Research Group 'Global Analysis'

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$\mathrm{SO}(3)_{ir}$-geometries in dimension five and seven – results and open problems

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Introduction

Fix a subgroup $\mathrm{G} \subset \mathrm{SO}(n)$ and consider a Riemannian manifold $(M^n, g, \mathcal{R})$ equipped with a G-structure $\mathcal{R}$.

Examples:

$\mathrm{G} = \mathrm{U}(n) \subset \mathrm{SO}(2n) \longrightarrow$ almost hermitian geometry.

$\mathrm{G} = \mathrm{U}(n) \subset \mathrm{SO}(2n+1) \longrightarrow$ contact geometry

$\mathrm{G} = \mathrm{G}_2 \subset \mathrm{SO}(7) \longrightarrow \mathrm{G}_2$-geometry in dimension 7

First Question: Does there exist a metric connection ∇^c preserving the structure $\mathcal{R}$ such that the torsion

$$\mathrm{T}^c(X, Y, Z) := g(\nabla^c_X Y - \nabla^c_Y X - [X, Y], Z)$$

is totally skew-symmetric ? $\rightarrow$ **characteristic connection**.

Second Question: Study the curvature and the spin geometry of the new connection ∇^c.

In particular, we look for solutions of type II string equations involving a spinor field (super-symmetry) Ψ and a non-trivial 'B-field' T^c:

$$\begin{aligned} \mathrm{Ric}^c &= \lambda \cdot g\,, \quad \delta(\mathrm{T}^c) \;=\; 0\,, \\ \nabla^c \Psi &= 0\,, \quad \mathrm{T}^c \cdot \Psi \;=\; \mu \cdot \Psi\,. \end{aligned}$$

Approach: We construct in a systematic way solutions in type II superstring theory starting from non-integrable geometric structures.

Results: In dimension $5 \leq n \leq 8$ for contact, almost hermitian, G_2- and $\mathrm{Spin}(7)$-geometries (Agricola, Friedrich, Ivanov, . . .).

Reference: the 'Srni lectures' of Ilka Agricola, 2006.

In dimension $n = 5$ one usually considers contact geometries; they are related to the subgroup $\mathrm{U}(2) \subset \mathrm{SO}(5)$ (Fr/Ivanov, Crelle J., 2004).

New Geometries: Consider the subgroup $\mathrm{SO}(3)_{ir} \subset \mathrm{SO}(5)$. Study the geometry of 5-manifolds with such a structure. Apply the described method to them.

First Results:

• M. Bobienski, P. Nurowski: started this program during their stay in Berlin (Crelle J., 2006)

• S. Chiossi, A. Fino: $\mathrm{SO}(3)_{ir}$-structures on Lie groups (J. Lie Th., 2007)

Aim of this lecture: discuss some topological and geometric problems for $\mathrm{SO}(3)_{ir}$-structures in dimensions $n = 5, 7$.

$\mathrm{SO}(3)_{ir}$-structures in dimension five

The group $\mathrm{SO}(5)$ contains two subgroups isomorphic to $\mathrm{SO}(3)$,

$$\mathrm{SO}(3)_{st} \ \subset \ \mathrm{SO}(5)\,, \quad \mathrm{SO}(3)_{ir} \ \subset \ \mathrm{SO}(5)\ .$$

The subgroup $\mathrm{SO}(3)_{ir}$ is the $\mathrm{SO}(3)$-action on $S^2_0(\mathbb{R}^3) = \mathbb{R}^5$.

The generators of the Lie algebra $\mathfrak{so}(3)_{ir} \subset \mathfrak{so}(5)$ are

$$\begin{aligned} X_1 &= e_{13} + \sqrt{3}\, e_{23} + e_{45}\,, \quad X_2 = 2\, e_{14} + e_{35} \\ X_3 &= -\, e_{15} + \sqrt{3}\, e_{25} - e_{34}\ . \end{aligned}$$

Question: Under which conditions a compact oriented 5-manifold M^5 admits an $\mathrm{SO}(3)_{st}$- or an $\mathrm{SO}(3)_{ir}$-structure ?

First case: $SO(3)_{st}$-structures

In order to formulate the condition, we need some invariants.

