NEUTRAL EINSTEIN HERMITIAN METRICS ON RULED SURFACES.

Włodzimierz Jelonek

Abstract. The aim of this talk is to give a family of new Einstein bi-Hermitian neutral metrics on ruled surfaces of genus $g > 1$.

0. Introduction. In the present talk we are concerned with the class of neutral semi-Riemannian 4-manifolds (M, g) whose Ricci tensor ρ satisfies the condition

$$(*) \qquad \nabla_X \rho(X, X) = \frac{1}{3} X\tau g(X, X)$$

where τ is the scalar curvature of (M, g). These class of manifolds was introduced by A. Gray (see [G],[Be]). For general facts and some results concerning neutral 4-manifolds we refer to [K], [Pe], [D],[M-L]. In [M] the condition for the existence of two opposite almost complex structures on 4-manifolds are studied. The main subject of the talk is the construction of new Einstein neutral metrics on ruled surfaces and is based on the paper [J-5].

In our papers [J-1]-[J-4] we have described Riemannian metrics g on compact complex surfaces (M, J) such that (M, g) satisfies the condition $*$ and has J-invariant Ricci tensor. In particular we have given a complete classification of bi-Hermitian Gray surfaces on ruled surfaces of genus $g \geq 1$ and surfaces which are of co-homogeneity 1 on ruled surfaces of genus $g = 0$. We denote by g also the Riemannian metric but it should not cause any misunderstandings. The technics used in [J-3] can be applied to describe a large class of neutral bi-Hermitian Gray surfaces. The equations, describing such surfaces whose Ricci tensor has exactly two real eigenvalues such that the corresponding two dimensional eigendistributions are space-like and time-like, are described by the same equations as in [J-3],[J-4]. In that way we present a large class of explicit examples of neutral bi-Hermitian

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Gray surfaces. In particular we get a large class of Einstein neutral bi-Hermitian surfaces on ruled surfaces of genus $g>1$.

1. Neutral $\mathcal{A}C^{\perp}$-surfaces. By an $\mathcal{A}C^{\perp}$- manifold (see [Be],[G]) we mean a semi-Riemannian manifold (M,g) satisfying the condition

$$(*)\qquad \mathfrak{C}_{XYZ}\nabla_X\rho(Y,Z)=\frac{2}{(\dim M+2)}\mathfrak{C}_{XYZ}X\tau g(Y,Z),$$

where ρ and τ are the Ricci tensor and the scalar curvature of (M,g) respectively and $\mathfrak{C}$ means the cyclic sum. In this paper, an $\mathcal{A}C^{\perp}$-manifold with neutral metric is also called a neutral Gray manifold. A Riemannian manifold (M,g) is an $\mathcal{A}C^{\perp}$ manifold if and only if the Ricci endomorphism Ric of (M,g) is of the form $Ric=S+\frac{2}{n+2}\tau Id$ where S is a Killing tensor and $n=\dim M$. Let us recall that a symmetric (1,1) tensor S on a semi-Riemannian manifold (M,g) is called a Killing tensor if $g(\nabla S(X,X),X)=0$ for all $X\in TM$ and that a semi-Riemannian manifold whose Ricci tensor is a Killing tensor is called an $\mathcal{A}$-manifold.

On a semi-Riemannian manifold (M,g) a distribution $\mathcal{D}\subset TM$ is called umbilical (see [J-3]) if $\mathcal{D}$ is non-degenerate (i.e. a metric g is non-degenerate on $\mathcal{D}$) and $\nabla_X X_{|\mathcal{D}^{\perp}}=g(X,X)\xi$ for every $X\in\Gamma(\mathcal{D})$, where $X_{|\mathcal{D}^{\perp}}$ is the $\mathcal{D}^{\perp}$ component of X with respect to the orthogonal decomposition $TM=\mathcal{D}\oplus\mathcal{D}^{\perp}$. The vector field ξ is called the mean curvature normal of $\mathcal{D}$. The foliation tangent to involutive distribution $\mathcal{D}$ is called totally geodesic if its every leaf is a totally geodesic (i.e $\nabla_X X\in\mathcal{D}$ for any $X\in\mathcal{D}$) non-degenerate submanifold of (M,g). In the sequel we shall not distinguish between $\mathcal{D}$ and a foliation tangent to $\mathcal{D}$ and we shall also say that $\mathcal{D}$ is totally geodesic in such a case.

