Strong Kaehler metrics with torsion

and

generalized Kaehler structures on torus bundles

Workshop on

"Special Geometries in Mathematical Physics"

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M^{2n} $2n$-dimensional (compact) manifold

• F non-degenerate 2-form on M^{2n}

• J *almost complex structure* on M^{2n}, i.e.
$J \in \mathrm{End}(TM^{2n})$ s.t. $J^2 = -\mathrm{id}_{TM^{2n}}$

• F is *symplectic* if $dF = 0$

• J is *integrable* if it is induced by a complex structure.

Newlander-Nirenberg

$$J \text{ is integrable} \qquad \Longleftrightarrow \qquad N_J = 0$$

where

$$N_J = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

$\forall X, Y$ vector fields on M.

• A Riemannian metric g on (M^{2n}, J) is said to be *J-Hermitian* if

$$g(JX, JY) := g(X, Y)\,, \quad \forall X, Y\,.$$

F symplectic form; J almost complex structure M is said to be *F-calibrated* if

$$g_J[x](X, Y) := F[x](X, JY)$$

is a J-Hermitian metric on M.

• (M, J, F, g_J) *Kähler*, if F is symplectic, J is complex and F-calibrated.

Weaker conditions

(I) $dF = 0$, J non-integrable.

(II) $dF \neq 0$, J integrable.

(I)

- Special symplectic manifolds,
- Geometry of Lagrangian submanifolds.

(II)

- *Geometry with Torsion,*
- *Generalized Kähler Geometry,*
- *Bi-Hermitian Structures,*
- *Special metrics on Complex manifolds e.g.* balanced, strong KT, astheno-Kähler

Special Symplectic Manifolds

Def 1. A *special symplectic Calabi-Yau manifold* (SSCY) is the datum of (M^6, F, J, ψ) where

- F is a symplectic structure
- J is a F-calibrated almost complex structure
- $g_J(\cdot,\cdot) := F(\cdot, J\cdot)$
- $\psi \in \wedge^{3,0}(M)$, $\psi \neq 0$,

s.t.

$$\begin{aligned} d\mathfrak{Re}\,\psi &= 0 \\ \psi \wedge \overline{\psi} &= \frac{4}{3} i\, F^3 \end{aligned}$$

Rem.

- If $d\mathfrak{Re}\,\psi = 0 = d\mathfrak{Im}\,\psi$, then J is a complex structure.

• $\mathfrak{Re}\,\psi$ is a *calibration* (see Harvey e Lawson).

Theorem (P. de Bartolomeis,—) There exists a compact complex manifold M such that

• M has a symplectic structure satisfying the Hard Lefschetz Condition;

• M admits a SSCY structure;

• M has no Kähler structures.

$$M = (\mathbb{C}^3, *)/\Gamma$$

where $*$ is defined by

$${}^t(z_1, z_2, z_3) *{}^t (w_1, w_2, w_3) =$$

$${}^t(z_1 + w_1, e^{-w_1} z_2 + w_2, e^{w_1} z_3 + w_3)$$

and Γ is a certain closed subgroup of $(\mathbb{C}^3, *)$ finitely generated.

• In [Conti,—](*Quarterly J.* '07) nilmanifolds carrying SSCY-structures are classified.

• For other results in higher dimensions [de Bartolomeis,—](*Inter. J. Math.* '06).

Generalized Complex Geometry

<u>*Indefinite metric*</u>

V real vector space of dimension n.

$$(,) : V \oplus V^* \to \mathbb{R}$$

$$(v + \xi, w + \eta) = \frac{1}{2}(\xi(w) + \eta(v))$$

$(,)$ is the *natural indefinite metric* on $V \oplus V^*$ with signature (n, n).

<u>*Twisted Courant bracket*</u>

M manifold, H closed 3-form.

Def.

$$[,] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$$

$$\begin{aligned} [X + \xi, Y + \eta] &= [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \\ &- \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + \iota_X \iota_Y H \,, \end{aligned}$$

$\mathcal{L}_X$ Lie derivative along X, ι_X contractions along X.

Generalized Complex Structures

M $2n$-dimensional manifold, $(,)$ indefinite metric on $TM \oplus T^*M$.

