Morse-Novikov cohomology of locally conformally Kähler manifolds

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Definitions

- $(M, J)$ complex manifold, $\text{dim}_\mathbb{C} M \geq 2$, connected.
- $(M, J)$ is LCK if it admits a Kähler covering
  \[ \Gamma \to (\tilde{M}, J, \Omega) \to (M, J) \]
  such that $\Gamma$ acts by holomorphic homotheties.
- Equivalent definition:
  $(M, J)$ admits a Hermitian metric $\omega$ on $M$ such that
  \[ d\omega = \theta \wedge \omega, \quad d\theta = 0 \]
  $\theta$ is called the Lee form.
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Real line bundle $L_{\mathbb{R}} \longrightarrow M$ associated to the representation

$$GL(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}.$$ 

The Lee form induces a connection $\nabla = d - \theta$ in $L_{\mathbb{R}}$.

$\nabla$ is associated to the Weyl covariant derivative determined on $M$ by the LCK metric and the Lee form.

the Weyl covariant derivative is uniquely defined by the properties $\nabla J = 0$, $\nabla g = \theta \otimes g$; in this context, $\theta$ is called the Higgs field.

As $d\theta = 0$, then $\nabla^2 = d\theta = 0$, and hence $L_{\mathbb{R}}$ is flat.
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Let $L = L \otimes \mathbb{C}$.

The Weyl connection extends naturally to $L$.

Its $(0, 1)$-part endows $L$ with a holomorphic structure.

As $L$ is flat, one can pick a nowhere degenerate section $\lambda$ satisfying

$$\nabla(\lambda) = \lambda \otimes (-\theta).$$

Hence, one chooses a Hermitian structure on $L$ such that $|\lambda| = 1$ and considers the associated Chern connection.

The curvature of the Chern connection on $L$ with respect to the above holomorphic and Hermitian structure is $-2\sqrt{-1}d^{c}\theta$.

$L$ determines a local system on $M$ associated to the character $\chi : \pi_1(M) \to \mathbb{R}^{>0}$. 
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Vaisman manifolds

- **LCK**: $\nabla^g \theta = 0$.

Properties:

1. $\theta^\#$ is Killing and real holomorphic ($\mathcal{L}_\theta J = 0$).
2. Conversely (Kamishima, O): A compact LCK manifold admits a LCK metric with parallel Lee form if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non-isometrically on its Kähler covering.
3. If $\mathcal{F} := \{\theta^\#, J\theta^\#\}$ has compact leaves, then $M/\mathcal{F}$ is Kähler orbifold.
4. If $\theta^\#$ has compact orbits, then $M/\theta^\#$ is Sasakian orbifold.
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Vaisman manifolds

LCK + ∇^g θ = 0.

Properties:

1. θ^♯ is Killing and real holomorphic (ℒ_{θ^♯} J = 0).
2. Conversely (Kamishima, O): A compact LCK manifold admits a LCK metric with parallel Lee form if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non-isometrically on its Kähler covering.

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\[ \nabla^g \theta = 0. \]

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**Structure Theorem**

- The monodromy of $L$ is $\mathbb{Z}$.
- Compact Vaisman manifolds are suspensions over $S^1$ with Sasakian fibre:
  - $\tilde{M}$ is a metric cone $N \times \mathbb{R}$.
  - $N$ is Sasaki.
  - $\Gamma$ is $\mathbb{Z}$ generated by $(x, t) \mapsto (\lambda(x), t + q)$ for some $\lambda \in \text{Aut}(N)$, $q \in \mathbb{R}_{>0}$. 
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- Compact Vaisman manifolds are suspensions over $S^1$ with Sasakian fibre:
  1. $\tilde{M}$ is a metric cone $N \times \mathbb{R}$
  2. $N$ is Sasaki.
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- Diagonal Hopf manifolds (generalizations of the rank 1 Hopf surfaces.): $H_A := \mathbb{C}^n/\langle A \rangle$ with $A = \text{diag}(\alpha_i)$, $|\alpha_i| > 1$, with:
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    - Then $A^* \varphi = C^{-1} \varphi$.
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- $(M, J)$ is **LCK with potential** if it admits a Kähler cover $(\tilde{M}, \Omega)$ with global potential $\varphi : \tilde{M} \to \mathbb{R}_+$ satisfying the following conditions:
  1. $\varphi$ is proper (i.e. it has compact level sets).
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- On compact manifolds, (1) is equivalent to the deck group being isomorphic to $\mathbb{Z}$ (a condition satisfied by compact Vaisman manifolds).
- All Vaisman manifolds are LCK with potential, but not conversely.
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Properties

- The class of compact LCK manifolds with potential is stable to small deformations.
  - Hence: the Hopf manifold \( (\mathbb{C}^N \setminus \{0\})/\Gamma \), with \( \Gamma \) cyclic group generated by a non-diagonal linear operator, is LCK with potential. This is a generalization of the (non–Vaisman) rank 0 Hopf surface.

