Weak Mirror Symmetry of Nilmanifolds

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Overview

Motivation

Differential Gerstenhaber algebras (DGA)
(Complex structures, symplectic structures)
Nilmanifolds.

Real 4-dimension as example.
Results in 6-dimension.

Algebraic structure: semi-direct product.

Geometric structure: special Lagrangian, torsion-free flat connections.

Motivation

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DGA, Differential Gerstenhaber Algebras. 1958

$\mathfrak{a} = \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n$ a graded algebra.
An associative product $\wedge$
A graded commutative product $[-,-]$
A differential d of degree $+1$.

- $(\mathfrak{a}, \wedge, d)$, graded (differential) associative algebra;
- $(\mathfrak{a}, [-,-], d)$, odd (differential) graded Lie algebra;
- $\wedge$ and bracket $[-,-]$ an odd distributive rule:

$$[a, b\wedge c] = [a,b]\wedge c + (-1)^{(\deg a+1)\deg b} b\wedge [a,c].$$

The cohomology of d, a Gerstenhaber algebra.

$\mathrm{DGA}(\mathfrak{a}, \wedge, [-,-], d) \approx \mathrm{DGA}(\mathfrak{b}, \bullet, \{-,-\}, \delta)$
quasi-isomorphic if and only if a DGA homomorphism induces
$(H^*_d, \wedge, [-,-]) \cong (H^*_\delta, \bullet, \{-,-\})$ isomorphism.

$\mathrm{DGA}(M, J)$

(M, J) a complex manifold. $\mathrm{DGA}(M, J)$.
$\wedge^*(T^{(1,0)}M \oplus T^{*(0,1)}M)$, $\wedge$ exterior product.
$[X, Y]$ Lie bracket. $[X, \alpha] = \mathcal{L}_X \alpha$. $[\alpha, \beta] = 0$.
$\overline{\partial}$, the usual $\overline{\partial}$-operator.

Remark: Maurer-Cartan equation. $\overline{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$.

The cohomology
$H^1 = H^0(M, \Theta) \oplus H^1(M, \mathcal{O})$.
Holomorphic tangent sheaf Θ. Structure sheaf $\mathcal{O}$.

$$H^2 = H^0(M, \wedge^2\Theta) \oplus H^1(M, \Theta) \oplus H^2(M, \mathcal{O}).$$

Classical: (1,1). Generalized: (0,2)+(2,0).
Extended: everything else.

$\mathrm{DGA}(N, \omega)$

(N, ω) a symplectic manifold. $\mathrm{DGA}(N, \omega)$.
$\wedge^*(T^*M)$, $\wedge$ exterior product.
$[\alpha, \beta]_\omega := \omega[\omega^{-1}\alpha, \omega^{-1}\beta]$. $\omega : TM \to T^*M$
d, exterior differential. Complexification.

Cohomology: De Rham (coefficients in complex numbers)

(M, J) and (N, ω) is a weak mirror pair if
$\mathrm{DGA}(M, J) \approx \mathrm{DGA}(N, \omega)$ Merkulov (2002)

Nilmanifolds

$M := \Gamma\backslash H$.
H simply connected, nilpotent.
Γ, co-compact lattice.
Example. Torus.
Kodaira-Thurston

$$T^2 \hookrightarrow M \longrightarrow T^2.$$

The algebra:

$$\mathfrak{h} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle, \quad [e_1, e_2] = -e_3.$$

Dual equation:

$$de^3 = e^1 \wedge e^2, \quad (0, 0, 12, 0)$$

$\mathrm{DGA}(\mathfrak{k}, \omega)$ and $\mathrm{DGA}(\mathfrak{h}, J)$

$\widehat{\Gamma}\backslash K$, invariant symplectic structure ω.
$\mathrm{DGA}(\mathfrak{k}, \omega) := (\wedge^* \mathfrak{k}^*, \wedge, [-,-]_\omega, d)$.
Nomizu (1958): $H^*_{\mathrm{DR}}(\widehat{\Gamma}\backslash K) \cong H^*_d(\mathfrak{h}^*)$ or

$$\mathrm{DGA}(\widehat{\Gamma}\backslash K, \omega) \approx \mathrm{DGA}(\mathfrak{k}, \omega).$$

$\Gamma\backslash H$, invariant J.
$\mathrm{DGA}(\mathfrak{h}, J) := (\wedge^\bullet \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}, \wedge, [-,-], \overline{\partial})$.

Rollenske (2007), after Fino, ..., Poon:
Often,

$$\mathrm{DGA}(\Gamma\backslash H, J) \approx \mathrm{DGA}(\mathfrak{h}, J).$$

The Issue

Find pseudo-Kählerian $(\mathfrak{h}, J, \omega)$ and $(\widehat{\mathfrak{h}}, \widehat{\omega}, \widehat{J})$ such that

$$\mathrm{DGA}(\mathfrak{h}, J) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\omega}), \quad \mathrm{DGA}(\mathfrak{h}, \omega) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J})$$

$\dim_R \mathfrak{h} = \dim_R \mathfrak{k}$.