It is not difficult to prove exactly as in the Riemannian case (see [J-3]) the following lemma:

Lemma 1. *Let $S\in End(TM)$ be a (1,1) tensor on a neutral semi-Riemannian 4-manifold (M,g). Let us assume that S has exactly two everywhere different eigenvalues λ,μ of the same multiplicity 2, i.e. dim $\mathcal{D}_\lambda=$ dim $\mathcal{D}_\mu=2$, where $\mathcal{D}_\lambda,\mathcal{D}_\mu$ are non-degenerate eigendistributions of S corresponding to λ,μ respectively. Then S is a Killing tensor if and only if both distributions $\mathcal{D}_\lambda$ and $\mathcal{D}_\mu$ are umbilical with mean curvature normal equal respectively*

$$\xi_\mu=\frac{\nabla\mu}{2(\lambda-\mu)},\ \xi_\lambda=\frac{\nabla\lambda}{2(\mu-\lambda)}.$$

We shall call a bi-Hermitian surface with bi-Hermitian Ricci tensor simply as bi-Hermitian surface If (M,g) is also an $\mathcal{A}C^{\perp}$ manifold, then we call it a neutral bi-Hermitian Gray surface.

Proposition 1. *Let us assume that (M,g) is a simply connected neutral Gray 4-manifold and the Ricci tensor S ($\rho(X,Y)=g(SX,Y)$) has exactly two real eigenvalues λ,μ and no null eigenvectors. Then there exist two Hermitian complex structures $J,\bar{J}$ commuting with S and (M,g) is a bi-Hermitian neutral Gray surface.*

Proof. Analogous to [J-1].◇

Let us recall that a Riemann surface Σ is a one-complex-dimensional connected complex analytic manifold which we shall assume to be compact. If J is a complex structure on Σ then there exists a a Riemannian metric g on Σ with constant Gauss curvature and such that J is orthogonal with respect to g.

Definition. A ruled surface of genus g is a complex surface X admitting a ruling, i.e. an analytically locally trivial fibration with fibre $\mathbb{C}P^1$ and structural group $PGL(2,\mathbb{C})$ over a smooth compact complex curve (a Riemannian surface) of genus g.

Now we give a theorem, whose proof is analogous as in the Riemannian case (see [M-S],[B],[S]).

Theorem 1. *Let us consider the manifold $U = (a,b) \times P$, where (P, g_0) is a 3-dimensional semi-Riemannian $\mathcal{A}$-manifold (a principle S^1 bundle $p : P \to \Sigma$) over a Riemannian surface (Σ, g_{can}) of constant sectional curvature K, with an $\mathcal{A}C^{\perp}$-metric*

$$g = dt^2 + f(t)^2\theta^2 - g(t)^2 p^* g_{can}, \tag{1.5}$$

where $g_0 = \theta^2 - p^* g_{can}$, $f, g \in C^{\infty}(a,b)$ and θ is the connection form of P. Then the metric g on U extends to a smooth $\mathcal{A}C^{\perp}$ metric on the ruled surface M which is a $\mathbb{C}P^1$-bundle over Σ and such that U is an open and dense subset of M if and only if the functions $f, g \in C^{\infty}(a,b)$ satisfy the conditions:
(a) $f(a) = f(b) = 0, f'(a) = 1, f'(b) = -1$,
(b) $g(a) \neq 0 \neq g(b), g'(a) = g'(b) = 0$.