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Associated to the operator $d - \theta$. Since $d\theta = 0$, $d^2\theta = 0$. Denote it $H^*_\theta(M)$.

- Some call it Lichnerowicz–Poisson (in Poisson and Jacobi geometry).

- Clearly $d_\theta \omega = 0$.

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- Analogue of the Kähler class.

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**Morse–Novikov cohomology of compact Vaisman manifolds**

- Morse–Novikov cohomology of compact Vaisman manifolds is trivial.
  - Follows from the Structure theorem.
  - Previously proven for locally conformally symplectic manifolds which admit a compatible metric for which the Lee form is parallel (de Leon, Lopez, Marrero, Padron).

- More generally: on compact Vaisman manifolds, the Morse–Novikov class of any LCK form vanishes. Precisely:
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More generally: on compact Vaisman manifolds, the Morse–Novikov class of any LCK form vanishes. Precisely:
Theorem 1

Let $M$ be a compact Vaisman manifold, $\dim_{\mathbb{C}} M \geq 3$, $\omega_1$ an LCK-form (not necessarily Vaisman), and $\theta_1$ its Lee form. Then $\theta_1$ is cohomologous with the Lee form of a Vaisman metric, and the Morse–Novikov class of $\omega_1$ vanishes.
Proof of Theorem 1

- Let $\rho$ be the Lee flow corresponding to the Vaisman structure $\omega$.
  - Modulo a deformation, it can be supposed with compact leaves.
  - By averaging over $\rho$, $\theta_1$ and $\omega_1$ can be supposed $\rho$-invariant. The cohomology class does not change.
  - Let $G_0$ be the closure of the group of holomorphic and conformal automorphisms of $M$ generated by $J(\theta^\#)$: compact and commutative.
  - As above, $\theta_1$ and $\omega_1$ can be supposed $G_0$-invariant.
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- Let $\tilde{M}$ be a Kähler covering on which $\tilde{\theta}$ is exact.

- **Fact:** If $\theta^\#$ and $J(\theta^\#)$ act conformally and holomorphically and $\theta^\#$ cannot be lifted to an isometry of $\tilde{M}$, then $M$ is Vaisman (K–O).

- Hence: suppose $\tilde{\omega}_1$ is $\tilde{\rho}$–invariant.

- Show that $\theta_1$ is **basic** wrt the foliation $\rho$.

- Hence: $d^c \theta_1 = 0$ (Tsukada), thus:
  - $0 = \int_M dd^c \omega_1^{n-1} = \int_M (n-1)^2 \theta_1 \wedge J(\theta_1) \wedge \omega_1^{n-1}$,
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- Show that $\theta_1$ is basic wrt the foliation $\rho$.
- Hence: $\partial^c \theta_1 = 0$ (Tsukada), thus:
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Main problem with non–Kähler manifolds: do not satisfy the global $\partial\bar{\partial}$-lemma.

One considers the Bott–Chern complex:

$$
\cdots \to \Lambda^{p-1,q-1}(M) \xrightarrow{\partial\bar{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial+\bar{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \to \cdots
$$

Its cohomology groups $H^{p,q}_{\partial\bar{\partial}}(M)$ are

$$
\ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \right)
$$

$$
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For compact manifolds, $H^{p,q}_{\partial\bar{\partial}}(M) = H^{p,q}_{\partial\bar{\partial}}(M) \iff$ global $\partial\bar{\partial}$-lemma.
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$$\cdots \to \Lambda^{p-1,q-1}(M) \xrightarrow{\partial} \Lambda^{p,q}(M) \xrightarrow{\partial + \bar{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \to \cdots$$

Its cohomology groups $H^{p,q}_{\partial \bar{\partial}}(M)$ are

$$\ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \right)$$

$$\operatorname{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p,q}(M) \right)$$

For compact manifolds, $H^{p,q}_{\partial \bar{\partial}}(M) \approx H^{p,q}_{\partial \bar{\partial}}(\overline{M}) \iff$ global $\partial \bar{\partial}$-lemma.
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Bott–Chern cohomology of LCK manifolds

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Meaning of the Bott–Chern class

\([\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M} \text{ admits an automorphic potential.}\)

- \(H^{1,1}_{\partial\bar{\partial}}(M, L) = 0\) is implied by \(H^1(M, L) = 0\) and \(H^2_\theta(M) = 0\) (easier to control).

