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G_2-HOLONOMY METRICS CONNECTED WITH A 3-SASAKIAN MANIFOLD

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Abstract: We construct complete noncompact Riemannian metrics with G_2-holonomy on noncompact orbifolds that are $\mathbb{R}^3$-bundles with the twistor space $\mathscr{Z}$ as a spherical fiber.

Keywords: exceptional holonomy group, 3-Sasakian manifold, twistor space

1. Introduction

This article addressing G_2-holonomy metrics is a natural continuation of the study of Spin(7)-holonomy metrics which was started in [1]. We consider an arbitrary 7-dimensional compact 3-Sasakian manifold M and discuss the existence of a smooth resolution of the conic metric over the twistor space $\mathscr{Z}$ associated with M.

Briefly speaking, a manifold M is 3-Sasakian if and only if the standard metric on the cone over M is hyper-Kähler. Each manifold of this kind M is closely related to the twistor space $\mathscr{Z}$ which is an orbifold with a Kähler–Einstein metric. We consider the metrics that are natural resolutions of the standard conic metric over $\mathscr{Z}$:

$$\bar{g} = dt^2 + A(t)^2\left(\eta_2^2 + \eta_3^2\right) + B(t)^2\left(\eta_4^2 + \eta_5^2\right) + C(t)^2\left(\eta_6^2 + \eta_7^2\right), \tag{$*$}$$

where η_2 and η_3 are the characteristic 1-forms of M, η_4, η_5, η_6, and η_7 are the forms that annul the 3-Sasakian foliation on M, and A, B, and C are real functions.

One of the main results of the article is the construction (in the case when $M/SU(2)$ is Kähler) of a G_2-structure which is parallel with respect to $(*)$ if and only if the following system of ordinary differential equations is satisfied:

$$A' = \frac{2A^2 - B^2 - C^2}{BC}, \quad B' = \frac{B^2 - C^2 - 2A^2}{CA}, \quad C' = \frac{C^2 - 2A^2 - B^2}{AB}. \tag{$**$}$$

In case $(**)$ we thus see that $(*)$ has holonomy G_2; hence, $(*)$ is Ricci-flat. The system of equations $(**)$ was previously obtained in [2] in the particular case $M = SU(3)/S^1$.

For a solution to $(**)$ to be defined on some orbifold or manifold, some additional boundary conditions are required at t_0 that we will state them later. These conditions cannot be satisfied unless $B = C$, which leads us to the functions that give rise to the solutions found originally in [3] when $M = S^7$ and $M = SU(3)/S^1$. If $B = C$ then $(*)$ is defined on the total space of an $\mathbb{R}^3$-bundle $\mathscr{N}$ over a quaternionic-Kähler orbifold $\mathscr{O}$. In general, $\mathscr{N}$ is not an orbifold except in the event that $M = S^7$ and $M = SU(3)/S^1$. Note that it is unnecessary for $\mathscr{O}$ to be Kähler in case $B = C$.

Finally, we consider the well-known examples of the 3-Sasakian manifolds constructed in [4] and describe the topology of the corresponding orbifolds $\mathscr{N}$.

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2. Construction of a Parallel G_2-Structure

The definition of 3-Sasakian manifolds, their basic properties, and further references can be found in [1]. We mainly take our notation from [1].

Let M be a 7-dimensional compact 3-Sasakian manifold with characteristic fields ξ^1, ξ^2, and ξ^3 and characteristic 1-forms η_1, η_2, and η_3. Consider the principal bundle $\pi : M \to \mathscr{O}$ with the structure group $Sp(1)$ or $SO(3)$ over the quaternionic-Kähler orbifold $\mathscr{O}$ associated with M. We are interested in the special case when $\mathscr{O}$ additionally possesses a Kähler structure.

