Morse-Novikov cohomology of locally conformally Kähler manifolds

Liviu Ornea   Misha Verbitsky

Kühlungsborn, April 2008

Definitions

- (M, J) complex manifold, $\dim_{\mathbb{C}} M \geqslant 2$, connected.
- (M, J) is LCK if it admits a Kähler covering

$$\Gamma \to (\widetilde{M}, J, \Omega) \to (M, J)$$

  such that Γ acts by holomorphic homotheties.
- Equivalent definition:
  (M, J) admits a Hermitian metric ω on M such that

$$d\omega = \theta \wedge \omega, \quad d\theta = 0$$

  θ is called the *Lee form*.

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The weight bundle

- Real line bundle $L_{\mathbb{R}} \longrightarrow M$ associated to the representation

$$\mathrm{GL}(2n,\mathbb{R}) \ni A \mapsto \mid \det A \mid^{\frac{1}{n}} .$$

- The Lee form induces a connection $\nabla = d - \theta$ in $L_{\mathbb{R}}$.
- ∇ is associated to the Weyl covariant derivative determined on M by the LCK metric and the Lee form.
- the Weyl covariant derivative is uniquely defined by the properties $\nabla J = 0$, $\nabla g = \theta \otimes g$; in this context, θ is called the Higgs field.
- As $d\theta = 0$, then $\nabla^2 = d\theta = 0$, and hence $L_{\mathbb{R}}$ is flat.

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The complexified weight bundle

- Let $L = L_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
- The Weyl connection extends naturally to L.
- Its $(0, 1)$-part endows L with a holomorphic structure.
- As L is flat, one can pick a nowhere degenerate section λ satisfying
$$\nabla(\lambda) = \lambda \otimes (-\theta).$$
- Hence, one chooses a Hermitian structure on L such that $\mid \lambda \mid = 1$ and considers the associated Chern connection.
- The curvature of the Chern connection on L with respect to the above holomorphic and Hermitian structure is $-2\sqrt{-1}d^c\theta$.
- L determines a local system on M associated to the character $\chi : \pi_1(M) \to \mathbb{R}^{>0}$.

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- LCK $\quad + \quad \nabla^g \theta = 0.$
- Properties:
  1. $\theta^\sharp$ is Killing and real holomorphic ($\mathcal{L}_{\theta^\sharp} J = 0$).
  2. Conversely (Kamishima, O): A compact LCK manifold admits a LCK metric with parallel Lee form if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non isometrically on its Kähler covering.
  3. If $\mathcal{F} := \{\theta^\sharp, J\theta^\sharp\}$ has compact leaves, then $M/\mathcal{F}$ is Kähler orbifold.
  4. If $\theta^\sharp$ has compact orbits, then $M/\theta^\sharp$ is Sasakian orbifold.
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Structure Theorem

- The monodromy of L is $\mathbb{Z}$.
- Compact Vaisman manifolds are suspensions over S^1 with Sasakian fibre:
  1. $\tilde{M}$ is a metric cone $N \times \mathbb{R}$
  2. N is Sasaki.
  3. Γ is $\mathbb{Z}$ generated by $(x,t) \to (\lambda(x), t+q)$ for some $\lambda \in \mathrm{Aut}(N)$, $q \in \mathbb{R}_+$.

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LCK manifolds with potential

- (M, J) is *LCK with potential* if it admits a Kähler cover $(\widetilde{M}, \Omega)$ with global potential $\varphi : \widetilde{M} \to \mathbb{R}_+$ satisfying the following conditions:

    1. φ is proper (i.e. it has compact level sets).
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- On compact manifolds, (1) is equivalent to the deck group being isomorphic to $\mathbb{Z}$ (a condition satisfied by compact Vaisman manifolds).
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Properties

- The class of compact LCK manifolds with potential is stable to small deformations.
  - Hence, the Hopf manifold $(\mathbb{C}^n \setminus 0)/\Gamma$, with Γ cyclic group generated by a *non-diagonal* linear operator, is LCK with potential. This is a generalization of the (non-Vaisman) rank 0 Hopf surface.
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- Clearly $d_\theta \omega = 0$.
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Theorem 1

Let M be a compact Vaisman manifold, $\dim_{\mathbb{C}} M \geqslant 3$, ω_1 an LCK-form (not necessarily Vaisman), and θ_1 its Lee form. Then θ_1 is cohomologous with the Lee form of a Vaisman metric, and the Morse–Novikov class of ω_1 vanishes.