Definition: The semi-characteristics (Kervaire) are defined by

$$k(M^5) := \sum_{i=0}^{2} \dim_{\mathbb{R}}\big(H^{2i}(M^5;\mathbb{R})\big) \mod 2 \ ,$$

$$\hat{\chi}_2(M^5) := \sum_{i=0}^{2} \dim_{\mathbb{Z}_2}\big(H_i(M^5;\mathbb{Z}_2)\big) \mod 2 \ .$$

Theorem:(Lusztig-Milnor-Peterson 1969)

$$k(M^5) - \hat{\chi}_2(M^5) = w_2(M^5) \cup w_3(M^5) \ .$$

In particular, if M^5 is spin, then $k(M^5) = \hat{\chi}_2(M^5)$.

Theorem: A compact, oriented 5-manifold admits an $\mathrm{SO}(3)_{st}$-structure (i.e. two vector fields) if and only if

$$w_4(M^5) = 0 \ , \quad k(M^5) = 0 \ .$$

Proof: E. Thomas in 1967 for spin manifolds ($w_4(M^5) = 0 = \hat{\chi}_2(M^5)$), M.F. Atiyah in 1969 for the general case.

Second case: $SO(3)_{ir}$-structures

Example 1: $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ has an $\mathrm{SO}(3)_{ir}$-structure.

Some topological properties of this space:

- M^5 is simply connected and a rational homology sphere.
- M^5 does not admit any Spin- or $\mathrm{Spin}^{\mathbf{C}}$-structure.
- $k(M^5) - \hat{\chi}_2(M^5) \;=\; w_2(M^5) \cup w_3(M^5) = 1$
- $k(M^5) = 1$ and $\hat{\chi}_2(M^5) = 0$

In particular, $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ does not admit any $\mathrm{SO}(3)_{st}$-structure!

Example 2: $M^5 = S^5$ has no $\mathrm{SO}(3)_{st}$- or $\mathrm{SO}(3)_{ir}$-structure.

- M^5 admits a Spin-structure
- $k(M^5) - \hat{\chi}_2(M^5) \;=\; w_2(M^5) \cup w_3(M^5) \;=\; 0$
- $k(M^5) = 1$ and $\hat{\chi}_2(M^5) = 1$

Example 3: Consider the subgroup $H = \{(A, A^2), A \in \mathrm{SO}(2)\} \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$ as well as the homogeneous space $M^5 = (\mathrm{SO}(3) \times \mathrm{SO}(3))/H$. Then M^5 has an $\mathrm{SO}(3)_{ir}$-structure and the following topological data:

- $H_1(M^5; \mathbb{Z}) = \mathbb{Z}_2\,,\ H_2(M^5; \mathbb{Z}) = \mathbb{Z}$.
- $k(M^5) = 0$ and $\hat{\chi}_2(M^5) = 0$.

More examples: Bobienski/Nurowski (Crelle Journal 2006): there is a 2-parameter family $\mathrm{G}^6(s,t)$ of 6-dimensional Lie groups containing $\mathrm{SO}(2)$ such that the isotropy representation of $M^5 = \mathrm{G}^6(s,t)/\mathrm{SO}(2)$ is the maximal torus $\mathrm{T}_{max} = \{(A, A^2, 1), A \in \mathrm{SO}(2)\} \subset \mathrm{SO}(3)_{ir} \subset \mathrm{SO}(5)$. The groups are, for example,

$$\begin{aligned} \mathrm{G}^6(s,t) &= \mathrm{SO}(3) \times \mathrm{SO}(3)\,, \quad \mathrm{SO}(3) \times \mathrm{SO}(1,2)\,, \\ &\quad \mathbb{R}^1 \times (\mathrm{SO}(2) \ltimes \mathbb{R}^4)\,, \quad (\mathrm{SO}(2) \ltimes \mathbb{R}^2) \times \mathrm{SO}(3)\ . \end{aligned}$$

The obstructions for $\mathrm{SO}(3)_{ir}$-structures:

The relevant space is $X^7 := \mathrm{SO}(5)/\mathrm{SO}(3)_{ir}$. Let us list some of its homotopy groups:

$$\pi_1(X^7) = 0\ , \quad \pi_2(X^7) = 0\ , \quad \pi_3(X^7) = \mathbb{Z}_{10}\ , \quad \pi_4(X^7) \ = \ \mathbb{Z}_2\ .$$

Consequence: The obstructions for the existence of an $\mathrm{SO}(3)_{ir}$-structure on a compact 5-manifold M^5 are in $H^4(M^5;\mathbb{Z}_{10}) = H^4(M^5;\mathbb{Z}_5) \oplus H^4(M^5;\mathbb{Z}_2)$ and in $H^5(M^5;\mathbb{Z}_2)$.