The field ξ^1 generates a locally free action of the circle S^1 on M, and the metric on the twistor space $\mathscr{Z} = M/S^1$ is a Kähler–Einstein metric. It is obvious that $\mathscr{Z}$ is topologically a bundle over $\mathscr{O}$ with fiber $S^2 = Sp(1)/S^1$ (or $S^2 = SO(3)/S^1$) associated with π. Consider the obvious action of $SO(3)$ on $\mathbb{R}^3$. The two-fold cover $Sp(1) \to SO(3)$ determines the action of $Sp(1)$ on $\mathbb{R}^3$, too. Now, let $\mathscr{N}$ be a bundle over $\mathscr{O}$ with fiber $\mathbb{R}^3$ associated with π. It is easy to see that $\mathscr{O}$ is embedded in $\mathscr{N}$ as the zero section, and $\mathscr{Z}$ is embedded in $\mathscr{N}$ as a spherical section. The space $\mathscr{N}\backslash\mathscr{O}$ is diffeomorphic to the product $\mathscr{Z} \times (0,\infty)$. Note that $\mathscr{N}$ can be assumed to be the projectivization of the bundle $\mathscr{M}_1 \to \mathscr{O}$ of [1]. In general, $\mathscr{N}$ is a 7-dimensional orbifold; however, if M is a regular 3-Sasakian space then $\mathscr{N}$ is a 7-dimensional manifold.

Let $\{e^i\}, i = 0, 2, 3, \dots, 7$, be an orthonormal basis of 1-forms on the standard Euclidean space $\mathbb{R}^7$ (the numeration here is chosen so as to emphasize the connection with the constructions of [1] and to keep the original notation wherever possible). Putting $e^{ijk} = e^i \wedge e^j \wedge e^k$, consider the following 3-form Ψ_0 on $\mathbb{R}^7$:

$$\Psi_0 = -e^{023} - e^{045} + e^{067} + e^{346} - e^{375} - e^{247} + e^{256}.$$

A differential 3-form Ψ on an oriented 7-dimensional Riemannian manifold N defines a G_2-structure if, for each $p \in N$, there exists an orientation-preserving isometry $\phi_p : T_pN \to \mathbb{R}^7$ defined in a neighborhood of p such that $\phi_p^*\Psi_0 = \Psi|_p$. In this case the form Ψ defines the unique metric g_Ψ such that $g_\Psi(v,w) = \langle \phi_p v, \phi_p w\rangle$ for $v, w \in T_pN$ [3]. If the form Ψ is parallel ($\nabla\Psi = 0$) then the holonomy group of the Riemannian manifold N lies in G_2. The parallelness of the form Ψ is equivalent to its closeness and cocloseness [5]:

$$d\Psi = 0, \quad d * \Psi = 0. \tag{1}$$

Note that the form $\Phi_0 = e^1 \wedge \Psi_0 - *\Psi_0$, where $*$ is the Hodge operator in $\mathbb{R}^7$, determines a Spin(7)-structure on $\mathbb{R}^8$ with the orthonormal basis $\{e^i\}_{i=0,1,2,\dots,7}$.

Locally choose an orthonormal system $\eta_4, \eta_5, \eta_6, \eta_7$ that generates the annihilator of the vertical subbundle $\mathscr{V}$ so that

$$\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7), \quad \omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5), \quad \omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6),$$

where the forms ω_i correspond to the quaternionic-Kähler structure on $\mathscr{O}$. It is clear that $\eta_2, \eta_3, \dots, \eta_7$ is an orthonormal basis for M annulling the one-dimensional foliation generated by ξ^1; therefore, we can consider the metric of the following form on $(0,\infty) \times \mathscr{Z}$:

$$\bar{g} = dt^2 + A(t)^2\bigl(\eta_2^2 + \eta_3^2\bigr) + B(t)^2\bigl(\eta_4^2 + \eta_5^2\bigr) + C(t)^2\bigl(\eta_6^2 + \eta_7^2\bigr). \tag{2}$$

Here $A(t)$, $B(t)$, and $C(t)$ are defined on the interval $(0,\infty)$.

