The Yamabe problem and conformal geometry of quaternionic contact structures

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Theorem (G.Folland & E.Stein)

Let **G** *be a Carnot group* **G** *of homogeneous dimension* Q*. For any* $1 < p < Q$ *there exists* $S_p = S_p(\mathbf{G}) > 0$ *such that for* $u \in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{p^*}\, dH(g)\right)^{1/p^*} \leq S_p \left(\int_\Omega |Xu|^p\, dH(g)\right)^{1/p}.$$

Theorem

The best constant is achieved.

- Euler-Lagrange (after scaling) is $\sum_{i=1}^m X_i(|Xu|^{p-2}X_i u) = -u^{p^*-1}$. Here, $|Xu|^2 = \sum_{i=1}^m |X_i u|^2$.
- When $p = 2$, $\sum_{i=1}^m X_i^2\, u = -\, u^{\frac{Q+2}{Q-2}}$ -the Yamabe equation.

Theorem (w/ Garofalo)

Let **G** *be a group of Heisenberg type. For every* $\epsilon \neq 0$ *the function*

$$K_\epsilon(g) \;=\; C_\epsilon \left((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2\right)^{-(Q-2)/4}, \qquad C_\epsilon = [m(Q-2)\epsilon^2]^{(Q-2)/4}$$

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Definition

- *U has cylindrical symmetry (w.r.t.* $g_o \in \mathbf{G}$*) if* $\tau_{g_o}U(g) \; = \; u(|x(g)|, |y(g)|)$.
- $U : \mathbf{G} \to \mathbb{R}$ *has partial symmetry (w.r.t.* $g_o \in \mathbf{G}$*) if* $\tau_{g_o}U(g) \; = \; u(|x(g)|, y(g))$.

The Yamabe equation on Iwasawa groups

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Theorem (w/ Garofalo)

Let $\mathbf{G}$ *be an Iwasawa group. Suppose* $U \not\equiv 0$ *is an entire solution of the Yamabe equation.*

a) *If U has partial symmetry, then U has cylindrical symmetry.*

b) *If* $U \not\equiv 0$ *is an entire solution of the the Yamabe equation with cylindrical symmetry. There exists* $\epsilon > 0$ *s.t.*

$$U(g) \;=\; \tau_{g_o}K_\epsilon(g).$$

Let (M, g) - compact, Riemannian manifold, $2^* = \frac{2n}{n-2}$. If $\bar{g} = u^{4/(n-2)}g$, then

$$4\frac{n-1}{n-2}\triangle u - \text{Scal}\cdot u = -\overline{\text{Scal}}\cdot u^{2^*-1}.$$

- **Yamabe functional**: $\Upsilon(u) = \int_M 4\frac{n-1}{n-2}|\nabla u|^2 + \text{Scal}\, u^2\, dv_g$.
- **Yamabe invariant**: $\Upsilon([g]) = \inf\{\Upsilon(u) : \int_M u^{2^*}\, dv_g = 1,\ u > 0\}$.

- For the round sphere $\Upsilon(S^n, [g_{st}]) = n(n-1)\omega_n^{2/n}$.

The classical Obata

Theorem (Aubin,Talenti, Obata)

Let (S^n, g_{st}) be the unit sphere in $\mathbb{R}^{n+1}$. If g is a Riem. metric, $g = \phi^2 g_{st}$, and $Scal_g = \ S = const$, then up to a homothety g is obtained from g_{st} by a conformal diffeo of the sphere, i.e.,

$$\exists \Phi \in Diff(S^n) \ s.t.\ Sg = \Phi^* g_{st}$$

Furthermore, $\Phi = exp(tX)$, $X = \nabla f$, $f = a_o x_o + \cdots + a_n x_n|_{S^n}$.