Problem: Compute the topological conditions for the existence in general.

The criterion given in Bobenski, math.dg/0601066 is wrong ($w_4 = 0\,,\ k = 0\,,\ p_1/5 \in \mathbb{Z}$). The space $M^5 = \mathrm{SU}(3)/\mathrm{SO}(3)$ does not satisfy it !

Proposition 1: M^5 admits an $\mathrm{SO}(3)_{ir}$-structure if and only if there exists a 3-dimensional bundle E^3 such that $T(M^5) = S_0^2(E^3)$.

Proposition 2: Suppose that $T(M^5) = S_0^2(E^3)$. Then

- $p_1(M^5) = 5 \cdot p_1(E^3)$.
- $w_1(M^5) = 0$ and $w_4(M^5) = 0$.
- $w_2(M^5) = w_2(E^3)$ and $w_3(M^5) = w_3(E^3)$.

Corollary: If M^5 admits an $\mathrm{SO}(3)_{ir}$-structure, then

- $w_4(M^5) = 0$ in $H^4(M^5; \mathbb{Z}_2)$;
- $p_1(M^5)/5 \in H^4(M^5; \mathbb{Z})$ is integral.

Conjecture: M^5 admits an $\mathrm{SO}(3)_{ir}$-structure if and only if

$$w_4(M^5) \;=\; 0 \;, \quad \hat{\chi}_2(M^5) = 0 \;, \quad \frac{p_1(M^5)}{5} \in H^4(M^5; \mathbb{Z}) \;.$$

$\mathrm{SO}(3)_{ir}$- and G_2-structures in dimension seven

The real, irreducible 7-dimensional representation $S^3_0(\mathbb{R}^3) = \mathbb{R}^7$ of $\mathrm{SO}(3)$ yield an embedding $\mathrm{SO}(3)_{ir} \subset \mathrm{G}_2 \subset \mathrm{SO}(7)$.

The sub-algebra $\mathfrak{so}(3)_{ir} \subset \mathfrak{g}_2 \subset \mathfrak{so}(7)$ is given by

$$\begin{aligned}
X_1 &= \sqrt{\frac{1}{5}}\Big(e_{12} + 2e_{34} - 3e_{56}\Big) \\
X_2 &= -\sqrt{\frac{6}{5}}\,e_{27} - \sqrt{\frac{1}{2}}\,e_{14} + \sqrt{\frac{1}{2}}\,e_{23} + \sqrt{\frac{3}{10}}\,e_{35} - \sqrt{\frac{3}{10}}\,e_{46} \\
X_3 &= -\sqrt{\frac{6}{5}}\,e_{17} - \sqrt{\frac{1}{2}}\,e_{13} - \sqrt{\frac{1}{2}}\,e_{24} + \sqrt{\frac{3}{10}}\,e_{36} + \sqrt{\frac{3}{10}}\,e_{45}\ .
\end{aligned}$$

The group G_2 preserves the 3-form

$$\omega^3 = e_{567} + e_{347} + e_{127} - e_{146} + e_{135} - e_{245} - e_{236}\ .$$

The following formula proves the inclusion $\mathrm{SO}(3)_{ir} \subset \mathrm{G}_2$:

$$*\big(X_1 \wedge X_1 \,+\, X_2 \wedge X_2 \,+\, X_3 \wedge X_3\big) \;=\; -\frac{6}{5}\,\omega^3 \;.$$

Theorem: $\mathrm{SO}(3)_{ir} \subset \mathrm{SO}(7)$ is the stabilizer of two symmetric tensors in $S^4(\mathbb{R}^7)$. The first polynomial is $(x_1^2 + \ldots + x_7^2)^2$ and the second polynomial is given by the formula

