Contact 5-manifolds with $SU(2)$-structure and its generalization in higher dimensions

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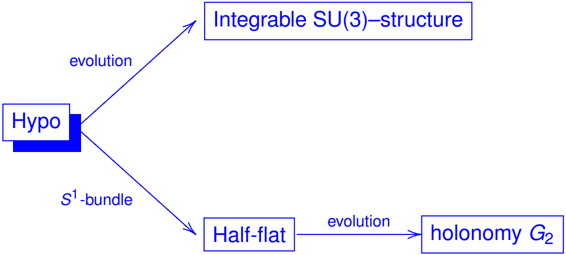

Hypo $\longrightarrow$ $SU(n)$-structures in $(2n+1)$-dimensions

$SU(2)$-structures in 5-dimensions

Definition

An $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 is given by a 1-form η and by three 2-forms ω_i such that

$$\omega_i \wedge \omega_j = \delta_{ij} v, \ v \wedge \eta \neq 0,$$
$$i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X, Y) \geq 0,$$

where i_X denotes the contraction by X.

Remark

The pair (η, ω_3) defines a $U(2)$-structure or an almost contact metric structure on N^5, i.e. (η, ξ, φ, g) such that

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{Id} + \xi \otimes \eta,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

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Sasaki-Einstein structures

Example

Sasaki-Einstein structure

$$d\eta = -2\omega_3,\ d\omega_1 = 3\eta \wedge \omega_2,\ d\omega_2 = -3\eta \wedge \omega_1.$$

On $S^2 \times S^3$ there exist an infinite family of explicit Sasaki-Einstein metrics [Gauntlett, Martelli, Sparks, Waldram].

Definition (Boyer, Galicki)

(N^{2n+1}, η, g) is Sasaki-Einstein if the conic metric $\tilde{g} = dr^2 + r^2 g$ on the symplectic cone $N^{2n+1} \times \mathbb{R}^+$ is Kähler and Ricci- flat (CY).

- $N^{2n+1} \times \mathbb{R}^+$ has an integrable $SU(n+1)$-structure, i.e. an Hermitian structure $(J, \tilde{g})$, with $F = d(r^2\eta)$, and a $(n+1, 0)$-form $\Psi = \Psi_+ + i\Psi_-$ of lenght 1 such that $dF = d\Psi = 0 \Rightarrow \tilde{g}$ has holonomy in $SU(n+1)$.
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Remark

An $SU(2)$-structure P on N^5 induces a spin structure on N^5 and P extends to $P \times_{SU(2)} Spin(5)$.

The spinor bundle is $P \times_{SU(2)} \Sigma$, where $\Sigma \cong \mathbb{C}^4$ and $Spin(5)$ acts transitively on the sphere in Σ with stabilizer $SU(2)$ in a fixed unit spinor $u_0 \in \Sigma$.

Then the $SU(2)$-structures are in one-to-one correspondence with the pairs $(P_{Spin(5)}, \psi)$, with ψ a unit spinor such that $\psi = [u, u_0]$ for any local section u of P, i.e. $\psi \in P \times_{Spin(5)} (Spin(5)u_0)$.

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An $SU(2)$-structure on N^5 is hypo if

$$d\omega_3 = 0,\ d(\eta \wedge \omega_1) = 0,\ d(\eta \wedge \omega_2) = 0.$$

Proposition (Conti, Salamon)

An $SU(2)$-structure P on N^5 is hypo if and only if the spinor ψ (defined by P_{SU}) is generalized Killing, i.e.

$$\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi,$$

where O is a section of $\mathrm{Sym}(TN^5)$ and $\cdot$ is the Clifford multiplication.

If N^5 is simply connected and Sasaki-Einstein, then $O = \pm \mathrm{Id}$ [Friedrich, Kath].

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• Any oriented hypersurface N^5 of (M^6, F, Ψ) with an integrable $SU(3)$-structure (F, Ψ) has in a natural way a hypo structure. The generalized Killing spinor on N^5 is the restriction of the parallel spinor on M^6 and O is just given by the Weingarten operator. If it is the restriction of a parallel spinor over the Riemannian cone then O is a constant multiple of the identity.