Def. *A* generalized complex structure *on* M *(*GC structure*) is the datum of a subbundle* $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ *such that*

- $E \oplus \overline{E} = (TM \oplus T^*M) \otimes \mathbb{C}$

- *the space of sections of* E *is closed with respect to the Courant bracket*

- E *is isotropic.*

Basic examples

• M complex manifold. Then

$$E = T^{0,1}M \oplus T^{1,0*}M$$

defines a GC structure on M.

i) $E \oplus \overline{E} = (TM \oplus T^*M) \otimes \mathbb{C}$.

ii) If $Z + \varphi$, $W + \psi \in E$, then

$$(Z + \varphi, W + \psi) = \frac{1}{2}(\varphi(W) + \psi(Z)) = 0\,.$$

iii)

$$\begin{aligned} [Z + \varphi, W + \psi] &= [Z, W] + \mathcal{L}_Z\psi - \mathcal{L}_W\varphi + \\ &\quad -\tfrac{1}{2}d\,(\iota_Z\psi - \iota_W\varphi) \\ &= [Z, W] + \mathcal{L}_Z\psi - \mathcal{L}_W\varphi\,. \end{aligned}$$

Since $\mathcal{L}_Z\psi$, $\mathcal{L}_W\varphi \in T^{1,0*}M$, then E is involutive.

• (M, ω) symplectic manifold. Then

$$E = \{Z - i\, Z \lrcorner \omega \mid Z \in TM \otimes \mathbb{C}\}$$

is a GC structure on M.

• *Equivalently, a GC structure can be viewed as an almost complex structure* $\mathcal{J} \in \mathrm{End}(TM \oplus T^*M)$*, which is* $(,)$*-orthogonal and integrable with respect to the Courant bracket*.

The previous examples shows that

Generalized Kähler structures

M $2n$-dimensional manifold.

Def. *A* generalized Kähler structure *on* M (GK structure) *is a pair of generalized complex structures* $(\mathcal{J}_1, \mathcal{J}_2)$ *on* M *such that*

- $\mathcal{J}_1$ *e* $\mathcal{J}_2$ *commute*
- $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *are compatible with the natural pairing* $(,)$ *on* $TM \oplus T^*M$
- $-(\mathcal{J}_1\mathcal{J}_2 \cdot, \cdot)$ *is positive definite*

$\square$

In terms of *bi-Hermitian geometry*

Theorem (Apostolov and Gualtieri, *Comm. Math. Phys.* '07)
A GK structure on M is equivalent to assign a triple (g, J_+, J_-) where:

- *g is a Riemannian metric on M*
- *J_+ and J_- are two complex structures on M, compatible with g and such that*

$$d^c_+ F_+ + d^c_- F_- = 0\,, \quad dd^c_+ F_+ = 0\,, \quad dd^c_- F_- = 0\,,$$

F_+, F_- fundamental forms of (g, J_+), (g, J_-),

$$d^c_+ = i(\overline{\partial}_+ - \partial_+)\,, \quad d^c_- = i(\overline{\partial}_- - \partial_-)\,.$$

□

$d^c_+ F_+$ *torsion form* of the GK structure.

Example (M, g, J) Kähler

$$J_+ = J\,, \quad J_- = \pm J$$

$\Rightarrow (g, J_+, J_-)$ GK structure on M.

□

Pb. When does a compact complex manifold (M, J) admit a *GK* structure (g, J_+, J_-) with $J = J_+$?

Interesting case: $J_+ \neq \pm J_-$, i.e. the GK structure is not induced by a Kähler metric on (M, J).

Strong KT geometry

(M, J, g) Hermitian manifold

∇ *Bismut connection*

$\nabla g = 0\,, \quad \nabla J = 0\,,$

$g(X, T^{\nabla}(Y, Z))$ totally skew-symmetric

The torsion form

$$T(X, Y, Z) = g(X, T^{\nabla}(Y, Z))$$

is JdF, where F is the fundamental form of g.

Def. *A Hermitian metric* g *on a complex manifold* (M, J) *is said to be* strong Kähler with torsion (strong KT) *if the fundamental form* F *is* $\partial\overline{\partial}$*-closed, i.e.*

$$\partial\overline{\partial}\, F = 0\,.$$

Rem. (M, J) $GK \Longrightarrow (M, J)$ has a strong KT metric.

• (M, J) compact complex surface $\Rightarrow$ any conformal class of a Hermitian metric has a strong KT representative (Gauduchon, *Math. Ann.* '84).

• $\dim_{\mathbb{R}} M > 4$ compact examples of strong KT metrics on nilmanifolds (Fino, Parton, Salamon, *Comm. Math. Helv. '04*).

Existence results

• (M, J) compact complex surface.