We suppose that $\mathscr{O}$ is a Kähler orbifold; therefore, $\mathscr{O}$ has the closed Kähler form that can be lifted to the horizontal subbundle $\mathscr{H}$ as a closed form ω. Without loss of generality we can assume that we locally have

$$\omega = 2(\eta_4 \wedge \eta_5 + \eta_6 \wedge \eta_7).$$

If we now put

$$e^0 = dt, \quad e^i = A\eta_i,\ i = 2,3, \quad e^j = B\eta_j,\ j = 4,5, \quad e^k = C\eta_k,\ k = 6,7,$$

then the forms Ψ_0 and $*\Psi_0$ become

$$\Psi_1 = -e^{023} - \frac{B^2+C^2}{4} e^0 \wedge \omega_1 - \frac{B^2-C^2}{4} e^0 \wedge \omega + \frac{BC}{2} e^3 \wedge \omega_2 - \frac{BC}{2} e^2 \wedge \omega_3,$$

$$\Psi_2 = C^2 B^2 \Omega - \frac{B^2+C^2}{4} e^{23} \wedge \omega_1 - \frac{B^2-C^2}{4} e^{23} \wedge \omega + \frac{BC}{2} e^{02} \wedge \omega_2 + \frac{BC}{2} e^{03} \wedge \omega_3,$$

where $\Omega = \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 = -\frac{1}{8}\omega_1 \wedge \omega_1 = -\frac{1}{8}\omega_2 \wedge \omega_2 = -\frac{1}{8}\omega_3 \wedge \omega_3$.

It is now obvious that Ψ_1 and Ψ_2 are defined globally and independently of the local choice of η_i; consequently, they uniquely define the metric $\bar{g}$ given locally by (2). Then the condition (1) that the holonomy group lies in G_2 is equivalent to the equation

$$d\Psi_1 = d\Psi_2 = 0. \tag{3}$$

Theorem. *If $\mathscr{O}$ possesses a Kähler structure then (2) on $\mathscr{N}$ is a smooth metric with holonomy G_2 given by the form Ψ_1 if and only if the functions A, B, and C defined on the interval $[t_0, \infty)$ satisfy the system of ordinary differential equations*

$$A' = \frac{2A^2 - B^2 - C^2}{BC}, \quad B' = \frac{B^2 - C^2 - 2A^2}{CA}, \quad C' = \frac{C^2 - 2A^2 - B^2}{AB} \tag{4}$$

with the initial conditions

(1) *$A(0) = 0$ and $|A_1'(0)| = 2$;*
(2) *$B(0), C(0) \neq 0$, and $B'(0) = C'(0) = 0$;*
(3) *the functions A, B, and C have fixed sign on the interval (t_0, ∞).*

PROOF. In [1] the following relations were obtained, closing the algebra of forms:

$$\begin{aligned} de^0 &= 0, \\ de^i &= \frac{A_i'}{A_i} e^0 \wedge e^i + A_i \omega_i - \frac{2A_i}{A_{i+1}A_{i+2}} e^{i+1} \wedge e^{i+2}, \quad i = 1,2,3 \bmod 3, \\ d\omega_i &= \frac{2}{A_{i+2}} \omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}} e^{i+1} \wedge \omega_{i+2}, \quad i = 1,2,3 \bmod 3. \end{aligned}$$

By adding the relation $d\omega = 0$ and carrying out some calculations to be omitted here, we obtain the sought system.

The smoothness conditions for the metric at t_0 are proven by analogy with the case of holonomy Spin(7) which was elaborated in [1]. We only note that, taking the quotient of the unit sphere S^3 by the Hopf action of the circle, we obtain the sphere of radius $1/2$, which explains the condition $|A'(0)| = 2$.

In case $B = C$ the system reduces to the pair of equations

$$A' = 2\left(\frac{A^2}{B^2} - 1\right), \quad B' = -2\frac{A}{B}$$

whose solution gives the metric

$$\bar{g} = \frac{dr^2}{1 - r_0^4/r^4} + r^2\left(1 - \frac{r_0^4}{r^4}\right)(\eta_2^2 + \eta_3^2) + 2r^2(\eta_4^2 + \eta_5^2 + \eta_6^2 + \eta_7^2).$$

The regularity conditions hold. This smooth metric was originally found in [3] in the event that $M = SU(3)/S^1$ and $M = S^7$ (observe that we need not require $\mathscr{O}$ to be Kähler when $B = C$).