Remark. Let us note that the metric (1.5) induces a semi-Riemannian metric on M if the functions f, g satisfy:
(i) f is positive on (a,b) , and f is odd at a and b , i.e. f is the restriction of a function f on $\mathbb{R}$ satisfying $f(a+t) = -f(a-t), f(b+t) = -f(b-t)$;
(ii) g is positive on $[a,b]$ and even at a and b which means that g is the restriction of a function g such that $g(a+t) = g(a-t), g(b+t) = g(b-t)$;
The proof is similar to the description of the metric in polar coordinates (see [Be] Lemma 9.114 and Theorem 9.125.)

Note that it is an easy exercise to prove that functions f, g satisfying the ODE characterizing $\mathcal{A}C^{\perp}$-metrics together with the initial conditions $(a),(b)$ extend to respectively odd and even (with respect to the points a, b) real analytic functions around a, b.

2. Neutral bi-Hermitian Gray surfaces. In this section we shall construct bi-Hermitian neutral metrics on ruled surfaces $M_{k,g}$ of genus g. The ruled surface $(M_{k,g}, g)$ is locally of co-homogeneity 1 with respect to the group of all local isometries of $(M_{k,g}, g)$ and an open, dense submanifold $(U_{k,g}, g) \subset (M_{k,g}, g)$ is isometric to the manifold $(a,b) \times P_k$ where (P_k, g_k) is a 3-dimensional semi-Riemannian $\mathcal{A}$-manifold (a principal circle bundle $p : P_k \to \Sigma_g$) over a Riemannian surface (Σ_g, g_{can}) of constant sectional curvature $K \in \{-4, 0, 4\}$ with a metric

$$g_{f,g} = dt^2 + f(t)^2\theta^2 - g(t)^2 p^* g_{can}, \tag{2.1}$$

where $g_k = \theta^2 - p^* g_{can}$ and θ is the connection form of P_k such that $d\theta = 2\pi k \; p^*\omega$, where the de Rham cohomology class $[\omega] \in H^2(\Sigma_g, \mathbb{R})$ defined by the form ω is an integral class corresponding to the class $1 \in H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$. Let $\theta^\sharp$ be a vector field dual to θ with respect to g_k. Let us consider a local orthonormal frame $\{X,Y\}$ on (Σ_g, g_{can}) and let X^h, Y^h be horizontal lifts of X,Y with respect to $p : M_{k,g} \to \Sigma_g$ (i.e. $dt(X^h) = \theta(X^h) = 0$ and $p(X^h) = X$) and let $H = \frac{\partial}{\partial t}$. Let us define two almost Hermitian structures $J, \bar{J}$ on M as follows

$$JH = \frac{1}{f}\theta^\sharp, JX^h = Y^h,\ \bar{J}H = -\frac{1}{f}\theta^\sharp, \bar{J}X^h = Y^h.$$

Proposition 3. *Let $\mathcal{D}$ be a distribution spanned by the fields $\{\theta^\sharp, H\}$. Then $\mathcal{D}$ is a non-degenerate totally geodesic foliation with respect to the metric $g_{f,g}$ where g is a non constant function. Both structures $J, \bar{J}$ are Hermitian and $\mathcal{D}$ is contained in the nullity of J and $\bar{J}$. The distribution $\mathcal{D}^\perp$ is umbilical with the mean curvature normal $\xi = -\nabla \ln g$. Let λ, μ be eigenvalues of the Ricci tensor S of $g_{f,g}$ corresponding to eigendistributions $\mathcal{D}, \mathcal{D}^\perp$ respectively. Then the following conditions are equivalent:*
(a) There exists $D \in \mathbb{R}$ such that $\lambda - \mu = Dg^2$,
(b) There exist $C, D \in \mathbb{R}$ such that $\mu = Dg^2 - C$,
(c) $\lambda - 2\mu$ is constant,
(d) $(U_{k,g}, g_{f,g})$ is a neutral bi-Hermitian Gray surface.