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"Proof" (Lee & Parker) $\bar{g}$ is Einstein. i.e., $0 = \overline{Ric_o} = Ric_o + \frac{n-2}{\phi}(\nabla^2 \phi)_o$. Thus, $(\nabla^2 \phi)_o = -\frac{\phi}{n-2} Ric_o$. Using $2\nabla^*(Ric_o) = \nabla S = 0$, from the contracted Bianchi and S=const, it follows

$$div\ Ric_o(\nabla \phi, .) = -\frac{\phi}{n-2}|Ric_o|^2.$$

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Theorem (N. Trudinger, Th. Aubin, R. Schoen; A. Bahri)

Let $(M^n, \bar{g})$, $n \geq 3$, *be a compact Riemannian manifold. There is a* $g \in [\bar{g}]$, *s.t.,* $Scal_g = const$.

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- (H. Yamabe, N. Trudinger, Th. Aubin). Always $\Upsilon([g]) \leq \Upsilon(S^n, st)$. The Yamabe problem can be solved on any compact manifold M with $\Upsilon([g]) < \Upsilon(S^n, [g_{st}])$.
- (Aubin) If $n > 6$ then $\Upsilon([g]) - \Upsilon(S^n, [g_{st}]) \geq c\|W^g\|^2$.
- (Schoen). If $3 \leq n \leq 5$, or if M is locally conformally flat, then $\Upsilon([g]) - \Upsilon(S^n, [g_{st}]) \geq cm_o$, where m_o is the mass of a one point blow-up (stereographic projection) of M.

$(M^{2n+1}, \theta) \subset \mathbb{C}^{n+1}$ - strongly pseudo-convex CR manifold.

Theorem (D. Jerison & J. Lee '88)

If θ is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar{\theta}$ on the unit sphere in $\mathbb{C}^{n+1}$ and $Scal_\theta =$const, then up to a multiplicative constant $\theta = \Phi^* \, \bar{\theta}$ with Φ a CR automorphism of the sphere.

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Theorem (J. Lee '88)

If $(M, \bar{\theta})$ is pseudo-Einstein, then $\theta = e^{2u}\bar{\theta}$ is pseudo-Einstein iff u is CR-pluriharmonic on M.

The CR Yamabe problem II

Theorem (D. Jerison & J. Lee '87-'89)

a) $\Upsilon([\theta]) \leq \Upsilon(S^{2n+1})$, *where* $S^{2n+1} \subset \mathbb{C}^{n+1}$ *is the sphere with its standard CR structure. If* $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$, *then the Yamabe equation has a solution. [D. Jerison & J. Lee '87]*

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b) *If* $n \geq 2$ *and* M *is not locally CR equivalent to* S^{2n+1}, *then* $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$. *[D. Jerison & J. Lee '89]*

$$Y(\theta_\epsilon) = \begin{cases} Y(S^{2n+1})\left(1 - c_n|S(q)|^2\epsilon^4\right) + \mathcal{O}(\epsilon^5), & n \geq 2; \\ Y(S^5)\left(1 + c_2|S(q)|^2\epsilon^4 \ln\epsilon\right) + \mathcal{O}(\epsilon^4), & n = 2. \end{cases}$$

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c) *If* $n = 1$ *or* M *is locally CR equivalent to* S^{2n+1}, *then the Yamabe equation has a solution. [R. Yacoub '01, N. Gamara & R. Yacoub, 01]*

- $\mathbb{H}$-quaternions, $q = t + ix + jy + kz$, where $t, x, y, z \in \mathbb{R}$ and $i,\ j,\ k$ satisfy the multiplication rules

$$i^2 \ = \ j^2 \ = \ k^2 \ = \ -1 \text{ and } ijk \ = \ -1.$$

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- $\mathbb{H}^n$-quaternionic space, $q = (q^1, \ldots, q^n), \quad q^\alpha \in \mathbb{H}, q^\alpha = t^\alpha + ix^\alpha + jy^\alpha + kz^\alpha$ for $\alpha = 1, \ldots, n$. Conjugation: $q^{\overline{\alpha}} = \overline{q^\alpha}$, i.e., $q^{\overline{\alpha}} = t^\alpha - ix^\alpha - jy^\alpha - kz^\alpha$.

