* Compact complex homogeneous spaces with vanishing first Chern class and heterotic string equations

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*work in progress

Characterization of compact complex homogeneous manifolds with vanishing first Chern class

C-spaces(Wang) : Compact simply complex homogeneous spaces

Such spaces are investigated first by Wang (1954):

Theorem 1 *If G is a compact semi-simple Lie group, and H is a closed connected subgroup whose semi-simple part coincides with the semisimple part of the centralizer of a toral subgroup of G, such that the coset space G/H is even-dimensional. Then G/H has a homogeneous complex structure and each C-space is homeomorphic to such coset.*

QUESTION: Which of the *C-spaces* have vanishing first Chern class?

Description of the invariant complex structures

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the (complexified) Lie algebras of G and H. Let $\mathfrak{t}$ is a Cartan subalgebra (maximal toral subalgebra) in $\mathfrak{g}$.

Now the above mentioned result of Wang provides $\mathfrak{h}^{ss} \subset \mathfrak{h} \subset \mathfrak{j}$, where $\mathfrak{j}$ is a parabolic subalgebra, $\mathfrak{t} \subset \mathfrak{j}$ and $\mathfrak{j}^{ss} = \mathfrak{h}^{ss}$. Here superscript "$ss$" means the semisimple part.

In particular

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{h}^{ss}$$

for a commutative subalgebra $\mathfrak{a}$ of the Cartan subalgebra $\mathfrak{t}$. The parabolic algebra $\mathfrak{j}$ is equal to the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Let $J \supset H$ is the subgroup of G corresponding to $\mathfrak{j}$.

The space G/J is a generalized flag manifold (or rational homogeneous space) and there is a fibration called the *Tits fibration* $G/H \to G/J$ with fibre a complex torus.

Now every complex structure on the flag manifold is determined by the choice of a system of simple roots Π which defines an ordering of the system of roots and a subset $\Pi_0 \subset \Pi$ which corresponds to $\mathfrak{j}$.

This correspondence determines the second cohomology and the first Chern class of G/H.

In general $\mathfrak{j}^{ss}$ is determined by the span of all roots in R which are positive with respect to Π_0. Then the compliment $\Pi - \Pi_0 = \Pi'$ provides a basis for the center ζ of $\mathbf{j}$ and there is an identification $span_{\mathbf{Z}}(\Pi') = H^2(G/J, \mathbf{Z})$.

The identification is:

$$\xi \to \frac{i}{2\pi} d\xi$$

where ξ is considered as a left invariant 1-form on G (in fact on G^c but we need it on G) and $d\xi$ is $ad(\mathbf{j})$-invariant, hence defines a 2-form on G/J. This form is obviously closed and in fact defines non-zero element in $H^2(G/J, \mathbf{Z})$. Moreover every class in $H^2(G/J, \mathbf{Z})$ has unique representative of this form.

Now we are interested in the first Chern class of G/H. It is determined by the so-called *Koszul form*:

$$\sigma_{G/H}(X) = Tr_{\frac{\mathfrak{g}}{\mathfrak{h}}}(ad(JX) - Jad(X)), X \in \mathfrak{g}$$

where J is extended as 0 on $\mathfrak{h}$. According to Alekseevsky-Perelomov, $\sigma_{G/H} = 2i(\sigma_G - \sigma_H)$, where σ_G is the sum of positive roots in G and σ_H is the sum of positive roots in G which are also in $\mathfrak{h}$. Then one has $\sigma_{G/J} = \sigma_{G/H}$ since the semisimple parts of $\mathfrak{j}$ and $\mathfrak{h}$ coincide.

As is proved by Koszul, the form $d\sigma$ descends to G/H and represents its first Chern class. Then the key point is that $d\sigma$ defines the zero element in $H^2(G/H, \mathbf{Z})$ iff σ descends to G/H itself. As it is proved by Tits σ is a sum with positive integer coefficients of elements of Π'. Then we have:

Theorem 2 *The first Chern class of G/H vanishes iff*

$$\sigma|_{\mathfrak{a}} = 0$$

i.e. the restriction of σ to $\mathfrak{a}$ vanishes.

Note that $d\sigma$ descends to G/J also and determines its first Chern class, which is positive.

Examples:

We begin with examples which include the following two extreme cases:

A) $\mathfrak{a} = 0$ i.e. $\mathfrak{h}$ is semisimple itself.

and

B) $H = U(1)$, where U(1) is appropriately embedded in odd-dimensional G,

For the case A) we start with an example from the A_ℓ-series. Consider

$$M = SU(n)/SU(n_1) \times SU(n_2) \times ... \times SU(n_k),$$

$$k - odd, n_i > 1$$

Here $SU(n_1) \times ... SU(n_k)$ is diagonally embedded as a matrix group in $SU(n)$ and $n_1 + n_2 = ... + n_k = n$.