Proof. The first assertion of Proposition 3 is a consequence similar to Proposition 3 in [J-2]. Note that $\nabla\lambda = H\lambda H, \nabla\mu = H\mu H$. Consequently $tr_g \nabla S = \frac{1}{2}\nabla\tau = (H\lambda + H\mu)H$. On the other hand one can easily check that $tr_g \nabla S = 2(\mu - \lambda)\xi + H\lambda H$. Thus

$$\nabla\mu = 2(\lambda - \mu)\nabla \ln g.$$

Now we prove that (a) $\Rightarrow$ (b). If (a) holds then $\nabla\mu = 2Dg^2\frac{\nabla g}{g} = D\nabla g^2$. Thus $\nabla(\mu - Dg^2) = 0$ which implies (b).

(b)$\Rightarrow$ (a). We have

$$-\frac{\nabla g}{g}2(\mu - \lambda) = \nabla\mu = 2Dg\nabla g,$$

and consequently $\nabla g(\frac{(\mu-\lambda)}{g} - Dg) = 0$ which is equivalent to (a).

(a)$\Rightarrow$(c). We have $\lambda - \mu = Dg^2$ and consequently $\nabla\mu = 2Dg\nabla g = D\nabla g^2$. Thus $\nabla\lambda = \nabla(\mu + Dg^2) = 2D\nabla g^2$ and $\nabla\lambda - 2\nabla\mu = 0$ which gives (c).

(c)$\Rightarrow$ (a). If $\nabla\lambda = 2\nabla\mu$ then $\nabla\lambda = 4(\lambda - \mu)\frac{\nabla g}{g}$. Consequently $\nabla\lambda - \nabla\mu = 2(\lambda - \mu)\frac{\nabla g}{g}$ and $\nabla \ln|\lambda - \mu| = 2\frac{\nabla g}{g} = 2\nabla \ln g$, which means that $\nabla \ln|\lambda - \mu|g^{-2} = 0$ or $\lambda = \mu$ on the whole of M. In the last case we obtain an Einstein metric. It follows that $\ln\frac{|\lambda-\mu|}{g^2} = \ln D$ for some $D \in \mathbb{R}_+$ or $\lambda - \mu = 0$ (D=0), which is equivalent to (a).

(d)$\Leftrightarrow$(c). This equivalence follows from [J-3]. $\diamondsuit$

Theorem 2. *On any ruled surface $M_{k,g}$ of genus g with $k > 0$ there exist a one-parameter family of neutral bi-Hermitian $\mathcal{AC}^\perp$-metrics $\{g_x : x \in (0, \epsilon_s)\}$, where $\epsilon_s > 0$ depends only on g and k, which consists of neutral bi-Hermitian Gray metrics on $M_{k,g}$.*

Proof. Note that for the first Chern class $c_1(\Sigma_g) \in H^2(\Sigma_g, \mathbb{Z})$ of the complex curve Σ_g we have the relation $c_1(\Sigma_g) = \chi\alpha$, where $\alpha \in H^2(\Sigma_g, \mathbb{Z})$ is an indivisible integral class and $\chi = 2 - 2g$ is the Euler characteristic of Σ_g. Let us write $s = \frac{2k}{|\chi|}$ if $g \neq 1$ and $s = k$ if $g = 1$. Then it is easy to show using O'Neill formulas for a semi-Riemannian submersion (see [ON],[B],[S]) that the manifold $(M_{k,g}, g)$ with the metric g given by (2.1) has the Ricci tensor with the following eigenvalues :

$$\lambda_0 = -2\frac{g''}{g} - \frac{f''}{f}, \tag{2.2a}$$

$$\lambda_1 = -\frac{f''}{f} - 2\frac{f'g'}{fg} + 2s^2\frac{f^2}{g^4}, \tag{2.2b}$$

$$\lambda_2 = -\frac{g''}{g} - \frac{f'g'}{fg} - (\frac{g'}{g})^2 - 2s^2\frac{f^2}{g^4} - \frac{K}{g^2}, \tag{2.2c}$$

