Workshop

Special geometries in mathematical physics

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Special structures on solvable Lie groups

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Lie algebras and groups in classical mechanics:

the AKS scheme.

[78-79: Adler, Kostant, Symes]

Basic geometric elements:

(i) a symplectic manifold,

(ii) an ad-invariant metric,

(iii) a product structure.

Goal: to study classical mechanical systems

. to solve Hamiltonian equations;

. to study integrability conditions for the Hamiltonian function.

(i) A *symplectic manifold* is a differentiable manifold M equipped with a symplectic structure ω, that is

ω is a non degenerate 2-form such that $d\omega = 0$.

In mechanical systems they appear as the phase space.

Examples.

• $\mathbb{R}^{2n}$ with its canonical symplectic structure given by

$$\omega = \sum_i dq_i \wedge dp_i.$$

global coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

In other words if (,) denotes the canonical inner product on $\mathbb{R}^{2n}$ and J the canonical complex structure

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

then

$$\omega(X, Y) = (X, JY)$$

• Coadjoint orbits:

G a Lie group with Lie algebra $\mathfrak{g}$.

$\mathfrak{g}^*$ the dual space of $\mathfrak{g}$.

coadjoint action: $G \times \mathfrak{g}^* \to \mathfrak{g}^*$:

$$g \cdot \varphi = \varphi \circ Ad(g^{-1}) \quad g \in G$$

The infinitesimal generator induced by $X \in \mathfrak{g}$ at $\varphi \in \mathfrak{g}^*$:

$$\tilde{X}(\varphi) = \frac{d}{dt}_{|t=0} \exp(tX) \cdot \varphi = -\varphi \circ ad(X)$$

Any coadjoint orbit $G \cdot \varphi$ is a symplectic manifold with

$$\omega_\varphi(\tilde{X}, \tilde{Y}) = -\varphi([X,Y]), \qquad \varphi \in \mathfrak{g}^*,\ X, Y \in \mathfrak{g}.$$

the Kirillov-Kostant-Souriau symplectic structure.

(M, ω) a symplectic manifold with

a differentiable $H : M \to \mathbb{R}$,

the *Hamiltonian vector field* X_H associated to H is

$$\omega_p(Y, X_H) = dH_p(Y) \qquad \text{for any } Y \in \chi(M)$$

ω symplectic on $M \to$ *Poisson* $\{\,,\,\}$ on $C^\infty M$:

$$\{f, g\}(p) = \omega_p(X_f, X_g) \qquad \text{for any } f, g \in C^\infty M$$

$$\begin{cases} C^\infty M \text{ is an associative algebra,} \\ \{\,,\,\} \text{ is a Lie bracket on } C^\infty M \\ f\{g,h\} = \{fg,h\} + \{g,fh\} \end{cases}$$

Example:

On $\mathbb{R}^{2n}$, the associated Poisson structure is given by

$$\{f, g\} = (\nabla f, J\nabla g) = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \qquad (1)$$

where ∇f is the gradient of f: $(\nabla f, Y) = df(Y)$ for any vector field Y.

The *Hamiltonian system* for $H : M \to \mathbb{R}$ (Hamiltonian) is:

$$x'(t) = X_H(x(t)) \quad x : \mathbb{R} \to M \text{ a curve.}$$

Example:

On $\mathbb{R}^{2n}$ a *quadratic Hamiltonian* is

$$H(x) = \frac{1}{2}(Ax, x) \quad A \text{ symmetric}$$

which yields the Hamiltonian system

$$x' = JAx \tag{2}$$

In classical mechanic this system describes "small oscillations": motion of a particle on $\mathbb{R}^n$ near an equilibrium position.

For instance for the motion of n-uncoupled harmonic oscillators, take $A = I$, then (2) implies

$$\begin{aligned} q_i'(t) &= p_i(t) \\ p_i'(t) &= -q_i(t) \end{aligned} \tag{3}$$

where $x(t) = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$.

A function f on a 2n-dimensional Poisson manifold $(M, \{\ ,\ \})$ is *completely integrable* if

there exist n functions $f_1, \ldots, f_n \in C^\infty M$ such that:

i) $\{f, f_i\} = 0$, $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq n$,

ii) The differentials $df_1, \ldots, df_n$ are linearly independent on a open set invariant under the flow of X_f.