The Tits fibration is $M \to SU(n)/S(U(n_1) \times U(n_2)... \times U(n_k))$ with fiber T^{k-1}. The existence of complex structure follows by Theorem 1. The vanishing of the Chern class follows by Theorem 2. These clearly could be generalized to the case

$$n_1 + n_2 ... + n_k \leq n, n_i > 1$$

where $n - (n_1 + n_2 + ... + n_k) + k$ is odd.

Examples for the other classical compact Lie groups are:

$$\begin{aligned}
M &= \frac{SO(2n)}{SU(n_1)\times ...\times SU(n_{2k})\times SO(2l)}, \\
& n_1 + ... + n_{2k} + l = n \\
M &= \frac{SO(2n)}{SU(n_1)\times ...\times SU(n_{2k})\times SU(n_{2k+1})}, \\
& n_1 + ... + n_{2k+1} = n \\
M &= \frac{SO(2n+1)}{SU(n_1)\times ...\times SU(n_{2k})\times SO(2l+1)}, \\
& n_1 + ... + n_{2k} + l = n \\
M &= \frac{Sp(n)}{SU(n_1)\times ... SU(n_{2k})\times Sp(l)}, \\
& n_1 + ... + n_{2k} + l = n
\end{aligned}$$

with at least one $n_i \neq 0$.

The other example of case B) is $G/U(1)$, but with U(1) appropriately embedded. We consider again $G = SU(n)$. The Tits fibration in this case is $SU(n)/U(1) \to F_{1,1...1}$ where $F_{1,1...,1} = SU(n)/S(U(1) \times ... \times U(1))$ is the standard flag manifold. In this case $\sigma_F = \sigma_{SU(n)} - \sigma_{S(U(1)\times...\times U(1))} = \sigma_{SU(n)}$, since $S(U(1)\times ... \times U(1))$ is abelian and $\sigma_{S(U(1)\times...\times U(1))} = 0$. To describe the sum of positive roots we need some notations. Let M_i be the matrix with i-th diagonal element equal to 1 and all others being 0. Then the set of all roots is $e_{i,j} = e_i - e_j$, where e_i are the duals of M_i. A set of simple roots is $e_{i,i+1}$ which also determines the positive roots $e_{i,j}, i < j$. Then the sum of all positive roots is:

$$\Sigma_{i<j} e_{i,j} = \Sigma_{k=1}^{n-1} k(n-k) e_{k,k+1} = \Sigma_{k=1}^{n} (n-2k+1) e_k$$

Proposition 1 *The space $SU(n)/U(1), n-$even is a complex homogeneous manifold with vanishing first Chern class iff $U(1)$ is embedded as a set of diagonal matrices:*

$$A = diag(e^{2\pi\theta_1 t}, e^{2\pi\theta_2 t}, ..., e^{2\pi\theta_n t})$$

with the property

$$\Sigma_{k=1}^{n-1}(n-2k+1)\theta_k = 0$$

Next we consider the general case of factors of $SU(n)$. In this case the Tits fibration is of the form:

$$\frac{SU(n)}{SU(n_1)\times...\times SU(n_k)\times T^l}$$
$$\downarrow$$
$$\frac{SU(n)}{SU(n_1)\times...\times SU(n_k)\times T^m}$$

with $n_1 + ... + n_k + m - k = n - 1$ and $l < m$.

At this point we notice that an invariant complex structure on the flag manifold $SU(n)/SU(n_1)\times ... \times SU(n_k) \times T^m$ is not unique but depends on the so called *painted Dynkin diagram*. In general painted diagrams are used to describe a flag manifolds and are popular for Grassmanians. For a general flag manifold painted Dynkin diagram is obtained by blackenning the vertices which correspond to Π'. More details are given in the paper

D.Alekseevski, A.Perelomov *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct.Anal.Appl.(3) 20 (1986)1–16.