where $\lambda_0, \lambda_1,$ correspond to eigenfields $T = \frac{d}{dt}, \theta^\sharp$ and λ_2 corresponds to a two-dimensional eigendistribution orthogonal to T and $\theta^\sharp$. Note that the equations we have got are practically the same as equations for Gray manifolds obtained in the Riemannian case. The only difference is that we now have $-K$ instead of K, so the cases $K > 0$ and $K < 0$ are now reversed, which give the different geometric meaning to the equations in Riemannian and semi-Riemannian neutral cases. We present here the method of solving equations (2.2) for the convenience of the reader and completeness, although the calculations are just the same as in [J-4]. If $(M, g) \in \mathcal{A}C^\perp$ is a bi-Hermitian Gray surface then $\lambda_0 = \lambda_1 = \lambda$, and if we denote $\mu = \lambda_2$, Proposition 3 b implies an equation

$$\mu = Dg^2 - C \tag{2.3}$$

for some $D, C \in \mathbb{R}$. Since $\lambda_0 = \lambda_1$ we get

$$f = \pm\frac{gg'}{\sqrt{s^2 + Ag^2}}. \tag{2.4}$$

Using a homothety of the metric we can assume that $A \in \{-1, 0, 1\}$. In the case $A = 0$ we get a neutral Kähler metric and these metrics on compact complex surfaces we shall describe in section 4. So we restrict our considerations to the case $A \in \{-1, 1\}$. Now we introduce a function h such that $h^2 = s^2 + Ag^2$. Note that $\mathrm{im}h \subset (-s, s)$ if $A = -1$, and $\mathrm{im}h \subset (s, \infty)$ if $A = 1$. Then $g = \sqrt{|s^2 - h^2|}$. Let us introduce a function z such that $h' = \sqrt{z(h)}$. Note that

$$f = h' \text{ and } f' = \frac{1}{2}z'(h). \tag{2.5}$$

It follows that equation (2.3) is equivalent to

$$z'(h) - z(h)\frac{s^2 + h^2}{h(s^2 - h^2)} = \frac{4\epsilon}{h} + \frac{D(s^2 - h^2)^2}{h} - \frac{C(s^2 - h^2)}{h}, \tag{2.6}$$

where $\epsilon = -\mathrm{sgn}KA \in \{-1, 0, 1\}$. It follows that

$$z(h) = (1 - (\frac{h}{s})^2)^{-1}(-4\epsilon(\frac{h}{s})^2 - \frac{Ds^4}{5}(\frac{h}{s})^6 + (Ds^4 - \frac{Cs^2}{3})(\frac{h}{s})^4 + \\ +(2Cs^2 - 3Ds^4)(\frac{h}{s})^2 - 4\epsilon + Cs^2 - Ds^4 + \frac{E}{s}\frac{h}{s}). \tag{2.7}$$

Let us denote again $C = Cs^2, D = Ds^4, E = \frac{E}{s}$ and let

$$(2.8)\quad z_0(t) = (1-t^2)^{-1}(-4\epsilon(1+t^2)+D(-\frac{1}{5}t^6+t^4-3t^2-1)+C(-\frac{1}{3}t^4+2t^2+1)+Et).$$

Write

$$(2.9)\qquad P(t) = -4\epsilon t^2 - \frac{D}{5}t^6 + (D-\frac{C}{3})t^4 + (2C-3D)t^2 + Et - 4\epsilon + C - D.$$

Then $z_0(t) = \frac{P(t)}{1-t^2}$. Note that $z(h) = z_0(\frac{h}{s})$ and $z'(h) = \frac{1}{s}z_0'(\frac{h}{s})$. In view of Th. 1. we are looking for real numbers $x, y \in \mathbb{R}, x > y$ such that

$$(2.10a)\qquad z_0(x) = 0, z_0'(x) = -2s,$$

$$(2.10b)\qquad z_0(y) = 0, z_0'(y) = 2s,$$

and $z_0(t) > 0$ for $t \in (y, x)$. Note that equations (2.10a) are equivalent to

$$(2.11a)\; -4\epsilon x^2 - \frac{D}{5}x^6 + (D-\frac{C}{3})x^4 + (2C-3D)x^2 - 4\epsilon + C - D + Ex = 0$$

$$(2.11b)\;\; -8\epsilon x - \frac{6D}{5}x^5 + 4(D-\frac{C}{3})x^3 + 2(2C-3D)x + E = -2s(1-x^2).$$