Example.

On $\mathbb{R}^{2n}$ for $H(x) = (x, x)$ the polynomials

$$f_i(x) = \frac{1}{2}(p_i^2 + q_i^2) \quad i = 1, \ldots, n$$

shows that H is completely integrable.

Moreover since the level sets $\{x \in \mathbb{R}^{2n} : H(x) = c\}$ are compact we have action angle coordinates (Liouville).

GOAL: To apply AKS-Theory for systems (2),

to model it and to study integrability.

The AKS-Theory was applied with *semisimple* Lie algebras and groups (Generalized Toda lattice).

We need:

(ii) *ad-invariant metric* on a Lie algebra $\mathfrak{g}$ is a bilinear map $\langle\, ,\, \rangle$ which is symmetric, non degenerate and satisfies

$$\langle [x,y], z\rangle + \langle [y,[x,z]\rangle = 0 \qquad \text{for all } x,y,z \in \mathfrak{g}$$

If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $\langle\, ,\, \rangle$ induces a bi-invariant metric on G.

Example.

a) semisimple Lie algebras with the Killing form.

b) semidirect products $\mathfrak{g} \ltimes_{coad} \mathfrak{g}^*$ with the canonical neutral metric

$$\langle (x_1, \varphi_1), (x_2, \varphi_2)\rangle = \varphi_1(x_2) + \varphi_2(x_1)$$

(iii) *Product structure* is a triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where $\mathfrak{g}_\pm$ is subalgebra of $\mathfrak{g}$ and such that

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

Example. On $\mathbb{R}^{2n}$ for $H(x) = \frac{1}{2}(x,x)$ (Hamiltonian), with $\{\,,\,\}$ one has

$$\{q_i, p_i\} = 1,\ \{H, q_i\} = p_i,\ \{H, p_i\} = q_i, \quad i = 1, \dots, n$$

therefore $(span\{1, q_i, p_i, H\}, \{\,,\,\})$ is a solvable Lie algebra, we take.

The construction of $\mathfrak{g}$, renaming

$$\begin{array}{ccc} 1 & \leftrightarrow & X_0 \\ q_i & \leftrightarrow & X_i \\ p_i & \leftrightarrow & Y_i \\ H & \leftrightarrow & X_{n+1} \end{array}$$

$\mathfrak{g} = span\{X_0, X_i, Y_i, X_{n+1}\}_{i=1,\dots,n}$ with $[\cdot,\cdot] \leftrightarrow \{\,,\,\}$ is

$\mathfrak{g} = \mathbb{R}X_{n+1} \ltimes \mathfrak{h}_n$,

$\mathfrak{h}_n = span\{X_0, X_i, Y_i\}_{i=1,\dots,n}$ Heisenberg Lie algebra

$$ad(X_{n+1})_{|\mathfrak{h}_n} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$ A block matrix: on $\{X_i, Y_i\}$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $\mathfrak{v} := span\{X_i, Y_i\}_{i=1,\dots,n}$.

$\mathfrak{g}$ can be equipped with the ad-invariant metric $\langle\, , \rangle$:

$$\langle X_0, X_{n+1}\rangle = 1,\ \langle\, , \rangle_{|\mathfrak{v}} = (\, ,),\ span\{X_0, X_{n+1}\} \perp \mathfrak{v} \tag{4}$$

Let $\mathfrak{g}_\pm$ denote the Lie subalgebras

$$\mathfrak{g}_+ = \mathbb{R}X_{n+1}, \qquad \mathfrak{g}_- = \mathbb{R}X_0 \oplus \mathfrak{v}.$$

then $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ direct sum of vector spaces and

$$\mathfrak{g} = \mathfrak{g}_-^\perp \oplus \mathfrak{g}_+^\perp \qquad \begin{aligned} \mathfrak{g}_-^\perp &= \mathbb{R}X_0, \\ \mathfrak{g}_+^\perp &= \mathfrak{v} \oplus \mathbb{R}X_{n+1}. \end{aligned}$$

the map

$$Y \in \mathfrak{g}_+^\perp \longrightarrow \ell_Y \in \mathfrak{g}_-^* : \ell_Y(U_-) = \langle Y, U_-\rangle,\ U_- \in \mathfrak{g}_-$$

is an isomorphism, thus $\mathfrak{g}_+^\perp \simeq \mathfrak{g}_-^*$.