We choose the flag manifold, which is the base of the Tits fibration, to be $SU(11)/SU(4) \times SU(3) \times SU(2) \times T^4 = SU(11)/S(T^2 \times U(4) \times U(3)\times U(2))$. It corresponds to a painted Dynkin diagram:

$$\bullet - - \bullet - - \circ - - \circ - - \circ - - \bullet - - \circ - - \circ - - \bullet - - \circ$$

Note that the diagram also determines the complex structure on the flag manifold. We need the Koszul form σ of this flag manifold. It is described by Alekseevsky and Perelomov:

$$\sigma = (2 + b_1)\overline{\alpha_1} + (2 + b_2)\overline{\alpha_2} + ... + (2 + b_m)\overline{\alpha_m}$$

where $\overline{\alpha_i}$ are the fundamental weights corresponding to the roots with black circles , defined as

$$\frac{(\overline{\alpha_k}, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta^i_j$$

Moreover the numbers b_i are equal the number of white circles of the Dynkin diagram, which are connected with the black circle corresponding to the root α_i by a series of white circles.

In particular in the above diagram we have:

$$\sigma = (2 + b_1)\overline{e_{1,2}} + (2 + b_2)\overline{e_{2,3}} + (2 + b_3)\overline{e_{6,7}}$$

$$+(2 + b_4)\overline{e_{9,10}} = 2\overline{e_{1,2}} + 5\overline{e_{2,3}} + 7\overline{e_{6.7}} + 5\overline{e_{9,10}}$$

Now to obtain explicit expression of the above element σ in terms of $e_{i,i+1}$ we need a description of the fundamental weights.

Since the product (,) is defined from the Killing form on $\mathfrak{g}$ then e_i are orthonormal matrices. Then one can check directly that the following elements satisfy the condition (1)

$$L_k = e_1 + ... + e_k - k/n(e_1 + ... + e_n) = \overline{e_{k,k+1}}$$

So in our case we have

$$
\begin{aligned}
\sigma &= 2e_1 - 2/11(e_1 + ... + e_{11}) + 5(e_1 + e_2) \\
&\quad -5.2/11(e_1 + ... + e_{11}) + 7(e_1 + ... + e_6) \\
&\quad -7.6/11(e_1 + ... + e_{11}) + 5(e_1 + ...e_9) \\
&\quad -5.9/11(e_1 + ... + e_{11}) \\
&= 10e_1 + 8e_2 + 3(e_3 + e_4 + e_5 + e_6 \\
&\quad -4(e_7 + ... + e_9) - 9(e_{10} + e_{11})
\end{aligned}
$$

The dimension count gives $dimSU(11)/S(T^2 \times U(4) \times U(3) \times U(2)) = 90$ and there could be 2 or 4 - dimensional fibers for a Tits fibration with this base. The 4-dimensional fiber case leads to an example of the previous type because the subgroup H will be semisimple.

So we consider the case of two dimensional fiber. At Lie-algebra level we have to add appropriate 2-dimensional space $\mathfrak{a}$ of diagonal matrices to the Lie algebra $\mathfrak{j}^{ss}$. It has to be of the form

$$\mathfrak{a} = diag(x_1, x_2, x_3, x_3, x_3, x_3, x_4, x_4, x_4, x_5, x_5)$$

and should obey the following conditions:

$$\begin{array}{rcl} x_1 + x_2 + 4x_3 + 3x_4 + 2x_5 & = & 0 \\ 10x_1 + 8x_2 + 12x_3 - 12x_4 - 18x_5 & = & 0 \end{array}$$

The first equation comes form the requirement that the matrices in $\mathfrak{a}$ are trace-free. The second follows from Theorem 2 and the form of σ above. Now we can fix two linearly independent solutions $(v_1, v_2, v_3, v_4, v_5)$ and $(w_1, w_2, w_3, w_4, w_5)$ with integer entries of the equations above. Then the Lie algebra $\mathfrak{h} = \mathfrak{j}^{ss} + \mathfrak{a}$ should be:

$$\mathfrak{h} = diag(v_1 t + w_s, v_2 t + w_2 s, (v_3 t + w_3 s)A,$$

$$(v_4 t + w_4 s)B, (v_5 t + w_5 s)C)$$

where A, B, C are trace-free skew-adjoint matrices of order 4,3 and 2 respectively and t, s are parameters. Then at the end we obtain that $SU(11)/H$ is a complex homogeneous manifold with vanishing first Chern class, if H is of the form:

$$H = diag(e^{2i\pi(v_1 t + w_1 s)}, e^{2i\pi(v_2 t + w_2 s)},$$

$$e^{2i\pi(v_3 t + w_3 s)}A, e^{2i\pi(v_4 t + w_4 s)}B, e^{2i\pi(v_2 t + w_2 s)}C)$$

where $A \in SU(4), B \in SU(3), C \in SU(2)$. Moreover any such manifold with a stationary subgroup H containing strictly $Id_2 \times SU(4) \times SU(3) \times SU(2))$ is of this form.