Equations (2.11) yield

$$(2.12a)\qquad D = \frac{5(-3E - 6s - 24\epsilon x + 3Ex^2 - 12sx^2 - 8\epsilon x^3 + 2sx^4)}{2(-1+x)x(1+x)(15+10x^2-x^4)},$$

(2.12b)

$$C = \frac{3(5E + 10s + 80\epsilon x + 30sx^2 - 10Ex^2 + 5Ex^4 - 10sx^4 - 16\epsilon x^5 + 2sx^6)}{2(-1+x)x(1+x)(-15-10x^2+x^4)}$$

Solving in a similar way equations (2.10b) one can see that there exists a function z_0 satisfying the equations (2.10) if

$$(2.13)\qquad \begin{aligned}&(x+y)(-4\epsilon(-5x + x^3 + 5y + 2x^2y - 2xy^2 - y^3)\\ &+s(5 + 2x^3y + 2xy^3 + 3y^2 + 3x^2 + x^2y^2 - 16xy)) = 0,\end{aligned}$$

where $x > y$, $x, y \in (-1, 1)$ in the case $A = -1$ and $x, y \in (1, \infty)$ in the case $A = 1$. Using standard methods one can check that in the case of the genus $g \leq 1$ (i.e. if $K = 4$ or $K = 0$) the only solutions of (2.13) giving a positive function z are $x = -y \in (0, 1)$. In the case of genus $g \geq 2$ we have $K = -4$ and apart from the solutions with $x = -y$ (see [J-3]) there is an additional family of solutions with $\epsilon = -1$. It follows that if $g \leq 1$ then $x = -y \in (0, 1), E = 0$ and $\epsilon = 1$ or $\epsilon = 0$. Now we give explicit formulas for the case $x = -y \in (0, 1)$. Then we get

$$(2.14)\qquad \begin{aligned}&P(t)\\ &= \frac{1}{x(15-5x^2-11x^4+x^6)}((t^2-x^2)(s(-15+10x^2-3x^4+t^2(10+12x^2-6x^4)\\ &+t^4(-3-6x^2+x^4)) + 4\epsilon x(x^2(-5+x^2) - t^4(3+x^2) + t^2(5+2x^2+x^4)))).\end{aligned}$$

Thus

$$P(0) = \frac{-4\epsilon x^4(x^2-5) + sx(15-10x^2+3x^4)}{15-5x^2-11x^4+x^6}$$

and, since $\lim_{x\to 0^+} \frac{P(0)}{x} = s > 0$, there exists $\epsilon_s > 0$ such that $P(t) > 0$ if $t \in (0,x)$ for all $x \in (0,\epsilon_s)$. In fact in the case $\epsilon = -1$ the real number ϵ_s is the first positive root of the polynomial $-4x^3(x^2-5) + s(-15+10x^2-3x^4)$ and $\epsilon_s = 1$ if $\epsilon = 1$. Note also that in both cases $\epsilon_s = 1$ if $s \geq 2$. Now the function $z_0(t) = \frac{1}{(1-t^2)}P(t)$ is positive on $(-x,x), x \in (0,\epsilon_s)$. If $x \in (0,\epsilon_s)$ then there exists a solution $h : (-a,a) \to (-sx, sx)$, where

$$a = \lim_{t\to sx^-} \int_0^t \frac{dh}{\sqrt{z_0(\frac{h}{s})}},$$

of an equation

$$h' = \sqrt{z_0(\frac{h}{s})},$$

such that $h(-a) = -sx, h(a) = sx, h'(-a) = h'(a) = 0, h''(-a) = 1, h''(a) = -1$. It follows that functions $f = h', g = \sqrt{s^2-h^2}$ are smooth on $(-a,a)$ and satisfy the boundary conditions described in Th.2. Consequently the metric

$$g_x = dt^2 + f(t)^2\theta^2 - g(t)^2 p^* g_{can},$$

on the manifold $(-a,a) \times P_k$ extends to the smooth metric on the compact ruled surface $M = P_k \times_{S^1} S^2$ which is a 2−sphere bundle over Riemannian surface Σ_g. Note that $g(-a) = g(a) = s\sqrt{1-x^2}$. ♢