The coadjoint representation from $\mathfrak{g}_-$ on $\mathfrak{g}_-^*$

$$(V_- \cdot \ell_Y)(U_-) = -\ell_Y \circ ad(V_-)(U_-) = -\langle Y, [U_-, V_-]\rangle$$

for $Y \in \mathfrak{g}_+^\perp, U_-, V_- \in \mathfrak{g}_-$. Since

$$-\langle Y, [U_-, V_-]\rangle = \langle [U_-, Y], V_-\rangle$$

one gets an action of $\mathfrak{g}_-$ on $\mathfrak{g}_+^\perp$:

$$U_- \cdot Y = \pi_{\mathfrak{g}_+^\perp}([U_-, Y]), \qquad U_- \in \mathfrak{g}_-, Y \in \mathfrak{g}_+^\perp$$

which corresponds to the action from G_- on $\mathfrak{g}_+^\perp$:

$$g_- \cdot Y = \pi_{\mathfrak{g}_+^\perp}(Ad(g_-)Y) \quad g_- \in G_-, Y \in \mathfrak{g}_+^\perp \tag{5}$$

In our example for $U_- \in \mathfrak{g}_-, Y \in \mathfrak{g}_+^\perp$

$$U_- \cdot Y = x_{n+1}(Y) \sum_i (y_i(U)X_i - x_i(U)Y_i),$$

$$\Longrightarrow \mathcal{M} = G_- \cdot Y = \begin{cases} G_-/\mathbb{R}X_0 & \text{if} \quad x_{n+1}(Y) \neq 0 \\ \{Y\} & \text{if} \quad x_{n+1}(Y) = 0. \end{cases} \tag{6}$$

The infinitesimal generator for $U_- \in \mathfrak{g}_-$ at $Y \in \mathfrak{g}_+^\perp$ is

$$\tilde{U}_-(Y) = \pi_{\mathfrak{g}_+^\perp}([U_-, Y]). \tag{7}$$

The symplectic form on the coadjoint orbits can be induced on these orbits, and we have

$$\omega_Y(\tilde{U}_-, \tilde{V}_-) = \langle Y, [U_-, V_-] \rangle \quad Y \in \mathfrak{g}_+^\perp, U_-, V_- \in \mathfrak{g}_- \tag{8}$$

Let $f : \mathfrak{g} \to \mathbb{R}$ smooth and let $H := f_{|_\mathcal{M}}$, its Hamiltonian vector field

$$X_H(Y) = -\pi_{\mathfrak{g}_+^\perp}([\nabla f_-(Y), Y]) \tag{9}$$

If f is ad-invariant, that is, $df_U[V,U]=0$ for all $U,V\in\mathfrak{g}$, then

$$X_H(Y)=-[\nabla f_-(Y),Y],\quad Y\in\mathfrak{g}_+^\perp \tag{10}$$

and the Hamiltonian equation follows

$$x'=-[\nabla f_-(x),x]=[\nabla f_+(x),x]\quad x \text{ a curve on } \mathfrak{g}_+^\perp \tag{11}$$

In our example, take $f(X)=\frac{1}{2}\langle X,X\rangle$, this is clearly ad-invariant, the restriction to an orbit $\mathcal{M}$ is explicitly

$$H(Y)=\frac{1}{2}\sum_i x_i(Y)^2+y_i(Y)^2\quad Y\in\mathfrak{g}_+^\perp$$

with Hamiltonian system in coordinates

$$\begin{aligned} x_i' &= -x_{n+1}y_i \\ y_i' &= x_{n+1}x_i \\ x_{n+1}' &= 0 \end{aligned}$$

for $x\subset\mathfrak{g}_+^\perp$ with coordinates x_i,y_i,x_{n+1}, $i=1,\ldots,n$.

Choosing $x_{n+1}=1$ and identifying

$$(q_i,p_i)\leftrightarrow(0,x_i,y_i,1)$$

one gets the Hamiltonian system on $\mathbb{R}^{2n}$.

General Case. $\mathbb{R}^{2n}$ with $H(x) = (Ax, x)$, where A symmetric and non singular.