CYT structures and homogeneous spaces

Let (M, J, g) be Hermitian manifold and $F(X, Y) = g(JX, Y)$ is the Kähler form. Using the notations of Gauduchon we have a one-parameter family of Hermitian connections ∇^t with the property

$$\nabla^t s - \nabla^u s = i\frac{t-u}{2}\delta F \otimes s$$

for any section s of the anti-canonical bundle K^{-1} where δ is the co-differential.

Let R^t be the curvature of ∇^t and $\rho^t(X, Y) = \Sigma g(R^t(X, Y)E_i, JE_i)$ be the corresponding trace. Then $i\rho^t$ is the curvature of K^{-1} and from the above relation we obtain:

$$\rho^t = \rho^u + \frac{t-u}{2} d\delta F$$

Now ∇^1 is the Chern connection, ∇^{-1} is the Bismut connection. Denote by ρ and ρ^B the Ricci forms of the Chern and Bismut connections respectively. We call a metric *CYT* if $\rho^B = 0$.

Theorem 3 *Every simply-connected compact complex homogeneous space with vanishing first Chern class admits a CYT structure.*

Another type of compact complex homogeneous manifolds with vanishing first Chern class are the complex parallelizable manifolds - i.e. the manifolds with holomorphic parallelization of its (holomorphic) tangent bundle. According to a well known theorem these are of the form G/Γ, where G is a complex Lie group and Γ is a cocompact lattice. For the Hermitian geometry of such manifolds there is the following result by Abbena and Grassi (1986):

Theorem 4 *For a compact complex parallelizable manifold any left invariant metric is a balanced metric.*

From this theorem we obtain:

Theorem 5 *For a compact complex parallelilzable manifold any left invariant metric is a CYT metric. Moreover all Ricci forms of the canonical Hermitian connections vanish.*

Moreover D.Guan proved the following in 2002:

Theorem 6 *Every compact complex homogeneous space with invariant volume form is a principal homogeneous complex torus bundle over the product of a projective rational homogeneous space and a parallelizable manifold.*

At this point we have:

Conjecture 1 *Every compact complex homogeneous space with invariant volume form admits a CYT structure*

Relation to the Strominger's equations in heterotic string theory

In 1986 A.Strominger analyzed heterotic superstring background with spacetime supersymmetry. His model is based on Hermitian manifolds which are generalization of the Calabi-Yau manifolds. In terms of Hermitian geometry it is about conformally balanced complex 3-manifold with holomorphic (3,0)-form with constant norm and an anomaly cancellation condition. The manifold is endowed with an auxiliary semistable bundle with Hermitian-Einstein connection A with curvature F_A and the anomaly cancellation condition is:

$$dH = 2i\partial\overline{\partial}F = \frac{\alpha'}{4}[tr(R \wedge R) - tr(F_A \wedge F_A)]$$

The first solutions on non-Kähler manifolds of this system were constructed only recently by J.Fu and S.T.Yau. We consider here solutions of the Strominger's system with $F_A = 0$ in the anomaly cancellation condition.

We begin with an example of nilmanifold which is not complex homogeneous, but satisfies the conditions of the Strominger's system with positive α'.

Let $e^1, Je^1, e^2, Je^2, e^3, Je^3$ be a unitary co-basis for a complex structure J and Hermitian metric g, so that the Kähler form is $F = 2(e^1 \wedge Je^1 + e^2 \wedge Je^2 + e^3 \wedge Je^3)$. Consider the Lie algebra defined via $d(Je^3) = e^1 \wedge Je^1 - e^2 \wedge Je^2$ and all other 1-forms of the basis are closed. It defines a Lie group which is the product $H^5 \times R^1$ of the 5-dimensional real Heisenberg group and a line. This admits a compact quotient M such that g and J descend to M. Then the structure J is integrable, since $d(e^3 + iJe^3) \in \Lambda^{(1,1)}$ so $d(\Lambda^{(1,0)}) \in \Lambda^{(1,1)}$ and balanced since $d(F^2) = 0$. Moreover $dd^cF = 2dd^c(e^3 \wedge Je^3) = 2d(J(-e^3 \wedge dJe^3)) = -2d(Je^3 \wedge dJe^3) = -2(dJe^3)^2$ since $JdJe^3 = dJe^3$.

Now we take a skew-symmetric connection matrix which defines a metric connection:

$$\omega = \begin{pmatrix} 0 & Je^3 & 0 & 0 & 0 & 0 \\ -Je^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The curvature of this connection is given by the matrix

$$R = d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & dJe^3 & 0 & 0 & 0 & 0 \\ -dJe^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we obtain $tr(R \wedge R) = -(dJe^3)^2 = +dd^cF$. Moreover by changing ω to $a\omega$ and choosing an anti-selfdual abelian instanton A on the base 4-torus, we can find a solution with non-vanishing field F_A.