3. Einstein neutral bi-Hermitian surfaces. Let us note that the solutions $(M_{k,g}, g_x)$ with $D(x) = 0$ correspond to Einstein neutral surfaces. Let us recall that

$$D = \frac{5(-6s-24\epsilon x-12sx^2-8\epsilon x^3+2sx^4)}{2(-1+x)x(1+x)(15+10x^2-x^4)} \tag{3.1}$$

Consequently there exist Einstein metrics in the family of Gray metrics $(M_{k,g}, g_x)$ if and only if the equation

$$-6s-24\epsilon x-12sx^2-8\epsilon x^3+2sx^4 = 0 \tag{3.2}$$

has a real root $x \in (0,1)$. Now it is not difficult to check that in the case $g = 0$ we have $\epsilon = 1$ and $s \in \mathbb{N}$ and that equation (3.2) does not have any solution in $(0,1)$. Similarly in the case $g = 1$. In the case $g > 1$ we have $s = \frac{k}{g-1} > 0$ and $\epsilon = -1$. In that case equation (3.2) has a real root in $(0,1)$ if and only if $s \in (0,2)$. To show this, let us denote

$$Q(x) = -6s-24\epsilon x-12sx^2-8\epsilon x^3+2sx^4.$$

Then

$$Q''(x) = -24(s+2\epsilon x-sx^2).$$

Consequently $Q''(x) < 0$ in $(0,1)$ for $\epsilon \geq 0$. In the case where $\epsilon = -1$, since $Q''(0) = -24s < 0$ and $Q''(1) = 48 > 0$, the equation $Q''(x) = 0$ has exactly real one root, say α_s. (In fact, $\alpha_s = s(\sqrt{1+s^2}+1)^{-1}$.) Then, we have $Q'(\alpha_s)(= 16(2-(1+s^2)(\sqrt{1+s^2}+1)^{-1} > 0$ if and only if $(0 <)s < (\sqrt{3}/2)^{\frac{1}{2}}(\sqrt{3}+1)$. In particular, for $0 < s < 2(< (\sqrt{3}/2)^{\frac{1}{2}}(\sqrt{3}+1)$, $Q'(x) \geq Q'(\alpha_s) > 0$, and hence $Q(x)$ is monotone increasing, in $(0,1)$. Since $Q(0) = -6s < 0$ and $Q(1) = 32 - 16s$, it follows that $Q(x)$ has exactly one root in $(0,1)$ if $s \in (0,2)$. On the other hand, if $s \geq 2$, then, since $3 + 6x^2 - x^4 > 0$ and $x^4 < x^3$ in $(0,1)$, we see that

$$
\begin{aligned}
Q(x) = 24x + 8x^3 - 2s(3+6x^2-x^4) \leq 24x + 8x^3 - 4(3+6x^2-x^4) \\
< 24x + 8x^3 - 12 - 24x^2 + 4x^3 = 12(x^3 - 2x^2 + 2x - 1) = \\
12(x^3 - 2x^2 + 2x - 1) = 12(x-1)(x^2 - x + 1) < 0.
\end{aligned}
$$

Thus the equation $Q(x) = 0$ has exactly one real root $x \in (0,1)$ if and only if $s \in (0,2)$. Since $s = \frac{k}{g-1}$ it follows that the real root $x \in (0,1)$ exists if and only if $k \in \{1,2,3,..,2g-3\}$. Note that for $D = 0$ the polynomial P is even and of degree 4 with exactly one positive root. Consequently we get an Einstein metric on $M_{k,g}$ for all $k \in \{1,2,..,2g-3\}$. Thus we obtain on the ruled surfaces $M_{k,g}$ of genus $g > 1$ constructed above exactly $2g-3$ different Einstein neutral bi-Hermitian metrics, exactly one Einstein metric on every surface $M_{k,g}$ for $k \in \{1,2,...,2g-3\}$. Note that for $g = 2$ we obtain only one metric corresponding to the Riemannian Einstein Bergery-Page metric on the first Hirzebruch surface F_1. Consequently we have proved

