Spin(7)-manifolds of cohomogeneity one

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Introduction

This talk is based on my doctoral thesis, which is supervised by Prof. Dr. L. Schwachhöfer.

The motivation for my work is the paper of Cleyton and Swann on cohomogeneity-one G_2-structures.

Spin(7)-manifolds

Definition 1. A Spin(7)-manifold is a pair of an 8-dimensional manifold M and a four-form Ω which is stabilized by Spin(7).

There exists a canonical metric g on M.

Definition 2. A Spin(7)-manifold is parallel if $\nabla^g \Omega = 0$ or equivalently $d\Omega = 0$.

Properties:

- Holonomy $\subseteq$ Spin(7).
- There is a parallel spinor ($\rightsquigarrow$ interesting for physicists).
- g is Ricci-flat.

Par. Spin(7)-manifolds - Construction

$d\Omega = 0$ is a linear PDE with a nonlinear restriction. ($\rightsquigarrow$ difficult to solve)

First examples by Bryant, Salamon, Joyce ...

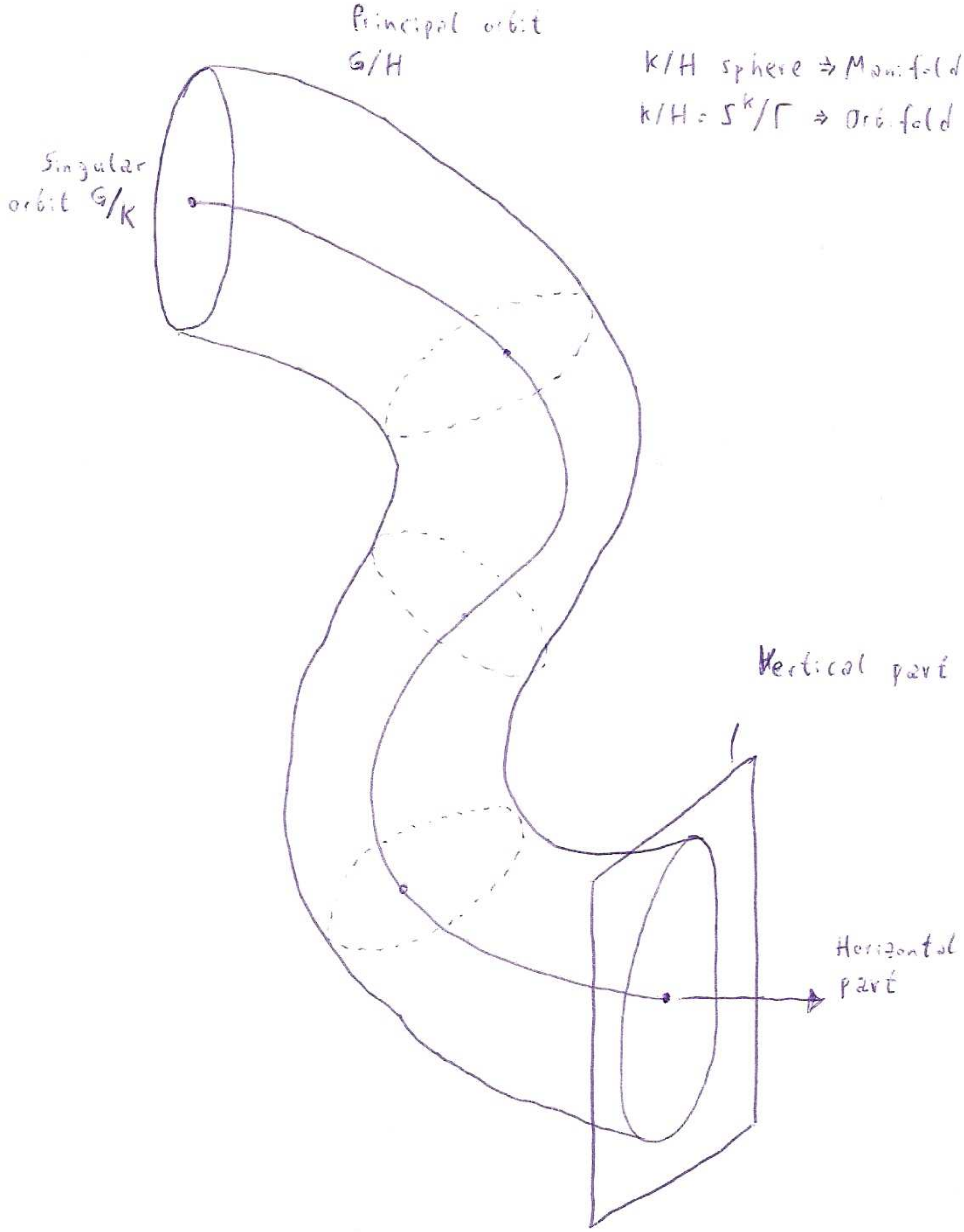
If there is a cohomogeneity-one action on M which preserves Ω, this becomes a non-linear system of ODEs:

$$\begin{aligned} \frac{\partial}{\partial t} * \omega &= d\omega \\ d * \omega &= 0 \end{aligned}$$

where ω is the canonical G_2-structure on the principal orbit. We will solve the ODEs around a singular orbit.

Remark: The above two equations have been considered in a more general context by Hitchin.

Structure of a cohom-one manifold



Possible principal orbits

Classify the principal orbits G/H:

$$\begin{aligned} & \exists \text{ Spin}(7)\text{-structure on } M \\ \Rightarrow & \exists \text{ canonical } G_2\text{-structure on } G/H \\ \Rightarrow & H \subseteq G_2 \end{aligned}$$

Classification result: There are two cases.

Reducible orbits: $G/H = G'/H' \times S^1$, G'/H' admits a $SU(3)$-structure. G'/H' classified by Cleyton and Swann.

Obvious parallel examples: Riemannian products of a parallel G_2-manifold and a circle. Are there other examples? At least not if $G'/H' = SU(3)/U(1)^2$.

Irreducible orbits: S^7, the Aloff-Wallach spaces, ... This list coincides with the list of Friedrich, Kath, Moroianu, and Semmelmann.

G_2-structures on G/H

In order to deduce the ODEs for the holonmy reduction, we have to classify the invariant metrics and G_2-structures on the principal orbit.

- The metrics can be classified by Schur's lemma. We only consider diagonal metrics.

  If the tangent space splits into pairwise inequivalent H-modules, all metrics are diagonal.

- The set of all G_2-structures which have the same associated metric and orientation is given by:

$$(\mathrm{Norm}_{SO(7)}H)/(\mathrm{Norm}_{G_2}H)$$

Problems

1. Sometimes there is no explicit solution of the ODEs.

2. On the singular orbit, there appear terms like $\frac{0}{0}$. We cannot apply Picard-Lindelöf's Theorem. In fact, there are metrics on the singular orbit which cannot be extended to a solution and there are examples where the solution depends on higher derivatives.

3. Do all solutions of the ODEs correspond to smooth metrics?

Power series ansatz

If there is no explicit solution of the ODEs, we make a power series ansatz. In our cases, there are only few G_2-structures with the same associated metric, and we have an ODE-system for g instead of ω:

$$g(t) = \sum_{t=0}^{\infty} g_k t^k$$

Advantages:

1. We find the g_0 for which $g_1, \ldots, g_m$ exist.

2. We find the derivatives of low order, which we can prescribe.

New Problems

1. Does there exist a formal power series solution, such that all g_k correspond to smooth objects?

2. Are there any higher derivatives, which we can prescribe?

3. Does the power series converge?

The results of Wang and Eschenburg

Since g is Ricci-flat, we can work with the results of Wang and Eschenburg on cohomogeneity-one Einstein metrics.

Technical condition: The tangent and the normal space of G/K have no equivalent H-modules in common.

With help of representation theoretical arguments, we can solve some of our problems:

1. We find the smoothness conditions on the coefficients of the metrics.
2. We find an upper bound on the number and order of the initial conditions which we can prescribe.
3. The power series converges **locally**.

As a by-product, we obtain new local examples of Einstein-metrics.

To do

We have to prove that

1. there exists a solution of the ODEs which satisfies the initial conditions.

   This can be proven with help of a variation of the Peano existence theorem.

2. the solution satisfies the smoothness conditions.

   This can be proven by symmetry considerations.

The principal orbit $Q^{1,1,1}$

Let $G/H = SU(2)^3/U(1)^2$. These spaces are denoted by $Q^{k,l,m}$ depending on the embedding $U(1)^2 \hookrightarrow SU(2)^3$.

1. G/H has to be $Q^{1,1,1}$ in order to admit a homogeneous G_2-structure.
2. The ODEs have explicit solutions: They have holonomy $SU(4)$ and are complete and noncompact.
3. Possible singular orbits: $S^2 \times S^2 \times S^2$ and $S^2 \times S^2$. In the first case, g is not smooth. In the second case, it is smooth.