Hence: If the Bott–Chern class of an LCK-manifold \(M\) vanishes and the monodromy of \(L\) is \(\mathbb{Z}\), then \(M\) is LCK with potential (will be generalized).

If \(\omega_1, \omega_2\) are LCK-metrics having the same Lee form \(\theta\), then the following conditions are equivalent:

- The Bott–Chern classes of \(\omega_1, \omega_2\) are equal.
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Analogy between Kähler and LCK

Kähler structures on a complex manifold are determined by:
1. a Kähler class in $\mathbb{H}^{1,1}(M)$;
2. a choice of a Kähler metric in this Kähler class, obtained by choosing an element in a cone locally modeled on $\mathcal{C}^\infty(M)/\text{const.}$

LCK-structures on a complex manifold with prescribed conformal structure are determined by:
1. a Bott–Chern class in $\mathbb{H}^{1,1}(\partial\partial(M), L)$;
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The space of LCK structures

Analogy between Kähler and LCK

- Kähler structures on a complex manifold are determined by:
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- LCK-structures on a complex manifold with prescribed conformal structure are determined by:
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The space of LCK structures

An analogy between Kähler and LCK structures:

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Theorem 2

Any compact LCK manifold with vanishing Bott–Chern class admits an LCK metric with potential.

Hence, if \( \dim_{\mathbb{C}} M \geq 3 \), it is embeddable in a Hopf manifold.

Our supposition, connected also with Theorem 1: Let \( M \) be a Vaisman manifold, equipped with an additional LCK-form \( \omega_1 \) (not necessarily Vaisman). Then the Bott–Chern class of \( \omega_1 \) vanishes; equivalently, \( \omega_1 \) is an LCK-structure with potential.
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Proof of Theorem 2

- \([\omega] = 0 \in H^{1,1}_{\partial \bar{\partial}}(M, L) \Leftrightarrow \tilde{M} \text{ admits an automorphic potential.}\)
- The weight bundle \(L\) is associated to the monodromy of this covering and the monodromy can be a priori a subgroup of \((\mathbb{R}^+, \cdot) \cong (\mathbb{R}, +)\), which is not necessarily discrete.
- Consider \(L\) as a trivial line bundle with connection \(\nabla_{\text{triv}} - \theta\) and deform \(L\) by adding a small term to \(\theta\) to obtain a bundle \(L'\) with monodromy \(\mathbb{Z}\).
- A local system on \(M\) is defined by a group homomorphism \(H_1(M, \mathbb{Z}) \to \mathbb{R}\). Its monodromy is \(\mathbb{Z}\) if this map is rational. Each real homomorphism from \(H_1(M, \mathbb{Z})\) can be approximated by a rational one.
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- Deforming the monodromy $\iff$ deforming $\theta = d \log \varphi \iff$ deforming the potential $\varphi$.

- We deform the pair $(L, \varphi)$ to a pair $(L', \varphi')$ in which $\varphi'$ is an automorphic function on $\tilde{M}$, with monodromy determined by $L'$.

- $\varphi'$ stays plurisubharmonic if $\theta'$ is sufficiently close to $\theta$ in the norm:

$$||\theta - \theta'||_{\text{PLS}} = \sup_{M} |\theta - \theta'| + \sup_{\tilde{M}} ||\nabla \theta - \nabla \theta'||.$$
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Problem 1
Determine all 1-forms $\theta$ for which there exists a Hermitian two-form $\omega$ having $\theta$ as its Lee form, and all the Morse–Novikov classes which can be realized by an LCK-form.

Problem 2
Let $M$ be a compact complex manifold, admitting an LCK-metric, and $[\theta] \in H^1(M)$ its Lee class. Determine the set of all classes $[\omega] \in H^1_{\partial\theta \bar{\partial}\theta}(M)$ such that $[\omega]$ is the Bott–Chern class of an LCK-structure with the Lee class $[\theta]$.

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Is there a global $\partial\theta \bar{\partial}\theta$–lemma?
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Unanswered questions

- **Problem 1**
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