• Nilmanifolds cannot admit Sasaki-Einstein structures but they can admit hypo structures.

Theorem (Conti, Salamon)

The nilpotent Lie algebras admitting a hypo structure are

$$\begin{array}{ll} (0,0,12,13,14), & (0,0,0,12,13+24), \\ (0,0,0,12,13), & (0,0,0,0,12+34), \\ (0,0,0,0,12), & (0,0,0,0,0). \end{array}$$

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Hypo evolution equations

Theorem (Conti, Salamon)

A real analytic hypo structure (η, ω_i) on N^5 determines an integrable $SU(3)$-structure on $N^5 \times I$, with I some open interval, if (η, ω_i) belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t))$ which satisfy the evolution equations

$$\begin{cases} \partial_t \omega_3(t) = -\hat{d}\eta(t), \\ \partial_t(\omega_2(t) \wedge \eta(t)) = \hat{d}\omega_1(t), \\ \partial_t(\omega_1(t) \wedge \eta(t)) = -\hat{d}\omega_2(t). \end{cases}$$

The $SU(3)$-structure on $N^5 \times I$ is given by

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η-Einstein structures

Definition

An almost contact metric manifold $(N^{2n+1}, \eta, \xi, \varphi, g)$ is η-Einstein if there exist $a, b \in \mathcal{C}^\infty(N^{2n+1})$ such that

$$\mathrm{Ric}_g(X, Y) = a\, g(X, Y) + b\, \eta(X)\eta(Y),$$

where $\mathrm{scal}_g = a(2n+1) + b$ and $\mathrm{Ric}_g(\xi, \xi) = a + b$.

If $b = 0$, a Sasaki η-Einstein is Sasaki- Einstein.

Theorem (Conti, Salamon)

A hypo structure on N^5 is η-Einstein $\Leftrightarrow$ it is Sasakian.

For a Sasaki η-Einstein structure on N^5 we have

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and for the associated generalized Killing spinor

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with a and b constants [Friedrich, Kim].

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In general, for a hypo structure the 1-form η is not a contact form.

A hypo structure is contact if and only if $d\eta = -2\omega_3$.

Problem

Find examples of manifolds N^5 with a hypo-contact structure.

Examples

- Sasaki η-Einstein structures.
- Contact Calabi-Yau structures, defined by the equations $d\eta = -2\omega_3$, $d\omega_1 = d\omega_2 = 0$ [Tomassini, Vezzoni].
An example is given by the nilmanifold associated to

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Classification

Theorem (De Andres, Fernandez, –, Ugarte)

A 5-dimensional solvable Lie algebra $\mathfrak{g}$ has a hypo-contact structure $\Leftrightarrow$ $\mathfrak{g}$ is isomorphic to one of the following:

$$\mathfrak{h}_1 : [e_1, e_4] = [e_2, e_3] = e_5 \text{ (nilpotent and } \eta\text{-Einstein)};$$

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$\Rightarrow$ Description of the 5-dimensional solvable Lie algebras which admit a hypo-contact structure.

Consequences

- All the 5-dimensional solvable Lie algebras with a hypo-contact structure are irreducible.
- $\mathfrak{h}_1$ is the unique nilpotent Lie algebra with a hypo-contact structure.
- The Lie algebras of the classification cannot be Einstein since they are solvable and contact [Diatta].
- The unique 5-dimensional solvable Lie algebras with a η-Einstein hypo-contact structure are $\mathfrak{h}_1$ and $\mathfrak{h}_3$.
- If $\mathfrak{g}$ is such that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and admits a contact Calabi-Yau structure then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1$.

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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups H_i $(1 \leq i \leq 5)$ with Lie algebra $\mathfrak{h}_i$:

Theorem (De Andres, Fernandez, –, Ugarte)

Any left-invariant hypo-contact structure on any H_i $(1 \leq i \leq 5)$ determines a Riemannian metric with holonomy $SU(3)$ on $H_i \times I$, for some open interval I.

For the nilpotent Lie group H_1 we get the metric found by Gibbons, Lu, Pope and Stelle.