$\mathfrak{v} := span\{X_i, Y_j\}_{i,j=1}^n \simeq \mathbb{R}^{2n}$ with metric $b(X, Y) = (AX, Y)$. Let

$$\mathfrak{g} = \mathbb{R}X_0 \oplus \mathfrak{v} \oplus \mathbb{R}X_{n+1}$$

where the Lie bracket is given by

$$[U, V] = b(JAU, V)X_0 \qquad [X_{n+1}, U] = JAU \qquad \text{for } U \in \mathfrak{v}, \tag{12}$$

Equip $\mathfrak{g}$ with the ad-invariant metric $\langle\, , \rangle$:

$$\langle X_0, X_{n+1}\rangle = 1,\ \langle\, , \rangle_{|\mathfrak{v}} = b,\ span\{X_0, X_{n+1}\} \perp \mathfrak{v} \tag{13}$$

Let $\mathfrak{g}_+ = \mathbb{R}X_{n+1}, \qquad \mathfrak{g}_- = \mathbb{R}X_0 \oplus \mathfrak{v}$.

$\mathfrak{g}_\pm$ is a Lie subalgebra, $\mathbb{R}X_0 \oplus \mathfrak{v} \simeq \mathfrak{h}_n$

and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a product structure.

$$\mathfrak{g}_+^\perp = \mathbb{R}X_{n+1} \oplus \mathfrak{v} \qquad \mathfrak{g}_-^\perp = \mathbb{R}X_0.$$

G_- acts on $\mathfrak{g}_+^\perp = span\{X_i, Y_i\}_{i=1,\ldots,n} \oplus \mathbb{R}X_{n+1}$.

For $U \in \mathfrak{g}_-$, $Y \in \mathfrak{g}_+^\perp$ we have

$$U \cdot Y = x_{n+1}(Y) JAU_{\mathfrak{v}} \tag{14}$$

The orbit $G_- \cdot Y = \begin{cases} \mathcal{M}_{x_{n+1}} \simeq \mathbb{R}^{2n} & x_{n+1}(Y) \neq 0 \\ \{Y\} & x_{n+1}(Y) = 0 \end{cases}$

The symplectic structure at an orbit is

$$\omega_Y(\tilde{U}_-, \tilde{V}_-) = x_{n+1}(Y) b(JAU_{\mathfrak{v}}, V_{\mathfrak{v}})$$

for $Y \in \mathfrak{g}_+^\perp$, $U_-, V_- \in \mathfrak{g}_-$.

Let $f : \mathfrak{g} \to \mathbb{R}$ be the ad-invariant function given by

$$f(X) = \frac{1}{2}\langle X, X \rangle.$$

The gradient of f at X is the position vector

$$\nabla f(X) = X.$$

where ∇ is the gradient of f defined by

$$\langle \nabla f(X), Y \rangle = df_X(Y) \qquad Y \in \mathfrak{g}.$$

The Hamiltonian equation for $x : \mathbb{R} \to \mathfrak{g}$ follows

$$\begin{aligned} x'(t) &= X_H(x) = [\nabla f_+(x), x] \\ &= x_{n+1} J A x_{\mathfrak{v}} \end{aligned} \tag{15}$$

and this can be solved by factorization. The solution passing through x_0 is

$$x(t) = x_{n+1} e^{JA} (x_0)_{\mathfrak{v}}$$

If we take $L, M \in M(2n+2, \mathbb{R})$ as

$$M = \begin{pmatrix} x_{n+1} JA & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} x_{n+1} JA & 0 & z \\ i\frac{1}{2} z^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $z^T = (x_1, x_2, \cdots, x_n, y_1, y_2, \ldots, y_n)$ then the Hamiltonian equation can be written

$$L' = [M, L].$$

The complete integrability of the function $H = f_{|_{\mathcal{M}_{x_{n+1}}}}$.

First: functions in involution.