  Inner product: $\langle q, q' \rangle = q \cdot \bar{q}' = \sum_{\alpha=1}^{n} q^\alpha \cdot q'^{\bar{\alpha}}$

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- Almost complex structures:

$$Iq = qi,\ Jq = qj,\ Jq = qj, \qquad \text{and} \qquad aI + bJ + cK,\ a^2 + b^2 + c^2 = 1.$$

Definition

M^{4n+3}-quaternionic contact if we have

i) *codimension three distribution H, locally, $H = \bigcap_{s=1}^{3} Ker\, \eta_s$, $\eta_s \in S^2_\eta$.*

ii) *a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures $I_s : H \to H$, $I_s^2 = -1$, satisfying $I_1 I_2 = -I_2 I_1 = I_3$ and $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 : a^2 + b^2 + c^2 = 1\}$;*

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Theorem (O. Biquard)

Under the above conditions and $n > 1$, there exists a unique supplementary distribution V of H in TM and a linear connection ∇ on M, s.t.,

1. *V and H are parallel*
2. *g and $\Omega = \sum_{j=1}^{3} (d\eta_j|_H)^2$ are parallel*
3. *torsion $T(A, B) = \nabla_A B - \nabla_B A - [A, B]$ satisfies*
    - $\forall X, Y \in H, \quad T_{X,Y} = -[X, Y]|_V \in V$
    - $\forall \xi \in V, \quad T_\xi := (X \mapsto (T_{\xi,X})_H) \in (sp(n) + sp(1))^\perp$

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- Note: V is generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$$

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- If the dimension of M is seven, $n = 1$, the above conditions do not always hold. Duchemin shows that if we assume, in addition, the existence of Reeb vector fields as above, then there is a connection as before. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying the Reeb conditions

- curvature: $\mathcal{R}(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$;
- (horizontal) Ricci tensor: $Ric(X,Y) = Ric^{\nabla}|_H = tr_H\{Z \mapsto \mathcal{R}(Z,X)Y\}$ for $X, Y \in H$
- scalar curvature: $\mathrm{Scal} = tr_H\, Ric$.

$\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\mathbb{H}, \quad (q,\omega) \in \boldsymbol{G}\,(\mathbb{H}),$

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$$\tilde{\Theta}_1 \;=\; \frac{1}{2}\, dx \;-\; x^\alpha dt^\alpha \;+\; t^\alpha dx^\alpha \;-\; z^\alpha dy^\alpha \;+\; y^\alpha dz^\alpha$$

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$$\tilde{\Theta}_2 = \frac{1}{2}\, dz - z^{\alpha} dt^{\alpha} - y^{\alpha} dx^{\alpha} + x^{\alpha} dy^{\alpha} + t^{\alpha} dz^{\alpha}.$$

ii) Left-invariant horizontal vector fields

$$T_{\alpha} = \frac{\partial}{\partial t_{\alpha}} + 2x^{\alpha}\frac{\partial}{\partial x} + 2y^{\alpha}\frac{\partial}{\partial y} + 2z^{\alpha}\frac{\partial}{\partial z} \quad X_{\alpha} = \frac{\partial}{\partial x_{\alpha}} - 2t^{\alpha}\frac{\partial}{\partial x} - 2z^{\alpha}\frac{\partial}{\partial y} + 2y^{\alpha}\frac{\partial}{\partial z}$$
$$Y_{\alpha} = \frac{\partial}{\partial y_{\alpha}} + 2z^{\alpha}\frac{\partial}{\partial x} - 2t^{\alpha}\frac{\partial}{\partial y} - 2x^{\alpha}\frac{\partial}{\partial z} \quad Z_{\alpha} = \frac{\partial}{\partial z_{\alpha}} - 2y^{\alpha}\frac{\partial}{\partial x} + 2x^{\alpha}\frac{\partial}{\partial y} - 2t^{\alpha}\frac{\partial}{\partial z}.$$

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Y_\alpha \;&=\; \frac{\partial}{\partial y_\alpha} + 2z^\alpha\frac{\partial}{\partial x} - 2t^\alpha\frac{\partial}{\partial y} - 2x^\alpha\frac{\partial}{\partial z} \quad & Z_\alpha \;&=\; \frac{\partial}{\partial z_\alpha} - 2y^\alpha\frac{\partial}{\partial x} + 2x^\alpha\frac{\partial}{\partial y} - 2t^\alpha\frac{\partial}{\partial z}\,.
\end{aligned}$$

iii) Left-invariant Reeb (vertical) vector fields ξ_1, ξ_2, ξ_3 are

$$\xi_1 = 2\frac{\partial}{\partial x} \qquad \xi_2 = 2\frac{\partial}{\partial y} \qquad \xi_3 = 2\frac{\partial}{\partial z}\,.$$

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I_i endomorphism on $\mathbb{H}^n$ using the right multiplication correspondingly by i, j and k.