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Definition (Chiossi, Salamon)

An $SU(3)$-structure $(F, \Psi = \Psi_+ + i\Psi_-)$ on M^6 is half-flat if

$$d(F \wedge F) = 0, \; d(\Psi_+) = 0.$$

Theorem (Hitchin)

If the half-flat structure (F, Ψ) belongs to a one-parameter family $(F(t), \Psi(t)$ of half-flat structures, with t in a open interval I, which satisfy the evolution equations

$$\begin{cases} \partial_t \Psi_+(t) = \hat{d} F(t), \\ F(t) \wedge \partial_t(F(t)) = -\hat{d}\Psi_-(t), \end{cases}$$

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From hypo to half-flat

Proposition (De Andres, Fernandez, –, Ugarte)

$(N^5, \eta, \omega_1, \omega_2, \omega_3)$ hypo

Ω: integer closed 2-form which annihilates both ω_3 and $\cos\theta\,\omega_1 + \sin\theta\,\omega_2$, for some θ

$\Rightarrow \exists$ a principal S^1-bundle $\pi : M^6 \longrightarrow N^5$ with connection form ρ such that Ω is the curvature of ρ and the $SU(3)$-structure

$$F^\theta = \pi^*(\cos\theta\,\omega_1 + \sin\theta\,\omega_2) + \pi^*(\eta) \wedge \rho,$$

$$\Psi_+^\theta = \pi^*((-\sin\theta\,\omega_1 + \cos\theta\,\omega_2) \wedge \eta) - \pi^*(\omega_3) \wedge \rho,$$

$$\Psi_-^\theta = \pi^*(-\sin\theta\,\omega_1 + \cos\theta\,\omega_2) \wedge \rho + \pi^*(\omega_3) \wedge \pi^*(\eta),$$

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$\Rightarrow$ We distinguish the S^1-bundles with a half-flat structure induced by a hypo structure.

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$(M^6, F, \Psi = \Psi_+ + i\Psi_-)$ *half-flat*

$\iota : N^5 \hookrightarrow M^6$ *oriented hypersurface with unit normal vector field* $\mathbb{U}$.

- *If* $g(\nabla_{\mathbb{U}}\mathbb{U}, X) = 0$ *and* $\mathcal{L}_{\mathbb{U}}\Psi_+ = 0$, *for any* X *on* N^5, *then the forms*

$$\eta = -i_{\mathbb{U}}F, \quad \omega_1 = -i_{\mathbb{U}}\Psi_-, \quad \omega_2 = \iota^* F, \quad \omega_3 = -i_{\mathbb{U}}\Psi_+,$$

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More in general

Theorem (De Andres, Fernandez, –, Ugarte)

M^6 with a $SU(3)$-structure $(F, \Psi = \Psi_+ + i\Psi_-)$ and a Killing vector field X preserving the $SU(3)$-structure, i.e.

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define an $SU(2)$-structure on N^5 (formed by the orbits of X), where t is the norm of X and the 1-form α is given by $\alpha(Z) = t^{-2} g(Z, X)$.

If X has constant lenght and $d\alpha \wedge i_X \Psi_+ = 0$, then the induced $SU(2)$-structure on N^5 is hypo-contact.

If in addition $i_X(dF - 2\Psi_+) = 0$, the $SU(2)$-structure is hypo-contact.

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New metrics with holonomy G_2

We consider S^1-bundles $K_i \to H_i$ with a half-flat structure induced by a hypo-contact structure on H_i.

Solving the Hitchin evolution equations

Theorem (De Andres, Fernandez, –, Ugarte)

The half-flat structure on K_i $(i = 1, 4, 5)$ gives rise to a Riemanian metric with holonomy G_2 on $K_i \times I$, for some open interval I.

- K_1 is nilpotent and the metric with holonomy G_2 on $K_1 \times I$ is known [Chiossi, –].
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$SU(n)$-structures in $(2n+1)$-dimensions

Definition

An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is determined by the forms

$$\eta = e^{2n+1}, \quad \phi = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^{2n},$$
$$\Omega = (e^1 + ie^2) \wedge \ldots \wedge (e^{2n-1} + ie^{2n}).$$

As for the case of $SU(2)$-structures in dimensions 5 we have that an $SU(n)$-structure P_{SU} on N^{2n+1} induces a spin structure P_{Spin} and if we fix a unit element $u_0 \in \Sigma = (\mathbb{C}^2)^{\otimes 2n}$ we have that

$$P_{SU} = \{u \in P_{Spin} \,|\, [u, u_0] = \psi\}.$$

The pair (η, ϕ) defines a $U(n)$-structure or an almost metric contact structure on N^{2n+1}.