$$\begin{aligned}
&4\sqrt{15}x_1^4 + 4\sqrt{15}x_2^4 + 120x_2^3x_5 - 120x_1^3x_6 + \\
&60x_1\big(10x_3x_4x_5 + (6x_2^2 + 5x_3^2 - 5x_4^2)x_6 - (4x_2x_4 + 2\sqrt{15}x_4x_5 + 2\sqrt{15}x_3x_6)x_7\big) - \\
&60x_2\big(5x_3^2x_5 - 5x_4^2x_5 - 10x_3x_4x_6 + 2\sqrt{15}x_3x_5x_7 - 2\sqrt{15}x_4x_6x_7\big) + \\
&\sqrt{15}\big(25x_3^4 + 25x_4^4 - 30x_4^2x_7^2 + 10x_3^2(5x_4^2 - 3x_7^2) + 9x_7^2(10x_5^2 + 10x_6^2 + x_7^2)\big) + \\
&2x_1^2\big(4\sqrt{15}x_2^2 + 10\sqrt{15}x_3^2 - 180x_2x_5 - 60x_3x_7 + \sqrt{15}(10x_4^2 + 30x_5^2 + 30x_6^2 + 9x_7^2)\big) \\
&+2x_2^2\big(10\sqrt{15}x_3^2 + 60x_3x_7 + \sqrt{15}(10x_4^2 + 30x_5^2 + 30x_6^2 + 9x_7^2)\big)
\end{aligned}$$

Some consequences:

- An $\mathrm{SO}(3)_{ir}$-structure on a Riemannian 7-manifolds is defined by a special symmetric tensor (polynomial) of degree 4.

- Any $\mathrm{SO}(3)_{ir}$-structure on a Riemannian 7-manifold (M^7, g) induces a unique G_2-structure ω^3.

- There are 4 basic classes $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4$ of G_2-structures on a 7-dimensional Riemannian manifold (Fernandez/Gray 1982). They are given by the G_2-components of the representation $\mathbb{R}^7 \otimes (\mathfrak{so}(7)/\mathfrak{g}_2)$.

- A G_2-manifold admits a characteristic connection if and only if it is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ ($d * \omega^3 = \theta \wedge *\omega^3$ – Fr/Ivanov 2002).

- Denote by V_k the real, $(2k+1)$-dimensional, irreducible representation of $\mathrm{SO}(3)$. The $\mathrm{SO}(3)_{ir}$-representations decompose into

$$\mathcal{W}_1 = V_0 \ , \ \mathcal{W}_2 = V_2 \oplus V_4 \ , \ \mathcal{W}_3 = V_2 \oplus V_4 \oplus V_6 \ , \ \mathcal{W}_4 = V_3 \ .$$

Theorem: There is a bijection between

- $\mathrm{SO}(3)_{ir}$-structures admitting a characteristic connection;

- G_2-structures of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ such that the holonomy $\mathfrak{hol}(\nabla^c) \subset \mathfrak{so}(3)_{ir}$ of their characteristic connection is contained in $\mathfrak{so}(3)_{ir}$.

Problem: Construct 7-dimensional Riemannian manifolds with $\mathrm{SO}(3)_{ir}$-structure such that the underlying G_2-structure is of type

$$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \;=\; V_0 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_6 \;.$$

Can any V_α-type be realized ?

An equivalent formulation: Construct G_2-structures on Riemannian 7-manifolds with characteristic connection such that $\mathfrak{hol}(\nabla^c) \subset \mathfrak{so}(3)_{ir}$ and of a fixed V_α-type.

Remark: A parallel G_2-manifold (i.e. $\nabla^c = \nabla^g, \mathrm{T}^c = 0$) cannot have a holonomy $\mathfrak{hol}(\nabla^c) = \mathfrak{hol}(\nabla^g) = \mathfrak{so}(3)_{ir}$ (Berger's holonomy theorem).

Theorem:

A compact, 7-dimensional $\mathrm{SO}(3)_{ir}$-manifold of type $\mathcal{W}_4 = V_3$ and $\mathfrak{hol}(\nabla^c) \subset \mathfrak{so}(3)_{ir}$, $\mathrm{T}^c \neq 0$ does not exist. Equivalently, a compact G_2-manifold of type $\mathcal{W}_4$ and $\mathfrak{hol}(\nabla^c) \subset \mathfrak{so}(3)_{ir}$, $\mathrm{T}^c \neq 0$ does not exist.

Sketch of the proof:

Up to a conformal change of the metric, the universal covering splits into $Y^6 \times \mathbb{R}^1$, where Y^6 is a nearly Kähler manifold (see Agricola/Friedrich, J. Geom. Phys. 2006). Then we conclude that

$$\mathfrak{hol}(\nabla^c) \subset \mathfrak{su}(3) \cap \mathfrak{so}(3)_{ir} = \mathfrak{so}(2) \ .$$

In particular, Y^6 is nearly Kähler with a reduced, 1-dimensional characteristic holonomy, a contradiction (Belgun/Moroianu, Ann. Glob. Anal. Geom. 2002).