In the general case $B \neq C$ system (4) can also be integrated [2]. However, the resulting solutions do not enjoy the regularity conditions.

3. Examples

Some interesting family of examples arises when we consider the 7-dimensional biquotients of the Lie group $SU(3)$ as 3-Sasakian manifolds. Namely, let p_1, p_2, and p_3 be pairwise coprime positive integers. Consider the following action of S^1 on the Lie group $SU(3)$:

$$z \in S^1 : A \mapsto \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \mathrm{diag}(1, 1, z^{-p_1-p_2-p_3}).$$

This action is free; moreover, it was demonstrated in [4] that there is a 3-Sasakian structure on the orbit space $\mathscr{S} = \mathscr{S}_{p_1,p_2,p_3}$. Moreover, the action of $SU(2)$ on $SU(3)$ by right translations

$$B \in SU(2) : A \mapsto A \cdot \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

commutes with the action of S^1 and can be pushed forward to the orbit space $\mathscr{S}$. The corresponding Killing fields will be the characteristic fields ξ_i on $\mathscr{S}$. Therefore, the corresponding twistor space $\mathscr{Z} = \mathscr{Z}_{p_1,p_2,p_3}$ is the orbit space of the following action of the torus T^2 on $SU(3)$:

$$(z, u) \in T^2 : A \mapsto \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \mathrm{diag}(u, u^{-1}, z^{-p_1-p_2-p_3}). \tag{5}$$

Lemma. *The space $\mathscr{Z}_{p_1,p_2,p_3}$ is diffeomorphic to the orbit space of $U(3)$ with respect to the following action of T^3:*

$$(z, u, v) \in T^3 : A \mapsto \mathrm{diag}(z^{-p_2-p_3}, z^{-p_1-p_3}, z^{-p_1-p_2}) \cdot A \cdot \mathrm{diag}(u, v, 1). \tag{6}$$

It suffices to verify that each T^3-orbit in $U(3)$ exactly cuts out an orbit of the T^2-action (5) in $SU(3) \subset U(3)$.

Action (6) makes it possible to describe the topology of $\mathscr{Z}$ and, consequently, the topology of $\mathscr{N}$ clearly. Here we use the construction of [6]. Consider the submanifold $E = \{(u, [v]) \mid u \perp v\} \subset S^5 \times \mathbb{C}P^2$. It is obvious that E is diffeomorphic to $U(3)/S^1 \times S^1$ (the "right" part of (6)) and is the projectivization of the $\mathbb{C}^2$-bundle $\widetilde{E} = \{(u, v) \mid u \perp v\} \subset S^5 \times \mathbb{C}^3$ over S^5. By adding the trivial one-dimensional complex bundle over S^5 to $\widetilde{E}$, we obtain the trivial bundle $S^5 \times \mathbb{C}^3$ over S^5.

The group S^1 acts from the left by the automorphisms of the vector bundle $\widetilde{E}$, and $\mathscr{Z} = S^1 \backslash E$ is the projectivization of the $\mathbb{C}^2$-bundle $S^1 \backslash \widetilde{E}$ over the weighted complex projective space $\mathscr{O} = \mathbb{C}P^2(q_1, q_2, q_3) = S^1 \backslash S^5$, where $q_i = (p_{i+1} + p_{i+2})/2$ for p_i all odd and $q_i = (p_{i+1} + p_{i+2})$ otherwise.

The above implies that the bundle $S^1 \backslash \widetilde{E}$ is stably equivalent to the bundle $S^1 \backslash (S^5 \times \mathbb{C}^3)$ over $\mathscr{O}$. The last bundle splits obviously into the Whitney sum $\sum_{i=1}^{3} \xi^{q_i}$, where ξ is an analog of the one-dimensional universal bundle of $\mathscr{O}$.

Corollary. *The twistor space $\mathscr{Z}$ is diffeomorphic to the projectivization of a two-dimensional complex bundle over $\mathbb{C}P^2(q_1, q_2, q_3)$ which is stably equivalent to $\xi^{q_1} \oplus \xi^{q_2} \oplus \xi^{q_3}$.*

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