The $U(n)$-structure is a contact metric structure if $d\eta = -2\phi$.

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Generalized Killing spinors

Example

$N^{2n+1} \hookrightarrow M^{2n+2}$ (with holonomy $SU(n+1)$).
Then the restriction of the parallel spinor defines an $SU(n)$-structure (η, ϕ, Ω) where the forms ϕ and $\Omega \wedge \eta$ are the pull-back of the Kähler form and the complex volume form on the CY manifold M^{2n+2}.

Proposition (Conti, –)

Let N^{2n+1} be a real analytic manifold with a real analytc $SU(n)$-structure P_{SU} defined by (η, ϕ, Ω). The following are equivalent:

1. The spinor ψ associated to P_{SU} is a generalized Killing spinor, i.e. $\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi$.
2. $d\phi = 0$ and $d(\eta \wedge \Omega) = 0$.
3. A neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$ has a CY structure which restricts to P_{SU}.

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The assumption of real analycity is certainly necessary to prove that (1) or (2) implies (3), but the fact that (1) implies (2) does not require this hypothesis.

(2) $\Rightarrow$ (3) can be described in terms of evolution equations in the sense of Hitchin. Indeed, suppose that there is a family $(\eta(t), \phi(t), \Omega(t))$ of $SU(n)$-structures on N^{2n+1}, with t in some interval I, then the forms

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define a CY structure on $N^{2n+1} \times (a, b)$ if and only if (2) holds for $t = 0$ and the evolution equations

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An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is contact if $d\eta = -2\phi$.

In this case N^{2n+1} is contact metric with contact form η and we may consider the symplectic cone over (N^{2n+1}, η) as the symplectic manifold $(N^{2n+1} \times \mathbb{R}^+, -\frac{1}{2}d(r^2\alpha))$.

If N^{2n+1} is Sasaki-Einstein, we know that the symplectic cone is CY with the cone metric $r^2 g + dr^2$ and the Kähler form equal to the conical symplectic form.

Problem

If one thinks the form ϕ as the pullback to $N^{2n+1} \cong N^{2n+1} \times \{1\}$ of the conical symplectic form, which types of contact $SU(n)$-structures give rise to a CY symplectic cone but not necessarily with respect to the cone metric?

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Contact $SU(n)$-structures

Definition

An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is contact if $d\eta = -2\phi$.

In this case N^{2n+1} is contact metric with contact form η and we may consider the symplectic cone over (N^{2n+1}, η) as the symplectic manifold $(N^{2n+1} \times \mathbb{R}^+, -\frac{1}{2} d(r^2\alpha))$.

If N^{2n+1} is Sasaki-Einstein, we know that the symplectic cone is CY with the cone metric $r^2 g + dr^2$ and the Kähler form equal to the conical symplectic form.

Problem

If one thinks the form ϕ as the pullback to $N^{2n+1} \cong N^{2n+1} \times \{1\}$ of the conical symplectic form, which types of contact $SU(n)$-structures give rise to a CY symplectic cone but not necessarily with respect to the cone metric?

The answer is given by the following

Proposition (Conti, –)

Let N^{2n+1} be a real analytic manifold with a real analytc contact $SU(n)$-structure P_{SU} defined by (η, ϕ, Ω). The following are equivalent:

1. The spinor ψ associated to P_{SU} is generalized Killing, i.e. $\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi$.
2. $d\eta = -2\phi$ and $\eta \wedge d\Omega = 0$.
3. A neighbourhood of $M \times \{1\}$ in the symplectic cone $M \times \mathbb{R}^+$ has a CY metric which restricts to P_{SU}.

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Examples

• 5-dimensional hypo-contact solvable Lie groups [De Andres, Fernandez, –, Ugarte].

• The $(2n+1)$-dimensional real Heisenberg Lie group

$$de^i = 0, \quad i = 1, \ldots, 2n,$$
$$de^{2n+1} = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^{2n}.$$

In this case the contact $SU(n)$-structure is contact Calabi-Yau [Tomassini, Vezzoni].