Consider $g_i, g_j \in C^\infty \mathbb{R}^{2n}$ associated to symmetric maps $A_i, A_j : \mathfrak{v} \to \mathfrak{v}$ resp.:

$$g_i(X) = \frac{1}{2}(A_i X, X) \qquad g_j(X) = \frac{1}{2}(A_j X, X).$$

take extensions of g_i, g_j on $\mathfrak{g}$ for instance as

$$g_i(X) = \frac{1}{2}(A_i X_{\mathfrak{v}}, X_{\mathfrak{v}}) + x_0 x_{n+1}$$

$$g_j(X) = \frac{1}{2}(A_j X_{\mathfrak{v}}, X_{\mathfrak{v}}) + x_0 x_{n+1}.$$

Let H_i, H_j denote the restrictions of g_i, g_j to $\mathcal{M}_{x_{n+1}}$ and let $X \in \mathcal{M}_{x_{n+1}} \subset \mathfrak{g}_+^\perp$.

The Poisson bracket of the functions H_i, H_j on the orbit follows:

$$\{H_i, H_j\}(X) = \langle X, [\nabla g_{i\_}(X), \nabla g_{j\_}(X)]\rangle$$

where the gradient of g_i is

$$\nabla g_i(X) = A^{-1}A_i X_{\mathfrak{v}} + x_0 X_0 + x_{n+1}X_{n+1}$$

(resp. for g_j)

THEOREM. *The functions H_i, H_j are in involution on the orbits $\mathcal{M}_{x_{n+1}}$ if and only if*

$$[JA_i, JA_j] = 0. \tag{16}$$

Proof.

$$\begin{aligned}
\{H_i, H_j\}(X) &= \langle X, [A^{-1}A_i X_{\mathfrak{v}}, A^{-1}A_j X_{\mathfrak{v}}]\rangle \\
&= \langle x_{n+1}[X_{n+1}, A^{-1}A_i X_{\mathfrak{v}}], A^{-1}A_j X_{\mathfrak{v}}\rangle \\
&= x_{n+1}(JA_i X_{\mathfrak{v}}, A_j X_{\mathfrak{v}}) \\
&= x_{n+1}(A_j JA_i X_{\mathfrak{v}}, X_{\mathfrak{v}})
\end{aligned}$$

Since $(A_j JA_i(u+v), u+v) = (A_j JA_i u, v) - (v, A_i JA_j u)$,

$$(A_j JA_i X_{\mathfrak{v}}, X_{\mathfrak{v}}) = 0, \quad \text{if and only if} \quad [JA_j, JA_i] = 0.$$

Relationship with $\mathfrak{h}_n$:

$$\{A \text{ is symmetric on } \mathbb{R}^{2n}\} \longleftrightarrow \left\{ \begin{array}{l} JA \text{ derivation on } \mathfrak{h}_n, \\ \text{trivial on } \mathfrak{z}(\mathfrak{h}_n) \end{array} \right\}$$

Let $\mathfrak{d}$ denote the derivations of $\mathfrak{h}_n$ acting trivially on the center $span\{X_0\}$.

COROLLARY. *If there exists an n-dimensional abelian subalgebra on $z(JA)_{\mathfrak{d}}$, where*

$$z(JA)_{\mathfrak{d}} = \{D \in \mathfrak{d} \text{ such that } [D, JA] = 0\}$$

then the Hamiltonian function $H = f_{|_{\mathcal{M}}}$, restriction of $f(X) = \frac{1}{2}(AX, X)$, is completely integrable on $\mathcal{M}_{x_{n+1}}$ for $x_{n+1} \neq 0$.

t is a derivation of $\mathfrak{h}_n$ acting trivially on $\mathfrak{z}(\mathfrak{h}_n)$ if and only if $Jt + t^*J = 0$, if and only if $t \in \mathfrak{sp}(n)$

THEOREM. [Adler, Kostant, Symes] Let $\mathfrak{g}$ be a Lie algebra with an ad-invariant metric $\langle\, , \rangle$. Assume $\mathfrak{g}_-, \mathfrak{g}_+$ are Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ as direct sum of vector subspaces. Then any pair of ad-invariant functions on $\mathfrak{g}$ Poisson commute on $\mathfrak{g}_+^\perp$ (resp. on $\mathfrak{g}_-^\perp$).

Adler, M., *On a trace for formal pseudodifferential operators and the symplectic structure for the KdV type equations*, Invent. Math., **50**, 219-248, (1979).

Kostant, B., *The solution to a generalized Toda lattice and representation theory*, Advances in Math., **39**, 195 - 338, (1979).

Symes, W., *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math., **59**, 13 - 53, (1978).