Remark: These metrics have already been considered by Cvetič, Gibbons, Lü, and Pope. The methods which we have applied and the proof that there are no other metrics of holonomy $\subseteq$ Spin(7) is new.

The principal orbit $M^{1,1,0}$

We consider spaces of type $G/H = (SU(3) \times SU(2))/(SU(2) \times U(1))$, where the $SU(2) \subseteq SU(2) \times U(1)$ shall be embedded into $SU(3)$.

1. The embedding of $U(1)$ has to be special $\rightsquigarrow$ $G/H = M^{1,1,0}$.
2. Again, we obtain explicit complete, non-compact metrics with holonomy $SU(4)$.
3. Possible singular orbits: $\mathbb{CP}^2 \times S^2$, $\mathbb{CP}^2$, and S^2. In the third case, we have an orbifold. Smoothness only for $\mathbb{CP}^2$.

Remark: These metrics have been considered by Cvetič, Gibbons, Lü, and Pope, too. Our methods and the proof that there are no further parallel Spin(7)-structures is new.

The Aloff-Wallach spaces I

The Aloff-Wallach space $N^{k,l}$ is defined as $SU(3)/\, U(1)_{k,l}$ where $U(1)_{k,l}$ consists of the matrices

$$\begin{pmatrix} e^{ikt} & 0 & 0 \\ 0 & e^{ilt} & 0 \\ 0 & 0 & e^{-i(k+l)t} \end{pmatrix}$$

We assume that $k \geq l \geq 0$. Any Aloff-Wallach space admits a cosymplectic G_2-structure.

$N^{1,1}$, $N^{1,0}$ are the exceptional Aloff-Wallach spaces. They admit non-diagonal metrics. The other ones are called generic.

The ODEs have explicit solutions only in special cases.

The Aloff-Wallach spaces II

Singular orbit $SU(3)/U(1)^2$: If the principal orbit is not $N^{1,1}$, there are no metrics with holonomy $\subseteq$ Spin(7). If the principal orbit is $N^{1,1}/\mathbb{Z}_2$, any metric on $SU(3)/U(1)^2$ can be extended to a local smooth metric with holonomy $\subseteq$ Spin(7).

Singular orbit S^5: Principal orbit $N^{1,0}$. Up to homothety, there is a one-parameter family of local metrics with holonomy Spin(7).

Singular orbit $SU(3)/SO(3)$: Principal orbit $N^{1,0}$. There are no diagonal examples but possibly non-diagonal ones.

$N^{k,l} \not\cong N^{1,1}$, singular orbit $\mathbb{CP}^2$: We have $K/H = S^3/\mathbb{Z}_{|k+l|}(\times\mathbb{Z}_2)$ (orbifold) and there are metrics of holonomy Spin(7), which depend on the metric on $\mathbb{CP}^2$ and a free parameter of third order.

Principal orbit $N^{1,1}$, singular orbit $\mathbb{CP}^2$: If $K \cong U(2)$ is special, there are **two** free parameters of third order.

The Aloff-Wallach spaces III

Remarks:

1. Let $N^{k,l} \not\cong N^{1,1}$ and the singular orbit be $\mathbb{CP}^2$. For a special choice of the free parameter, there are explicit solutions. (Cf. Cvetič et al., Gukov/Sparks, Kanno/Yasui.)

2. Work on the smoothness and completeness of some of the metrics with principal orbit $N^{1,1}$ has been done by Bazaikin.

3. Power series expansion up to third order for the $SU(3)/U(1)^2$-case and the final $N^{1,1}$-case has been done by Kanno/Yasui.

4. The S^5-example is, as far as the author knows, new.

5. The free parameter(s) now can be explained. We have proven the smoothness and convergence of the power series and that there are no further free parameters.

Thank you for your attention!

ODEs for the Aloff-Wallach space $N^{k,l}$ as principal orbit:

$$\begin{aligned}
\frac{a'}{a} &= \frac{b^2+c^2-a^2}{abc} + \frac{-k-l}{2\Delta}\frac{f}{a^2}\\
\frac{b'}{b} &= \frac{c^2+a^2-b^2}{abc} + \frac{l}{2\Delta}\frac{f}{b^2}\\
\frac{c'}{c} &= \frac{a^2+b^2-c^2}{abc} + \frac{k}{2\Delta}\frac{f}{c^2}\\
\frac{f'}{f} &= -\frac{-k-l}{2\Delta}\frac{f}{a^2} - \frac{l}{2\Delta}\frac{f}{b^2} - \frac{k}{2\Delta}\frac{f}{c^2}
\end{aligned}$$

where $\Delta = k^2 + lk + l^2$.