Complex parallelizable manifolds and Strominger's equations

Let $\mathfrak{g}$ be a complex 3-dimensional Lie algebra with a basis of left-invariant holomorphic (1,0)-forms α, β, γ. Let G be the simply connected complex Lie group with algera $\mathfrak{g}$ and consider the following examples of compact complex parallelizable manifolds G/Γ where Γ is a cocompact lattice (these exhaust the 3-dimensional compact complex parallelizable manifolds):

1. Complex Iwasawa manifold. It is determined by the complex Heisenberg algebra:

$$d\alpha = \beta \wedge \gamma, d\beta = d\gamma = 0$$

2. A solvmanifold determined by

$$d\alpha = \alpha \wedge \gamma, d\beta = -\beta \wedge \gamma, d\gamma = 0$$

3. The space $SL(2,\mathbb{C})/\Gamma$ determined by:

$$d\alpha = \beta \wedge \gamma, d\beta = \gamma \wedge \alpha, d\gamma = -\alpha \wedge \beta$$

In each example we consider the metric given by $g = \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma}$. Then the Kähler form is $F = i(\alpha \wedge \overline{\alpha} + \beta \wedge \overline{\beta} + \gamma \wedge \overline{\gamma})$ and it is easy to see that $dF^2 = 0$, so the metric is balanced as follows from the result by Abbena and Grassi.

Let $\alpha^r, \beta^r, \gamma^r$ be the real parts of α, β, γ and the imaginery parts are $J\alpha^r, J\beta^r, J\gamma^r$ accordingly. Then

$$dd^cF = 2(d\alpha^r \wedge d\alpha^r + dJ\alpha^r \wedge dJ\alpha^r$$

$$+d\beta^r \wedge d\beta^r + dJ\beta^r \wedge dJ\beta^r + d\gamma^r \wedge d\gamma^r + dJ\gamma^r \wedge dJ\gamma^r)$$

Then we can check easily that for all 3 examples above

$$d\alpha^r \wedge d\alpha^r = dJ\alpha^r \wedge dJ\alpha^r, d\beta^r \wedge d\beta^r$$

$$= dJ\beta^r \wedge dJ\beta^r, d\gamma^r \wedge d\gamma^r = dJ\gamma^r \wedge dJ\gamma^r$$

Now we choose a connection on the tangent bundle given by the matrix of 1-forms for each case as follows:

Case 1.

$$\omega = \begin{pmatrix} 0 & \alpha^r & 0 & 0 & 0 & 0 \\ -\alpha^r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 2.

$$\omega = \begin{pmatrix} 0 & \alpha^r & 0 & 0 & 0 & 0 \\ -\alpha^r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^r & 0 & 0 \\ 0 & 0 & -\beta^r & 0 & & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 3.

$$\omega = \begin{pmatrix} 0 & \alpha^r & 0 & 0 & 0 & 0 \\ -\alpha^r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^r & 0 & 0 \\ 0 & 0 & -\beta^r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma^r \\ 0 & 0 & 0 & 0 & -\gamma^r & 0 \end{pmatrix}$$

These are skew-symmetric matrices which in the fixed unitary bases define metric connections.

Then it is easy to calculate $R = d\omega + \omega \wedge \omega$ and we see for example that in the first case:

$$R \wedge R = \begin{pmatrix} -d\alpha^r \wedge d\alpha^r & 0 & 0 & 0 & 0 & 0 \\ 0 & -d\alpha^r \wedge d\alpha^r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so $tr(R \wedge R) = -2(d\alpha^r \wedge d\alpha^r)$ Since in case 1, $d\beta^r \wedge d\beta^r = d\gamma^r \wedge d\gamma^r = 0$, then $dd^cF = -2tr(R \wedge R)$ for the Iwasawa manifold. Similarly $tr(R \wedge R) = -[2(d\alpha^r)^2 + 2(d\beta^r)^2] = -1/2dd^cF$ in the second case and $tr(R \wedge R) = -2(d\alpha^r)^2 - 2(d\beta^r)^2 - 2(d\gamma)^2 = -1/2dd^cF$ in the third.

Note that if instead of the above skew-symmetric matrices for ω we choose a symmetric ones with the same entries above the diagonal, then we have a non-metric connections ω which satisfy $dd^cF = 2tr(R \wedge R)$.