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$$\tilde{\Theta}_1 \;=\; \frac{1}{2}\, dx \;-\; x^\alpha dt^\alpha \;+\; t^\alpha dx^\alpha \;-\; z^\alpha dy^\alpha \;+\; y^\alpha dz^\alpha$$
$$\tilde{\Theta}_2 \;=\; \frac{1}{2}\, dy \;-\; y^\alpha dt^\alpha \;+\; z^\alpha dx^\alpha \;+\; t^\alpha dy^\alpha \;-\; x^\alpha dz^\alpha$$
$$\tilde{\Theta}_2 \;=\; \frac{1}{2}\, dz \;-\; z^\alpha dt^\alpha \;-\; y^\alpha dx^\alpha \;+\; x^\alpha dy^\alpha \;+\; t^\alpha dz^\alpha .$$

ii) Left-invariant horizontal vector fields

$$T_\alpha \;=\; \frac{\partial}{\partial t_\alpha} \;+2x^\alpha \frac{\partial}{\partial x} + 2y^\alpha \frac{\partial}{\partial y} + 2z^\alpha \frac{\partial}{\partial z} \quad X_\alpha \;=\; \frac{\partial}{\partial x_\alpha} \;-2t^\alpha \frac{\partial}{\partial x} - 2z^\alpha \frac{\partial}{\partial y} + 2y^\alpha \frac{\partial}{\partial z}$$
$$Y_\alpha \;=\; \frac{\partial}{\partial y_\alpha} \;+2z^\alpha \frac{\partial}{\partial x} - 2t^\alpha \frac{\partial}{\partial y} - 2x^\alpha \frac{\partial}{\partial z} \quad Z_\alpha \;=\; \frac{\partial}{\partial z_\alpha} \;-2y^\alpha \frac{\partial}{\partial x} + 2x^\alpha \frac{\partial}{\partial y} - 2t^\alpha \frac{\partial}{\partial z} .$$

iii) Left-invariant Reeb (vertical) vector fields ξ_1, ξ_2, ξ_3 are

$$\xi_1 = 2\frac{\partial}{\partial x} \qquad \xi_2 = 2\frac{\partial}{\partial y} \qquad \xi_3 = 2\frac{\partial}{\partial z} .$$

On $\boldsymbol{G}\left(\mathbb{H}\right)$ let ∇ be the left-invariant connection - this is the Biquard connection. It is flat!

- Contact 3-form on the sphere $S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$,

$$\tilde{\eta} \; = \; dq \cdot \bar{q} \; + \; dp \cdot \bar{p} \; - \; q \cdot d\bar{q} - \; p \cdot d\bar{p}.$$

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- Identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma \;=\; \{(q', p') \in \mathbb{H}^n \times \mathbb{H} \;:\; \mathrm{Re}\, p' \;=\; |q'|^2\},$$

  by using the map $(q', \omega') \mapsto (q', |q'|^2 - \omega')$.

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- Cayley transform, $\mathcal{C} : S \setminus \{\mathrm{pt.}\} \to \Sigma$,

$$(q', p') \;=\; \mathcal{C}\left((q, p)\right) = ((1+p)^{-1}\, q, (1+p)^{-1}\,(1-p)).$$

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- $\mathcal{C}^* \,\tilde{\Theta} \;=\; \frac{1}{2\,|1+p\,|^2}\,\lambda\,\tilde{\eta}\,\bar{\lambda}$, $\quad \lambda$-unit quaternion (eg. of *conformal quaternionic contact map*).