The basic example: $X^7 = \mathrm{SO}(5)/\mathrm{SO}(3)_{ir}$.

Geometric properties of the basic example:

- X^7 admits an $\mathrm{SO}(3)_{ir}$-structure. It is not symmetric.

- The underlying G_2-structure is of type $\mathcal{W}_1$ (nearly parallel). In particular, it realizes the type V_0. Moreover, X^7 admits one real Killing spinor. This spinor field is ∇^c-parallel.

Theorem: (TF 2006)

X^7 is the unique G_2-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ such that

$$\mathfrak{hol}(\nabla^c) \subset \mathfrak{so}(3)_{ir} , \quad \nabla^c \mathrm{T}^c = 0 , \quad \mathrm{T}^c \neq 0 .$$

Topological properties of the basic example:

- X^7 is simply connected and a rational homology sphere,

$$
\begin{aligned}
H_1(M^7;\mathbb{Z}) \;&=\; 0\;, \quad H_2(M^7;\mathbb{Z}) = 0\;, \quad H_3(M^7;\mathbb{Z}) = \mathbb{Z}_{10}\;, \\
H_4(M^7;\mathbb{Z}) \;&=\; 0\;, \quad H_5(M^7;\mathbb{Z}) = 0\;, \quad H_6(M^7;\mathbb{Z}) = 0\;.
\end{aligned}
$$

- $k(X^7) = 1$ and $\hat{\chi}_2(X^7) = 0$.

- All Stiefel-Whitney classes $w_i(X^7)$ and the Pontrjagin class $p_1(X^7)$ are trivial.

- X^7 is not parallelizable, but admits 4 vector fields (Goette/Kitchloo/Shankar 2002). In particular, there is no further reduction of the frame bundle to a subgroup of $\mathrm{SO}(3)_{ir}$.

Existence of $\mathrm{SO}(3)_{ir}$-structures in dimension seven

Problem: Study the conditions for the existence of a topological $\mathrm{SO}(3)_{ir}$-structure on a compact 7-manifold. The relevant space is

$$Z^{18} = \mathrm{SO}(7)/\mathrm{SO}(3)_{ir} \ .$$

The homotopy groups of Z^{18}:

$$\begin{aligned} \pi_1(Z^{18}) &= \mathbb{Z}_2 \ , \quad \pi_2(Z^{18}) = \mathbb{Z}_2 \ , \quad \pi_3(Z^{18}) = \mathbb{Z}_{28} \ , \\ \pi_4(Z^{18}) &= 0 \ , \quad \pi_5(Z^{18}) = \mathbb{Z}_2 \ , \quad \pi_6(Z^{18}) = \mathbb{Z}_2 \ . \end{aligned}$$

Consequence: Let M^7 be a compact, oriented 7-manifold. Then the obstructions for the existence of an $\mathrm{SO}(3)_{ir}$-structure are in

$$H^2(M^7;\mathbb{Z}_2) \ , \ H^3(M^7;\mathbb{Z}_2) \ , \ H^4(M^7;\mathbb{Z}_{28}) \ , \ H^6(M^7;\mathbb{Z}_2) \ , \ H^7(M^7;\mathbb{Z}_2) \ .$$

Proposition: M^7 admits an $\mathrm{SO}(3)_{ir}$-structure if and only if there exists a 3-dimensional bundle E^3 such that $T(M^7) = S_0^3(E^3)$.

Now we compute again characteristic classes.

Theorem: Let F^7 be a real, oriented, 7-dimensional vector bundle over some space Y and suppose that there exists a 3-dimensional bundle E^3 such that $F^7 = S_0^3(E^3)$. Then we have:

- $w_2(F^7) = w_3(F^7) = w_5(F^7) = w_7(F^7) = 0$;
- $w_4(F^7) = w_2^2(E^3)$ and $w_6(F^7) = w_3^2(E^3)$;
- $p_1(F^7)/14 = p_1(E^3)$ is an integral cohomology class and $p_1(F^7)/14 = w_4(F^7)$ mod 2.