Proof of Theorem 1

- Let ρ be the Lee flow corresponding to the Vaisman structure ω.
  - Modulo a deformation, it can be supposed with compact leaves.
- By averaging over ρ, θ_1 and ω_1 can be supposed ρ-invariant. The cohomology class does not change.
- Let G_0 be the closure of the group of holomorphic and conformal automorphisms of M generated by $J(\theta^\sharp)$: compact and commutative.
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- Main problem with non–Kähler manifolds: do not satisfy the global $\partial\overline{\partial}$-lemma.
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$$\longrightarrow \Lambda^{p-1,q-1}(M) \xrightarrow{\partial\overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial\oplus\overline{\partial}} \Lambda^{p+1,q}(M)\oplus\Lambda^{p,q+1}(M) \longrightarrow$$

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  - For compact manifolds, $H^{p,q}_{\partial\overline{\partial}}(M) \cong H^{p,q}_{\overline{\partial}}(M) \Longleftrightarrow$ global $\partial\overline{\partial}$-lemma.

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- Same complex, but for d_θ:

$$\Lambda^{p-1,q-1}(M) \xrightarrow{\partial_\theta \overline{\partial}_\theta} \Lambda^{p,q}(M) \xrightarrow{\partial_\theta \oplus \overline{\partial}_\theta} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)$$

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Meaning of the Bott–Chern class

- $[\omega] = 0 \in H^{1,1}_{\partial\bar\partial}(M, L) \Leftrightarrow \widetilde{M}$ admits an *automorphic* potential.
  - $H^{1,1}_{\partial\bar\partial}(M, L) = 0$ is implied by $H^1(M, L) = 0$ and $H^2_\theta(M) = 0$ (easier to control).
- Hence: If the Bott–Chern class of an LCK-manifold M vanishes and the monodromy of L is $\mathbb{Z}$, then M is LCK with potential (will be generalized).
- If ω_1, ω_2 are LCK-metrics having the same Lee form θ, then the following conditions are equivalent:
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The space of LCK structures

Analogy between Kähler and LCK

- Kähler structures on a complex manifold are determined by:

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- Kähler structures on a complex manifold are determined by:
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Meaning of the Bott–Chern class: Main result

Theorem 2

Any compact LCK manifold with vanishing Bott–Chern class admits an LCK metric with potential.

Hence, if $\dim_{\mathbb{C}} M \geqslant 3$, it is embeddable in a Hopf manifold.

- Our supposition, connected also with Theorem 1: Let M be a Vaisman manifold, equipped with an additional LCK-form ω_1 (not necessarily Vaisman). Then the Bott–Chern class of ω_1 vanishes; equivalently, ω_1 is an LCK-structure with potential.

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Proof of Theorem 2

- $[\omega] = 0 \in H^{1,1}_{\partial\bar\partial}(M, L) \Leftrightarrow \widetilde{M}$ admits an *automorphic* potential.
- The weight bundle L is associated to the monodromy of this covering and the monodromy can be a priori a subgroup of $(\mathbb{R}^{>0}, \cdot) \cong (\mathbb{R}, +)$, which is not necessarily discrete.
- Consider L as a trivial line bundle with connection $\nabla_{triv} - \theta$ and deform L by adding a small term to θ to obtain a bundle L' with monodromy $\mathbb{Z}$.
  - A local system on M is defined by a group homomorphism $H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}$. Its monodromy is $\mathbb{Z}$ if this map is rational. Each real homomorphism from $H_1(M, \mathbb{Z})$ can be approximated by a rational one.

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  - A local system on M is defined by a group homomorphism $H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}$. Its monodromy is $\mathbb{Z}$ if this map is rational. Each real homomorphism from $H_1(M, \mathbb{Z})$ can be approximated by a rational one.

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Unanswered questions

- **Problem 1**
  Determine all 1-forms θ for which there exists a Hermitian two-form ω having θ as its Lee form, and all the Morse–Novikov classes which can be realized by an LCK-form.

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  Let M be a compact complex manifold, admitting an LCK-metric, and $[\theta] \in H^1(M)$ its Lee class. Determine the set of all classes $[\omega] \in H^{1,1}_{\partial_\theta \overline{\partial}_\theta}(M)$ such that $[\omega]$ is the Bott–Chern class of an LCK-structure with the Lee class $[\theta]$.

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