Conformal transformations

$\eta = (\eta_1, \eta_2, \eta_3)$, $\mu \in \mathcal{C}^\infty(M)$, $\mu > 0$, $\Psi \in \mathcal{C}^\infty(M : SO(3))$.

$$\bar{\eta} = \mu \Psi \eta$$

Lemma (O. Biquard '99)

If $\bar{\eta} = u^{4/(Q-2)} \eta$, *then*

$$4 \frac{Q+2}{Q-2} \triangle u - u\, Scal = -u^{2^*-1} \overline{Scal},$$

where $\triangle u = tr_H (\nabla du)$, $Q = 4n + 6$, $2^* = 2Q/(Q-2)$.

- Yamabe functional is

$$\Upsilon(u) = \int_M 4 \frac{Q+2}{Q-2} |\nabla_H u|^2 + \text{Scal}\, u^2\, dv_g.$$

- The Yamabe invariant is the infimum

$$\Upsilon([\eta]) = \inf_u \{\Upsilon(u) : \int_M u^{2^*} dv_g = 1,\ u > 0\}.$$

Theorem (W. Wang '06)

a) $\Upsilon_M([\eta]) \leq \Upsilon_{S^{4n+3}}([\tilde{\eta}])$.

b) *If* $\Upsilon_M([\eta]) < \Upsilon_{S^{4n+3}}([\tilde{\eta}])$, *then the Yamabe problem has a solution.*

- Sp(1)$=\{$unit quaternions$\} \subset$ SO(4n), $\lambda q = q \cdot \lambda^{-1}$.
- Sp(n)-quaternionic unitary $\subset SO(4n)$.
- Sp(n)Sp(1)-product in $SO(4n)$.

Let $\Psi \in$End(H).

- $Sp(n)$-invariant parts as follows

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Explicitly, $4\Psi^{+++} = \Psi - I_1 \Psi I_1 - I_2 \Psi I_2 - I_3 \Psi I_3$, etc.

- The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Using End$(H) \overset{g}{\cong} \Lambda^{1,1}$ the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the operator

$$\Upsilon \; = \; I_1 \otimes I_1 \; + \; I_2 \otimes I_2 \; + \; I_3 \otimes I_3.$$

The Torsion Tensor. $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$.

$T^0_{\xi_j}$-symmetric, $I_j U$-skew-symmetric.

Theorem (w/ St. Ivanov, I. Minchev)

Define $T^0 = T^0_{\xi_j} I_j \in \Psi_{[-1]}$. *We have* $Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g$.

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Definition

M is called qc-Einstein if $T^0 = 0$ and $U = 0$. M is called qc-pseudo-Einstein if $U = 0$.

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Definition

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Theorem (w/ St. Ivanov, I. Minchev)

a) *If M is qc-Einstein then Scal=const.*

b) *Suppose* $Scal \neq 0$. *The next conditions are equivalent:*
   - i) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold;*
   - ii) *M is locally 3-Sasakian: locally there exists a matrix* $\Psi \in \mathcal{C}^\infty(M : SO(3))$, *s.t.*, $(\frac{16n(n+2)}{Scal}\Psi \cdot \eta, Q)$ *is 3-Sasakian;*
   - iii) *The torsion of the Biquard connection is identically zero.*

Transformation Of Torsion Under Conformal Transformation

The components of the torsion tensor transform according to the following formulas: if $\bar{\eta} = \frac{1}{2h}\eta$

- $\overline{T}^0(X, Y) \;=\; T^0(X, Y) \;+\; h^{-1}\,[\nabla dh]_{[sym][-1]}$, where the symmetric part is given by

$$[\nabla dh]_{[sym]}(X, Y) = \nabla dh(X, Y) + \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X, Y).$$

- $\bar{U}(X, Y) \;=\; U(X, Y) \;+\; (2h)^{-1}[\,\nabla dh - 2h^{-1} dh \otimes dh]_{[3][0]}$ or if $f = \frac{1}{2h}$, $\bar{\eta} = f\eta$, then

$$\bar{U}(X, Y) \;=\; U(X, Y) \;-\; (2f)^{-1}[\,\nabla df]_{[3][0]}.$$

Einstein deformations

Theorem (w/ St. Ivanov, I. Minchev)

Let $\Theta = \frac{1}{2h}\tilde{\Theta}$ be a conformal deformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. If Θ is also qc-Einstein, then up to a left translation the function h is given by

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\,(x^2 + y^2 + z^2)\right],$$

where c and ν are positive constants. All functions h of this form have this property.