If $F^7 = T(M^7)$ is the tangent bundle of some 7-manifold, then $w_1(M^7) = w_2(M^7) = w_3(M^7) = 0$ implies the vanishing of $w_4(M^7)$ and $w_6(M^7)$ (use the Wu formulas !).

Corollary: Let M^7 be an oriented, compact 7-manifold. If it admits an $\mathrm{SO}(3)_{ir}$-structure, then

- all Stiefel-Whitney classes are trivial.
- The Pontrjagin class $p_1(M^7)$ is divisible by 28.

Remark: The criterion is only necessary, but not sufficient. Again the highest obstruction in $H^7(M^7; \mathbb{Z}_2)$ is missing.

Remark: This obstruction cannot be $\hat{\chi}_2(M^7)$. Indeed, the sphere S^7 is parallelizable and $\hat{\chi}_2(S^7) = 1$. On the other hand, $\mathrm{SO}(5)/\mathrm{SO}(3)_{ir}$ admits an $\mathrm{SO}(3)_{ir}$-structure, but $\hat{\chi}_2(\mathrm{SO}(5)/\mathrm{SO}(3)_{ir}) = 0$. However, both manifolds have $k = 1$.

Bundles over twistor spaces

Let X^4 be a compact, oriented 4-manifold such that

$$w_3(X^4) = 0\,, \quad 6\sigma(X^4) - 2\chi(X^4) \equiv 0 \bmod 28$$

holds. Denote by Z^6 its twistor space and consider a principal S^1-bundle $M^7 \to Z^6$. Then all Stiefel-Whitney classes of M^7 vanish and the Pontrjagin class $p_1(M^7)$ is divisible by 28.

Remark: The condition $0 = w_3(X^4) \in H^3(X^4;\mathbb{Z}_2) = H_1(X^4;\mathbb{Z}_2)$ is satisfied for example if

- X^4 is simply connected.
- X^4 is a spin manifold (apply Wu's formula !).

Example: Consider $X^4 = A\cdot(S^1\times S^3)\,\#\,B\cdot(S^2\times S^2)$. X^4 is spin and

$$\sigma(X^4) = 0, \quad \chi(X^4) = 2(B+1-A)\,.$$

Example: Consider $X^4 = \mathbb{CP}^2 \# k \cdot (-\mathbb{CP}^2)$ = blow up of $\mathbb{CP}^2$ in k points. X^4 is simply connected and

$$\sigma(X^4) = 1 - k, \quad \chi(X^4) = 3 + k, \quad 6\sigma(X^4) - 2\chi(X^4) = -8k\,.$$

In particular, del Pezzo surfaces ($k = 7$) satisfy the necessary conditions. These surfaces admit Kähler-Einstein metrics with positive scalar curvature.

More example: $X^4 = A \cdot (S^2 \times S^2) \# B \cdot (K3)$,
$X^4 = A \cdot (S^1 \times S^3) \# B \cdot (K3)$, . . .

Construction of $\mathrm{T}_{max} \subset \mathrm{SO}(3)_{ir}$-structures

The algebraic fact

$$\mathrm{T}_{max} = \mathrm{diag}(A, A^2, A^{-3}, 1) = \mathrm{SO}(3)_{ir} \cap \mathrm{SU}(3) \subset \mathrm{G}_2\,, \quad A \in \mathrm{SO}(2)$$

yields the following

Theorem: Let M^5 be a 5-manifold with an $\mathrm{T}_{max} = \{(A, A^2, 1), A \in \mathrm{SO}(2)\} \subset \mathrm{SO}(3)_{ir} \subset \mathrm{SO}(5)$-structure $\mathcal{R}$ and denote by ρ the 2-dimensional representation of T_{max} given by $\rho(A) = A^{-3}$, $A \in \mathrm{SO}(2)$. Then $M^7 := \mathcal{R} \times_\rho \mathbb{R}^2$ admits an $\mathrm{T}_{max} \subset \mathrm{SO}(3)_{ir} \subset \mathrm{SO}(7)$-structure. M^7 is a complex vector bundle over M^5.

Remark: In a similar way we can consider 5 manifolds with an $\{(A, A^{-3}, 1), A \in \mathrm{SO}(2)\} \subset \mathrm{SO}(5)$- or an $\{(A^2, A^{-3}, 1), A \in \mathrm{SO}(2)\} \subset \mathrm{SO}(5)$-structure, Then a vector bundle M^7 over M^5 again admits an $\mathrm{T}_{max} \subset \mathrm{SO}(3)_{ir} \subset \mathrm{SO}(7)$-structure.

