n_{P_i} > \kappa_i$. (Hence σ' is $\Sigma_0^{(n)}$ - preserving for $n \leq k$ and $\omega\rho^n_{P_{i+1}} > \lambda_i$.) The proof of this is contained in the initial part of the proof of Lemma 3.