Classification theorem of generalized Kähler structures

(Apostolov and Gualtieri, *Comm. Math. Phys.* '07)

• $\dim_{\mathbb{R}} M = 6$.
By [Cavalcanti and Gualtieri, *J. of Sympl. Geom.* '05]

any nilmanifold carries a GC structure

• $\dim_{\mathbb{R}} M = 2n$

there are no nilmanifolds (*different from Tori*) *admitting an invariant GK structure.*

(Cavalcanti, *Topol. and its Applic.* '06)

Compact example

- $\mathfrak{s}_{a,b}$ solvable Lie algebra defined by:

$$\begin{cases} de^1 = a\, e^1 \wedge e^2\,, \\ de^2 = 0\,, \\ de^3 = \frac{1}{2}a\, e^2 \wedge e^3\,, \\ de^4 = \frac{1}{2}a\, e^2 \wedge e^4\,, \\ de^5 = b\, e^2 \wedge e^6\,, \\ de^6 = -b\, e^2 \wedge e^5\,, \end{cases} \tag{1}$$

a, b real parameters different from zero.

- $S_{a,b}$ simply-connected Lie group whose Lie algebra is $\mathfrak{s}_{a,b}$

$(t, x_1, x_2, x_3, x_4, x_5)$ global coordinates on $\mathbb{R}^6$.

- Product on $S_{a,b}$

$$(t, x_1, x_2, x_3, x_4, x_5) \cdot (t', x_1', x_2', x_3', x_4', x_5') =$$

$$(t + t', e^{a\,t} x_1' + x_1, e^{\frac{a}{2}t} x_2' + x_2, e^{\frac{a}{2}t} x_3' + x_3,$$

$$x_4' \cos(b\,t) - x_5' \sin(b\,t) + x_4,$$

$$x_4' \sin(b\,t) + x_5' \cos(b\,t) + x_5).$$

- $S_{a,b}$ unimodular semidirect product

$$\mathbb{R} \ltimes_\varphi (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2),$$

$\varphi = (\varphi_1, \varphi_2)$ diagonal action of $\mathbb{R}$ on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$.

Theorem (A. Fino, —)(to appear in *J. of Sympl. Geom.*)

- $S_{1,\frac{\pi}{2}}$ has a compact quotient

$$M^6 = S_{1,\frac{\pi}{2}}/\Gamma\,.$$

- M^6 is the total space of a $\mathbb{T}^2$-bundle over the Inoue surface.

- $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$ has a non-trivial left invariant GK structure.

- $b_1(M^6) = 1 \Rightarrow M^6$ has no Kähler metrics.

$\square$

• GK structure on $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$

$$\varphi_+^1 = e^1 + ie^2, \quad \varphi_+^2 = e^3 + ie^4, \quad \varphi_+^3 = e^5 + ie^6,$$
$$\varphi_-^1 = e^1 - ie^2, \quad \varphi_-^2 = e^3 + ie^4, \quad \varphi_-^3 = e^5 + ie^6.$$

$(\varphi_\pm^1, \varphi_\pm^2, \varphi_\pm^3)$ $(1,0)$-forms associated with $J_\pm$.

• $J_\pm$ integrable.

• $g = \sum_{\alpha=1}^{6} e^\alpha \otimes e^\alpha$ $\qquad J_\pm$-Hermitian.

Then

$$d_+^c F_+ + d_-^c F = 0\,, \quad dd_+^c F_+ = 0\,, \quad dd_-^c F_- = 0\,,$$

(g, J_+, J_-) defines a left-invariant GK structure on M^6.

$d^c_+ F_+ = e^1 \wedge e^3 \wedge e^4$ closed non-exact

<u>*Uniform subgroup*</u>

- $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ where

$$\begin{aligned}(t,u,z,w) * (t',u',z',w') \;\; = \;\; & (t+t', c^t u' + u, \\ & \alpha^t z' + z, e^{i\frac{\pi}{2}t} w' + w),\end{aligned}$$

$\forall t, t', u, u' \in \mathbb{R}$ e $z, z', w, w' \in \mathbb{C}$.

- Γ is isomorphic to $\mathbb{Z} \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^2)$

$g_0 : (t,u,z,w) \mapsto (t+1, cu, \alpha z, iw)$,
$g_j : (t,u,z,w) \mapsto (t, u + c_j, z + \alpha_j, w)$, $j = 1,2,3$,
$g_4 : (t,u,z,w) \mapsto (t,u,z,w+1)$,
$g_5 : (t,u,z,w) \mapsto (t,u,z,w+i)$.

It can be checked that

i) Γ acts freely and in a properly discontinuos way on $S_{1,\frac{\pi}{2}}$

ii) $S_{1,\frac{\pi}{2}}/\Gamma$ is compact. Furthermore

$$\begin{aligned} \pi: \ \ \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}) &\to \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C}), \\ (t, u, z, w) &\mapsto (t, u, z) \end{aligned}$$

M^6 is a $\mathbb{T}^2$-bundle over the Inoue surface.