Construction of $\mathrm{SO}(3)_{ir}$-structures via twistor theory

The algebraic fact

$$\mathrm{T}_{max} = \mathrm{diag}(z, z^2, z^{-3}) = \mathrm{SO}(3)_{ir} \cap \mathrm{SU}(3) \subset \mathrm{G}_2$$

yields the following

Proposition: Let Y^6 be a 6-manifold such that its tangent bundle splits $TY^6 = (E \oplus E^2 \oplus E^{-3})_{\mathbb{R}}$, where E is a complex, 1-dimensional bundle. Then any S^1-bundle $M^7 \to Y^6$ admits a topological $\mathrm{T}_{max} \subset \mathrm{SO}(3)_{ir}$-structure.

Consider a compact spin 4-manifold X^4. Then the twistor space $Z^6 = P(S^-)$ is the projective spin bundle and there exists the tautological bundle $H \to Z^6$. The tangent bundle is given by

$$T(Z^6) = T^v \oplus T^h , \quad T^v = H^{-2} .$$

Ansatz: $T^h = H^{-1} \oplus H^3$.

Then any S^1-bundle over Z^6 admits a $\mathrm{T}_{max} \subset \mathrm{SO}(3)_{ir}$-structure.

Theorem: If $T^h = H^{-1} \oplus H^3$, then $9\,\sigma(X^4) = \ 10\,\chi(X^4)$.

Conversely, if $9\,\sigma(X^4) = \ 10\,\chi(X^4)$, then

- $c_1(T^h) = c_1(H^{-1} \oplus H^3)\,,\ c_2(T^h) = c_2(H^{-1} \oplus H^3)$.

Moreover, the following conditions are equivalent:

- T^h splits into $H^{-1} \oplus H^3$.
- T^h splits into the sum of two line bundles.

Remark: If $9\,\sigma(X^4) = 10\,\chi(X^4) = 0$, then Z^6 is parallelizable. Consequently, the interesting case is $9\,\sigma(X^4) = 10\,\chi(X^4) \neq 0$

Problem to handle: Suppose that $9\,\sigma(X^4) = 10\,\chi(X^4) \neq 0$. Then we have to decide whether or not the complex 2-dimensional bundle T^h over Z^6 splits. These bundles are basically given by the homotopy classes

$$[\mathbf{Z^6}\ ,\ \mathbf{P(H)^\infty}] = [\mathbf{Z^6}\ ,\ \mathbf{S^4}]$$

Theorem: (Steenrod classification theorem)
Let Z^6 be a compact 6-dimensional manifold and consider two $\mathrm{SU}(2)$-principal fiber bundles with the same Chern class, $c_2(P_1) = c_2(P_2) \in H^4(Z^6\,;\,\mathbb{Z})$. Then there exist a cohomology class

$$\delta(P_1, P_2) \in H^5(Z^6\,;\,\mathbb{Z}_2)/Sq^2\big(H^3(Z^6\,;\,\mathbb{Z}_2)\big)$$

such that $\delta(P_1, P_2) = 0$ if and only if P_1 and P_2 are isomorphic over $Z^6 - \{\mathrm{point}\}$. In this case, the last obstruction to $P_1 = P_2$ over Z^6 is in $H^6(Z^6\,;\,\pi_5(\mathrm{SU}(2)) = H^6(Z^6\,;\,\mathbb{Z}_2) = \mathbb{Z}_2$.

Corollary: Let X^4 be a compact, oriented 4-manifold such that

- X^4 is spin.
- $9\,\sigma(X^4) = 10\,\chi(X^4) \neq 0$.

Then the twistor space Z^6 of X^4 is not parallelizable. The tangent bundle splits into

$$T(Z^6) = H^{-1} \oplus H^{-2} \oplus H^3 \ .$$

over the 4-skeleton of Z^6

Examples:

The spaces

$$
\begin{aligned}
X^4 &= 20 \cdot (S^1 \times S^3) \# 5 \cdot (K3) \quad \text{and} \\
X^4 &= (1 + A) \cdot (S^1 \times S^3) \# A \cdot (S^2 \times S^2)
\end{aligned}
$$

satisfy the condition $9\,\sigma(X^4) = 10\,\chi(X^4)$.