$\mathfrak{h}$ and $\mathfrak{k}$ are nilpotent.

pseudo-Kähler

Hasegawa. Benson-Gordon. More than twenty years ago.

Very low dimension

$\dim_R \mathfrak{h} = 2$. No kidding.
$\dim_R \mathfrak{h} = 4$. There is a choice: trivial or non-trivial.
On Kodaira-Thurston "surface",

$$DGA(M, \text{Kodaira's } J) \approx DGA(M, \text{Thurston's } \omega).$$

(Poon 2006 in Crelle)
Proof:
$DGA(M, \omega) \approx DGA(\mathfrak{h}, \omega)$. (Nomizu's)
$DGA(M, J) \approx DGA(\mathfrak{h}, J)$.
A spectral sequence computation over elliptic fibrations.

$$T^2 \hookrightarrow M \longrightarrow T^2.$$

Do the algebra by hands.
Find the isomorphism by eyes.

Six-dimensional

Classifications.
Symplectic structures (Goze-Khakimdjanov, 1996) A table.
Complex structures (Salamon, 2001) Another table.
Pseudo-Kählerian nilpotent algebra. (Cordero, Fernandez, Ugarte, 2006) Yet another table.
Task: Identify $DGA(\mathfrak{h}, J)$ up to (quasi-)isomorphism when $\dim_{\mathbb{R}} \mathfrak{h} = 6$, J is pseudo-Kählerian.
Step 1. Nilpotence $\Longrightarrow$ quasi-isomorphic if and only if isomorphic.
Step 2. $DGA(\mathfrak{h}, J) = \Lambda^\bullet(\mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)})$.
$\mathfrak{k}^1 := \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$. A Lie algebra.
Identify this Lie algebra.
Step 3. Compute all $\mathfrak{k}^1$ for all such $(\mathfrak{h}, J)$, by finding appropriate "invariants" from the structure equations.

Even more tables. (AJM with R. Cleyton)

$\mathfrak{g}\backslash\mathfrak{f}^1(\mathfrak{g},J)$	$\mathfrak{h}_1$	$\mathfrak{h}_3$	$\mathfrak{h}_4$	$\mathfrak{h}_6$	$\mathfrak{h}_7$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$	$\mathfrak{h}_{17}$
$\mathfrak{h}_1$	✓									
$\mathfrak{h}_2$				✓	✓					
$\mathfrak{h}_3$				✓						
$\mathfrak{h}_4$				✓	✓					
$\mathfrak{h}_5$				✓	✓	✓				
$\mathfrak{h}_6$				✓						
$\mathfrak{h}_7$			✓							
$\mathfrak{h}_8$						✓				
$\mathfrak{h}_9$							✓			
$\mathfrak{h}_{10}$								✓		
$\mathfrak{h}_{11}$									✓	
$\mathfrak{h}_{12}$									✓	
$\mathfrak{h}_{13}$									✓	
$\mathfrak{h}_{14}$									✓	
$\mathfrak{h}_{15}$		✓	✓	✓			✓	✓	✓	✓
$\mathfrak{h}_{16}$			✓							

Algebra vs geometry, back to general theory

Special Lagrangian geometry.

$$T^3 \hookrightarrow M \to B^3. \quad \text{Totally real. Lagrangian.}$$

$$\left(T^3\right)^* \hookrightarrow \widehat{M} \to B^3. \quad \text{Same.}$$

On algebra level. Semi-direct product

Abelian ideal and subalgebra. $\mathfrak{h} = \mathfrak{g} \ltimes V$. $\mathfrak{g} \to \mathrm{End}(V)$

Dual semi-direct product. $\widehat{\mathfrak{h}} = \mathfrak{g} \ltimes V^*$.

$0 \to V \to \mathfrak{h} \to \mathfrak{g} \to 0$. $0 \to V^* \to \widehat{\mathfrak{h}} \to \mathfrak{g} \to 0$.

Semi-direct product also appears in $\mathfrak{k}^1 := \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$,

$$[\mathfrak{h}^{(1,0)}, \mathfrak{h}^{(1,0)}] \subseteq \mathfrak{h}^{(1,0)}, \quad [\mathfrak{h}^{(1,0)}, \mathfrak{h}^{*(0,1)}] \subseteq \mathfrak{h}^{*(0,1)}, [\mathfrak{h}^{*(0,1)}, \mathfrak{h}^{*(0,1)}] = \{0\}.$$

$\mathfrak{h}^{(1,0)}$ sub-algebra. $\mathfrak{h}^{*(0,1)}$ abelian ideal.