Theorem 4. *On a ruled surface $M_{k,g}$ of genus $g \geq 2$ for $k \in \{1,2,...,2g-3\}$ there exists an Einstein bi-Hermitian non-Kähler neutral metric.*

Note that if two Riemannian surfaces (Σ_g, J), (Σ_g, J') of the same genus g are not bi-holomorphic then the corresponding Einstein metrics on $M_{k,g}, M'_{k',g}$ are not isometric. In the Riemannian case we have only one Einstein metric-the Page metric on the Hirzebruch surface F_1 which is given by this construction (see [P],[B],[LeB],[S])
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References.

[B] Bérard-Bergery, L.*Sur de nouvelles variétés riemanniennes d'Einstein*, Institut Élie Cartan,6, Univ. Nancy, (1982), 1-60.

[Be] A. Besse, *Einstein manifolds*, Springer Verlag, 1987.

[D] M. Dunajski, *Anti-self-dual four manifolds with a parallel real spinor*, Proc. R. Soc. London. A(2002) 458, 1205-1222

[G] A. Gray *Einstein-like manifolds which are not Einstein* Geom. Dedicata **7** (1978) 259-280.

[J-1] W. Jelonek, *Compact Kähler surfaces with harmonic anti-self-dual Weyl tensor*, Diff. Geom. and Appl.16 (2002),267-276

[J-2] W. Jelonek, *Extremal Kähler $\mathcal{AC}^{\perp}$-surfaces*, Bull. Belg. Math. Soc. - Simon Stevin, 9,(2002), 561-571.

[J-3] W. Jelonek, *Bi-Hermitian Gray surfaces* , Pacific Math. J,222 no. 1, 57-68 (2005)

[J-4] W. Jelonek, *Bi-Hermitian Gray surfaces II*, (to appear in Diff. Geom and Appl.)

[J-5] W. Jelonek, *Neutral bi-Hermitian Gray surfaces*, Topics in Almost Hermitian geometry and related fields, 113-127 (2005)

[K] H. Kamada, *Self-dual Kähler metrics of neutral signature on complex surfaces* Tohoku Mathematical Publications, September 2002.

[LeB] C. Lebrun *Einstein metrics on complex surfaces*, in Geometry and Physics (Aarhus 1995) eds. J. Andersen, J. Dupont, H. Pedersen and A. Swann, Lect. notes in Pure Appl. Math., Marcel Dekker, 1996.

[M] Y. Matsushita *Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds* Math. Zeitschrift **207** (1991), 281-291. par

[M-L] Y. Matsushita and P. Law *Hitchin-Thorpe Type Inequalities for Pseudo-Riemannian 4-Manifolds of Metric Signature* $(++--)$ Geom. Dedicata **87** (2001), 65-89.

[M-S] B.Madsen, H. Pedersen, Y. Poon, A. Swann *Compact Einstein-Weyl manifolds with large symmetry group.* Duke Math. J. **88** (1997), 407-434.

[ON] B. O'Neill *Semi-Riemannian Geometry*, Academic Press, 1983.

[P] D. Page *A compact rotating gravitational instanton*, Phys. Lett.79 **B** (1978), 235-238.

[Pe] D. Petean *Indefinite Kähler-Einstein Metrics on Compact Complex Surfaces*, Comm. Math. Phys. 189 (1997), 227-235.

[S] P. Sentenac ,*Construction d'une métriques d'Einstein sur la somme de deux projectifs complexes de dimension 2*, Géométrie riemannienne en dimension 4 (Séminaire Arthur Besse 1978-1979) Cedic-Fernand Nathan, Paris (1981), pp. 292-307.

Institute of Mathematics
Technical University of Cracow
Warszawska 24
31-155 Kraków,POLAND.
E-mail address: wjelon@usk.pk.edu.pl