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where c and ν are positive constants. All functions h of this form have this property.

Lemma (w/ St. Ivanov, I. Minchev)

Let $(M, \bar{\eta})$ be a compact qc-Einstein manifold of dimension $(4n+3)$. Let $\eta = \frac{1}{2h}\,\bar{\eta}$ be a conformal deformation with $Scal_{\eta}$ =const.

a) *If $n > 1$, then any one of the following conditions implies that η is a qc-Einstein structure.*
   i) *the vertical space of η is integrable;*
   ii) *the QC structure η is qc-pseudo Einstein, $U = 0$; ($\nabla^* U = 0$ is enough)*
   ii) *the QC structure η has $\nabla^* T^0 = 0$.*

b) *If $n = 1$ and the vertical space of η is integrable than η is a qc-Einstein structure.*

The Bianchi Identities

$$\sigma_{X,Y,Z}\Big\{R(X,Y,Z,V) - g((\nabla_X T)(Y,Z),V) - g(T(T_{X,Y},Z),V)\Big\} = 0$$

$$\sigma_{X,Y,Z}\Big\{g((\nabla_X R)(Y,Z)V,W) + g(R(T_{X,Y},Z)V,W)\Big\} = 0$$

Theorem (w/ St. Ivanov, I. Minchev)

The divergences of the curvature tensors satisfy the system $B\,b \;=\; 0$*, where*

$$\mathbf{B} \;=\; \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16n(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$$\mathbf{b} \;=\; \Big(\;\nabla^* T^o, \quad \nabla^* U, \quad A, \quad dScal, \quad Ric(\xi_j, I_j\,.)\;\Big)^t$$

and $A = I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2]$.

Note: Horizontal divergence $\nabla^* P$ of a (0,2)-tensor field P is a (0,1)-tensor

$$\nabla^* P(.) = -\sum_{\alpha=1}^{4n} (\nabla_{e_\alpha} P)(e_\alpha, .),$$

where $e_\alpha, \alpha = 1, \ldots, 4n$ is an orthonormal basis of H.

The Divergence Formula

Proposition

Let $(M^{4n+3},\ \eta,\ g_H)$ be a compact closed manifold with a contact quaternionic structure and σ a horizontal 1-form, $\sigma \in \Lambda^1(H)$. Then we have

$$\int_M (\nabla^*\sigma)\, \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_1^{2n} \;=\; 0,$$

where $\nabla^*\sigma \;=\; -(\nabla\sigma)(e_\alpha; e_\alpha)$ and $\{e_\alpha\}_\alpha$ is an ONB frame on H, $\alpha = 1, \ldots, 4n$.

$$[Ric_0]_{[-1]}(X,Y) \;=\; (2n+2)T^0(X,Y) \;=\; -(2n+2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)$$
$$[Ric_0]_{[3]}(X,Y) \;=\; 2(2n+5)U(X,Y) \;=\; -(2n+5)h^{-1}[\nabla dh - 2h^{-1}dh \otimes dh]_{[3][0]}(X,Y).$$

$$\int_M h \,|\, [Ric_o]_{[-1]} \,|^2 \,\eta \wedge \omega^{2n} \;=\; (2n+2) \int \langle [Ric_o]_{[-1]}, \nabla dh] \rangle\, \eta \wedge \omega^{2n}$$
$$= (2n+2) \int_M \langle \nabla^* \, [Ric_o]_{[-1]}, \nabla h] \rangle\, \eta \wedge \omega^{2n} \;=\; 0.$$

Obata type Theorem

Theorem (w/ St. Ivanov, I. Minchev)

Let $\eta = f\tilde{\eta}$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the quaternionic sphere S^{4n+3}. Suppose η has constant qc-scalar curvature and in addition

a) *if $n > 1$*
   i) *the vertical space of η is integrable; or*
   ii) *the function f is the real part of an anti-CRF function;*

b) *if $n = 1$ the vertical space of η is integrable,*

then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism.