Changing the ad-invariant condition for another one as follows.

THEOREM. [Ratiu] With the data as above + $\mathfrak{g}_+$ is an ideal. If f, h are smooth Poisson commuting functions on $\mathfrak{g}$, then the restrictions of f and h to $\mathfrak{g}_+^\perp$ are in involution in the Poisson structure of $\mathfrak{g}_+^\perp$.

Ratiu, T., *The motion of the free n-dimensional rigid body*, Indiana Univ. Math. Journal, **29**, 609 - 629, (1980).

On the Lie algebras $\mathfrak{g}$ as before.

$\mathfrak{g}$ with ad-invariant metric $\langle\, ,\, \rangle$ called: metric, quadratic.

Constructed by a *double extension procedure*

[85, Medina - Revoy] [87, Favre- Santharoubane] (Kac 79)

$\mathfrak{g} = L(G)$, $\langle\, ,\, \rangle$ ad-invariant metric:

Any $\langle\, ,\, \rangle$ induces a right and left invariant metric on G.

the geodesics are $exptX$, with $X \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathbb{R}X_0 \oplus \mathfrak{v} \oplus \mathbb{R}X_{n+1}$, whith Lie bracket

$$[U, V] = b(JAU, V)X_0 \quad [X_{n+1}, U] = JAU \quad \text{for } U \in \mathfrak{v}, \tag{17}$$

has a ad-invariant metric $\langle\, ,\, \rangle$:

$$\langle X_0, X_{n+1}\rangle = 1,\ \langle\, ,\, \rangle_{|\mathfrak{v}} = b,\ span\{X_0, X_{n+1}\} \perp \mathfrak{v} \tag{18}$$

• How many ad-invariant metrics are on $\mathfrak{g}$?

THEOREM. With respect to any ad-invariant metric $\mathfrak{g}$ decomposes into a orthogonal direct sum

$$\mathfrak{g} = span\{X_0, X_{n+1}\} \oplus \mathfrak{v}$$

Any ad-invariant metric on $\mathfrak{g}$ is

$$g = \alpha \langle \, , \rangle + \beta \, dx_{n+1}^2 \quad \alpha \in \mathbb{R} - \{0\}, \, \beta \in \mathbb{R}$$

Denoting by ∇ the Levi Civita connection, $R(x, y)$ the curvature tensor and Ric the Ricci curvature,

explicitly, for $x, y \in \mathfrak{g}$:

$$\begin{aligned} \nabla_x y &= \tfrac{1}{2}[(AJAx_{\mathfrak{v}}, y_{\mathfrak{v}})_0 + x_{n+1} JAy_{\mathfrak{v}} - y_{n+1} JAx_{\mathfrak{v}}] \\ R(x, y)z &= \tfrac{1}{4}[(x_{n+1} A(JA)^2 y_{\mathfrak{v}} - y_{n+1} A(JA)^2 x_{\mathfrak{v}}) X_0 \\ &\quad -z_{n+1}(x_{n+1}(JA)^2 y_{\mathfrak{v}} - y_{n+1}(JA)^2 x_{\mathfrak{v}})] \\ Ric(x, y) &= -\tfrac{1}{4} x_{n+1} y_{n+1} tr((JA)^2) \end{aligned}$$

[FS] G. Favre, L.J. Santharoubane, *Symmetric, invariant, non-degenerate bilinear form on a Lie algebra*, J. Algebra **105** (1987), 451–464.

[MR] Medina, A., Revoy, Ph., *Algèbres de Lie et produit scalaire invariant*, Ann. scient. Éc. Norm. Sup., 4^e série, **t. 18**, 391–404, (1985).

The isometry group, $I(G) = F(G)L(G)$.

1989 Müller [M] proved

f is an isometry of $(G, \langle\, , \rangle)$ such that $f(e) = e$ if and only if

for all $x, y \in \mathfrak{g}$, $df_e := A$ satisfies

i) $\langle Ax, Ay \rangle = \langle x, y \rangle$ ii) $A[x, [x, y]] = [Ax, [Ax, Ay]]$

[M] D. Müller, *Isometries of bi-invariant pseudo-Riemannian metrics on Lie groups*, Geom. Dedicata **29**, (1989), 65-96.