• A two-parameter family of examples in the sphere bundle in $T\mathbb{CP}^2$ [Conti].

• A 7-dimensional compact example, quotient of the Lie group $SU(2) \ltimes \mathbb{R}^4$, which has a weakly integrable generalized G_2-structure [–, Tomassini].

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Link with generalized G_2-structures

If N^7 has a contact $SU(3)$-structure $(\eta, \phi, \Omega = \Omega_+ + i\Omega_-)$, then the differential form of mixed degree and even type

$$\rho = \phi + \Omega_+ \wedge \eta - \frac{1}{6}\phi^3 \wedge \eta$$

defines a generalized G_2-structure in the sense of Witt. Its companion is given by $\hat{\rho} = \eta - \Omega_- - \frac{1}{2}\phi^2 \wedge \eta$.

Given a closed 3-form H on N^7, the generalized G_2-structure is weakly integrable with respect to H if and only if

$$d_H\rho = d\rho + H \wedge \rho = \lambda\hat{\rho},$$

for a non-zero constant λ.

The previous equation is equivalent to

$$d\eta = \lambda\phi, \quad d\Omega_- = (H - \lambda\Omega_+) \wedge \eta, \quad H \wedge \Omega_- = -\frac{1}{3}\lambda\phi^3.$$

which imply that the associated spinor is generalized Killing.

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Proposition (Conti, –)

H: compact Lie group
$\rho : H \to V$ a representation of V.
Thene $H \ltimes_\rho V$ has a left-invariant contact structure if and only if $H \ltimes_\rho V$ is either $SU(2) \ltimes \mathbb{R}^4$ or $U(1) \ltimes \mathbb{C}$.

Then, if H is compact, the example $SU(2) \ltimes \mathbb{R}^4$ is unique in dimensions > 3.

If H is solvable we have

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There exists $H \ltimes \mathbb{R}^4$ admitting a contact $SU(3)$-structure whose associated spinor is generalized Killing if and only if the Lie algebra of H is isomorphic to one of the following

$$(0,0,0), \quad (0, \pm 13, 12),$$
$$(0,12,13), \quad (0,0,13).$$

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Contact reduction

Let N be a $(2n+1)$-dimensional manifold endowed with a contact metric structure (η, ϕ, g) and a spin structure compatible with the metric g and the orientation.

We say that a spinor ψ on N is compatible if

$$\eta \cdot \psi = i^{2n+1}\psi, \quad \phi \cdot \psi = -ni\psi.$$

Suppose that S^1 acts on N preserving both metric and contact form, so that the fundamental vector field X satisfies

$$\mathcal{L}_X \eta = 0 = \mathcal{L}_X \phi.$$

and denote by t its norm.

The moment map is given by $\mu = \eta(X)$.

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Assume that 0 is a regular value of μ and consider the hypersurface $\iota : \mu^{-1}(0) \to N$.

Then the contact reduction is given by $N//S^1 = \mu^{-1}(0)/S^1$ [Geiges].

• The contact $U(n)$-structure on N induces a contact $U(n-1)$-structure on $N//S^1$.

Let ν be the unit normal vector field, dual to the 1-form $i_{t^{-1}X}\phi$.

• The choice of an invariant compatible spinor ψ on N determines a spinor

$$\psi^\pi = \iota^*\psi + i\nu \cdot \iota^*\psi.$$

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Theorem (Conti, –)

N with a contact $U(n)$-structure (g, η, ϕ) and a compatible generalized Killing spinor ψ.
Suppose that S^1 acts on N preserving both structure and spinor and acts freely on $\mu^{-1}(0)$ with 0 regular value. Then the induced spinor ψ^π on $N//S^1$ is generalized Killing if and only if at each point of $\mu^{-1}(0)$ we have

$$dt \in span < i_X F, \eta >,$$

where X is the fundamental vector field associated to the S^1-action, and t is the norm of X.

Example

If we apply the previous theorem to $SU(2) \ltimes \mathbb{R}^4$ we get a new hypo-contact structure on $S^2 \times \mathbb{T}^3$.

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