$$\phi \in Diff(M), \quad \phi^*\tilde{\eta} = \mu\Psi\tilde{\eta}, \quad \Psi \in \mathcal{C}^\infty(M : SO(3)),$$

$$\eta = \phi^*\tilde{\eta}.$$

Note: On a 3-Sasakian, $df = d_1 w + d_2 u + d_3 v \quad mod \quad \tilde{\eta}$ implies $[\tilde{\nabla} df]_{[3][0]} = 0$.

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Note: On a 3-Sasakian, $df = d_1 w + d_2 u + d_3 v \quad mod \quad \tilde{\eta}$ implies $[\tilde{\nabla} df]_{[3][0]} = 0$. Recall, $U(X,Y) \ = \ \tilde{U}(X,Y) \ - \ (2f)^{-1}[\ \tilde{\nabla} df]_{[3][0]}$..

Seven Dimensional Case. Recall: $T^0 = T^0_{\xi_j} I_j$

Theorem (w/ St. Ivanov, I. Minchev)

Let $\tilde\eta = \frac{1}{2h}\eta$, $\tilde\eta$ standard quaternionic contact structure on the quaternionic unit sphere S^7. If η has constant qc-scalar curvature, then up to a multiplicative constant η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism. Furthermore, $\lambda(S^7) = \Upsilon(\tilde\eta) = 48\,(4\pi)^{1/5}$ and this minimum value is achieved only by $\tilde\eta$ and its images under conformal quaternionic contact automorphisms.

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Theorem

Suppose (M^7,η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^7,\bar\eta)$, $\tilde\eta \;=\; \frac{1}{2h}\,\eta$. If $Scal_\eta = Scal_{\tilde\eta} = 16n(n+2)$, $f = \frac12 \;+\; h \;+\; \frac14 h^{-2}|\nabla h|^2$ we have

$$div\Big\{ fD + \sum_{s=1}^{3}\Big(dh(\xi_s)\,F_s + 4dh(\xi_s)I_sA_s - \frac{10}{3}dh(\xi_s)\,I_sA\Big)\Big\} = f|T^0|^2 + h\,\langle QV,\ V\rangle .$$

Here, Q is a positive definite matrix, $V = (D_1,D_2,D_3,A_1,A_2,A_3)$, $A_i \;=\; I_i[\xi_j,\xi_k]$, $A = A_1 + A_2 + A_3$.

$$D_1(X) \;=\; -h^{-1}T^{0^{+---}}(X,\nabla h),\ D_2(X) \;=\; -h^{-1}T^{0^{-+-}}(X,\nabla h),\ D_3(X) \;=\; -h^{-1}T^{0^{--+}}(X,\nabla h),$$

$$F_s(X) \;=\; -h^{-1}\;T^0(X,I_s\nabla h), \quad s = 1,2,3.$$

Theorem (Folland and Stein)

Let $\mathbf{G} = \mathbb{H} \times Im\,\mathbb{H}$ *and* $\Omega \subset \mathbf{G}$. *There is* $S_2 = S_2(\mathbf{G}) > 0$, *such that, for* $u \in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{2^*} \, dH(g) \right)^{1/2^*} \leq S_2 \left(\int_\Omega |\nabla u|^2 \, dH(g) \right)^{1/2}, \qquad 2^* = 5/4.$$

Folland-Stein inequality In Dimension Seven

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Theorem (w/ St. Ivanov, I. Minchev)

Let $\mathbf{G} = \mathbb{H} \times Im\,\mathbb{H}$. *The best constant in the* L^2 *Folland-Stein embedding theorem is*

$$S_2 = \frac{15^{1/10}}{\pi^{2/5}\, 2\sqrt{2}}.$$

An extremal is given by the function ($\gamma = \ldots$)

$$F(g) = \gamma \left[(1 + |q|^2)^2 + 16|\omega|^2) \right]^{-2}.$$

Any other non-negative extremal is obtained from F by translations and dilations.