$J \Rightarrow \widehat{\omega}$, $\omega \Rightarrow \widehat{J}$

Given $(\mathfrak{g} \ltimes V, J, \omega)$, pseudo-Kähler.
get $\mathfrak{g} \ltimes V^*$. Find $(\widehat{J}, \widehat{\omega})$ pseudo-Kähler.
(Fukuya 1999, Ben-Bassat 2006)

$$\mathfrak{h} = \mathfrak{g} \ltimes V, \quad J \quad \Longrightarrow \widehat{\mathfrak{h}} = \mathfrak{g} \ltimes V^*, \quad \widehat{\omega} \quad ?$$

$$\text{Given } J\mathfrak{g} = V, \quad JV = \mathfrak{g}.$$

$$\text{Define } \widehat{\omega}(X, \alpha) := \alpha(JX), \quad X \in \mathfrak{g}, \quad \alpha \in V^*.$$

$$\mathfrak{k} = \mathfrak{g} \ltimes W, \quad \omega \quad \Longrightarrow \widehat{\mathfrak{k}} = \mathfrak{g} \ltimes W^*, \quad \widehat{J} \quad ?$$

$$\text{Given } \omega : \mathfrak{g} \to W^*. \quad \omega^{-1} : W^* \to \mathfrak{g}.$$

$$\text{Define } \widehat{J}(X) = \omega(X), \quad \widehat{J}(\alpha) := -\omega^{-1}(\alpha)$$

$$X \in \mathfrak{g}, \quad \alpha \in W^*.$$

The isomorphism

$$(\mathfrak{g} \ltimes V, J) \Longrightarrow (\mathfrak{g} \ltimes V^*, \widehat{\omega}) \Longrightarrow (\mathfrak{g} \ltimes V, J).$$
$$(\mathfrak{g} \ltimes V, \omega) \Longrightarrow (\mathfrak{g} \ltimes V^*, \widehat{J}) \Longrightarrow (\mathfrak{g} \ltimes V, \omega).$$

Wanted:
$\mathrm{DGA}(\mathfrak{g} \ltimes V, J) \cong \mathrm{DGA}(\mathfrak{g} \ltimes V^*, \widehat{\omega})$
$\mathrm{DGA}(\mathfrak{g} \ltimes V, \omega) \cong \mathrm{DGA}(\mathfrak{g} \ltimes V^*, \widehat{J})$

That is to find a homomorphism
$\phi : \mathrm{DGA}(\mathfrak{g} \ltimes V^*, \widehat{\omega}) \to \mathrm{DGA}(\mathfrak{g} \ltimes V, J)$
i.e. $\phi : (\mathfrak{g} \ltimes V^*)_{\mathbf{C}} \to (\mathfrak{g} \ltimes V)^{(1,0)} \oplus (\mathfrak{g} \ltimes V)^{*(0,1)}$

Explicitly, $\phi(X + \alpha) := (1 - iJ)X + (1 - iJ)\alpha.$

Construction of J, ω in our context

Given the general theory, do it on six-dimensional nilpotent algebras.

1. Construct all six-dimensional pseudo-Kähler $(\mathfrak{g} \ltimes V, \quad J)$.
2. Find $(\mathfrak{g} \ltimes V^*, \quad \widehat{\omega})$.

Step 1. Lie theoretic computation of all possible semi-direct products.
Step 2. Lie theoretic computation of all dual semi-direct products.
Step 3. Impose geometric condition to identify compatible algebraic structures.

That is to start from algebra to geometry.

Alternative: Yoga

Begin with the geometric requirement, find the algebras on $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$.
Given J, totally real. $\gamma(X)Y := -J\rho(X)JY$.
Linear map $\gamma : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Make a connection. $\nabla_X Y = \gamma(X)Y$, for $X, Y \in \mathfrak{g}$.
Totally real, integrable $\leftrightarrow$ torsion-free, flat connection over $\mathfrak{g}$.

Given ω, Lagrangian. Make $\gamma(X)Y := \omega^{-1}\rho^*(X)\omega(Y)$.
Lagrangian, symplectic $\leftrightarrow$ torsion-free, flat connection over $\mathfrak{g}$.

Special Lagrangian $\leftrightarrow$ both γ and γ^* are torsion free, flat.

Results, Tables after tables

Special Lagrangian semi-direct product six-dimensional nilpotent algebras:

$$\mathfrak{h}_1, \quad \mathfrak{h}_4 = (0,0,0,0,12,14+23), \mathfrak{h}_7 = (0,0,0,12,13,23),$$
$$\mathfrak{h}_8 = (0,0,0,0,0,12), \quad \mathfrak{h}_9 = (0,0,0,0,12,14+25),$$
$$\mathfrak{h}_{10} = (0,0,0,12,13,14), \quad \mathfrak{h}_{11} = (0,0,0,12,13,14+23).$$

The mirror pairs:

$(\mathfrak{h}, J, \omega)$	$\mathfrak{h}_1$	$\mathfrak{h}_4$	$\mathfrak{h}_7$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$
$(\widehat{\mathfrak{h}}, \widehat{\omega}, \widehat{J})$	$\mathfrak{h}_1$	$\mathfrak{h}_7$	$\mathfrak{h}_4$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$

Self mirror on $\mathfrak{h}_1$, $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$.
On $\mathfrak{h}_{11}$, same space, different geometry.
$\mathfrak{h}_4$ and $\mathfrak{h}_7$ form a non-trivial pair.

That's all, folks!

Any questions?

What about solvable?

Work in progress. 33.333% done. Kählerian