More consequences of the Bianchi identities

Theorem (Ivanov,V.)

The following tensors

- $R(X,Y,Z,V) - R(Z,V,X,Y)$
- $4R_{[-1]}(X,Y,Z,V) = 3R(X,Y,Z,V) - R(I_1X,I_1Y,Z,V) - R(I_2X,I_2Y,Z,V) - R(I_3X,I_3Y,Z,V)$
- $R(\xi_i,X,Y,Z)$
- $R(\xi_i,\xi_j,X,Y)$

are determined by the (horizontal!) torsion tensor, i.e., T^0, U and $Scal$.

Corrolary

A QC manifold is locally isomorphic to the quaternionic Heisenberg group exactly when the curvature of the Biquard connection restricted to H vanishes, $R_{|_H} = 0$.

- ”Schouten” tensor $L(X,Y) = \frac{1}{2}T^0(X,Y) + U(X,Y) + \frac{Scal}{32n(n+2)}\,g(X,Y)$.
- Conformal curvature

$$\begin{aligned} W^{qc}(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_s L)(X,Y,Z,V) \\ &- \frac{1}{2}\sum_{(i,j,k)} \omega_i(X,Y)\Big[L(Z,I_iV) - L(I_iZ,V) + L(I_jZ,I_kV) - L(I_kZ,I_jV)\Big] \\ &- \sum_{s=1}^{3}\omega_s(Z,V)\Big[L(X,I_sY) - L(I_sX,Y)\Big] + \frac{1}{2n}(trL)\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V), \end{aligned}$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum.

Properties of the Conformal Curvature.

Proposition

a) *The [-1]-part w. r. t. the first two arguments of W^{qc} vanishes,*
$W^{qc}_{[-1]}(X, Y, Z, V) = \frac{1}{4}\Big[3W^{qc}(X, Y, Z, V) - \sum_{s=1}^{3} W^{qc}(I_s X, I_s Y, Z, V)\Big] = 0.$

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b) *The [3]-part w. r. t. the first two arguments of W^{qc} is determined by the torsion and the scalar curvature*

$$
\begin{aligned}
W^{qc}_{[3]}(X,Y,Z,V) &= \frac{1}{4}\Big[R(X,Y,Z,V) + \sum_{s=1}^{3} R(I_sX,I_sY,Z,V)\Big] \\
&\quad - \frac{1}{2}\sum_{s=1}^{3}\omega_s(Z,V)\Big[T^0(X,I_sY) - T^0(I_sX,Y)\Big] \\
&\quad + \frac{Scal}{32n(n+2)}\Big[(g \owedge g)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge \omega_s)(X,Y,Z,V)\Big] \\
&\quad + (g \owedge U)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_sU)(X,Y,Z,V),
\end{aligned}
$$

where

$$
W^{qc}_{[3]}(X,Y,Z,V) = \frac{1}{4}\Big[W^{qc}(X,Y,Z,V) + \sum_{s=1}^{3} W^{qc}(I_sX,I_sY,Z,V)\Big].
$$

Conformal Flatness and Ferrand-Obata Type Theorem

Theorem (w/ St. Ivanov)

a) The qc conformal curvature W^{qc} is invariant under quaternionic contact conformal transformations, i.e., if

$$\bar\eta = \phi\Psi\eta \quad \textit{then} \qquad W^{qc}_{\bar\eta} = \phi\, W^{qc}_{\eta},$$

for any smooth positive function ϕ and any $SO(3)$-matrix Ψ.
b) A qc structure on a (4n+3)-dimensional smooth manifold is locally quaternionic contact conformal to the standard flat qc structure on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.

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Corrolary

A qc manifold is locally quaternionic contact conformal to the quaternionic sphere S^{4n+3} if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.

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Corrolary

A qc manifold is locally quaternionic contact conformal to the quaternionic sphere S^{4n+3} if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.

Theorem (w/ St. Ivanov)

Let (M, η) be a compact quaternionic contact manifold and G a connected Lie group of conformal quaternionic contact automorphisms of M. If G is non-compact then M is qc conformally equivalent to the unit sphere S in quaternionic space.