Harmonicity of sections of sphere bundles

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$$(M, \langle \cdot, \cdot \rangle_M), \qquad (N, \langle \cdot, \cdot \rangle_N)$$

$$f : M \to N$$

M compact and oriented

$$\mathcal{E}(f) = \frac{1}{2} \int_M \|f_*\|^2 dv$$

Tension field

$$\tau(f) = \widetilde{\nabla}_{e_i}(f_* e_i) - f_* \nabla_{e_i} e_i,$$

$\widetilde{\nabla}$ is the induced connection by ∇^N on the pullback bundle $f^* TN$

$$(m, \widetilde{X}), \quad m \in M \text{ and } \widetilde{X} \in \mathrm{T}_{f(m)} N$$

f harmonic map if and only if $\tau(f) = 0$

J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.

$$(M, \langle\cdot,\cdot\rangle), \qquad (T_1M, \langle\cdot,\cdot\rangle^S)$$

Sasaki metric

$$\langle A, B\rangle^S = \langle \pi_* A, \pi_* B\rangle + \langle K(A), K(B)\rangle,$$

K is the connection map determined by ∇

$X \in \mathfrak{X}_1(M)$

$$X : M \to T_1M$$

M compact and oriented

harmonic unit vector field

$$\nabla^*\nabla X = \|\nabla X\|^2 X$$

G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. 303 (1995), 325-344.

$$\nabla^*\nabla X = -(\nabla^2)_{e_i,e_i}X = \nabla_{e_i}(\nabla_{e_i}X) - \nabla_{\nabla_{e_i}e_i}X.$$

Vector bundle $\pi : \mathbb{E} \to M$, $\langle \cdot, \cdot \rangle$ fibre metric on $\mathbb{E}$,

Γ a metric connection on $L(\mathbb{E})$,

$$X\langle \sigma_1, \sigma_2 \rangle = \langle \nabla_X \sigma_1, \sigma_2 \rangle + \langle \sigma_1, \nabla_X \sigma_2 \rangle,$$

For $\pi(u) = m$ and $u = \xi^\alpha e_\alpha(x_i)_m \in \mathbb{E}$,

$$\eta = X^j \frac{\partial}{\partial x_j}_{|u} + \eta^\alpha \frac{\partial}{\partial \xi^\alpha}_{|u} \in T_u\mathbb{E}$$

$$\iota : T\mathbb{E} \to \mathbb{E}, \qquad \iota(\eta) = \eta^\alpha e_\alpha(x_i)_m$$

$$\begin{aligned} \eta^{vert} &= \left(\eta^\alpha + \xi^\beta \left(\nabla_{\pi_{*u}\eta} e_\beta(x_i)\right)^\alpha\right) \frac{\partial}{\partial \xi^\alpha}_{|u}, \\ \eta^{hor} &= X^j \frac{\partial}{\partial x_j}_{|u} - \xi^\beta \left(\nabla_{\pi_{*u}\eta} e_\beta(x_i)\right)^\alpha \frac{\partial}{\partial \xi^\alpha}_{|u} = (\pi_{*u}\eta)^{\mathcal{H}}. \end{aligned}$$

The connection map $K : T\mathbb{E} \to \mathbb{E}$ is defined by $K(\eta) := \iota\, \eta^{vert}$

$$\mathcal{V} = \ker \pi_* \qquad \mathcal{H} = \ker K$$

$$\langle \eta_1, \eta_2 \rangle^S = \langle \pi_*(\eta_1), \pi_*(\eta_2) \rangle + \langle K(\eta_1), K(\eta_2) \rangle$$

M compact and oriented, $\sigma : M \to \mathbb{E}$, $K(\sigma_* X) = \nabla_X \sigma$

$$\mathcal{E}(\sigma) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\nabla\sigma\|^2 dv.$$

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \langle R_{X,\, e_i}\sigma, \nabla_{e_i}\sigma \rangle$$

$$\tau(\sigma) = \left((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp \right)^{hor} \circ \sigma - (\nabla^*\nabla\sigma)^{vert} \circ \sigma$$

O. Gil-Medrano, J. C. González-Dávila and L. Vanhecke, Harmonicity and minimality of oriented distributions, *Israel Journal of Math.* 143 (2004), 253-279.

$\mathcal{E} : \Gamma^\infty(\mathbb{E}) \to I\!R$

$\sigma_t \in \Gamma^\infty(\mathbb{E})$ a smooth variation of $\sigma_0 = \sigma$

variation vector field $x \in M \mapsto V(x) = \frac{d}{dt}_{|t=0}\sigma_t(x)$ is a section of the pullback bundle $\sigma^*\mathcal{V} \subset T\mathbb{E}$.

First variation formula

$$\frac{d}{dt}_{|t=0}\mathcal{E}(\sigma_t) = -\int_M \langle V, \tau(\sigma)\rangle^S dv,$$

σ is a harmonic section if and only if the vertical part of $\tau(\sigma)$ is zero if and only if σ is ∇-parallel if and only if $\sigma : M \to \mathbb{E}$ is a harmonic map

$$\tau(\sigma) = \left((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp\right)^{hor} \circ \sigma - (\nabla^*\nabla\sigma)^{vert} \circ \sigma$$

$S_{\mathbb{E}}(r)$

$\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$, $\quad \frac{1}{r}\sigma^{vert} \circ \sigma$ is a unit normal vector field to $S_{\mathbb{E}}(r)$

$$\Gamma^\infty(\mathbb{E}) = \mathcal{V}(\sigma)^\perp \oplus \mathcal{V}(\sigma)$$

Proposition

Let $\pi : \mathbb{E} \to M$ be a vector bundle with a metric connection over a closed and oriented Riemannian manifold and let $\mathcal{E} : \Gamma^\infty(S_{\mathbb{E}}(r)) \to \mathbb{R}$ be the energy functional on $\Gamma^\infty(S_{\mathbb{E}}(r))$. We have

(i) $d\mathcal{E}_\sigma(\varphi) = \int_M \langle \nabla^* \nabla \sigma, \varphi \rangle \, dv$, for each $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$ and $\varphi \in \mathcal{V}(\sigma)^\perp$.

(ii) If σ is a harmonic section of $S_{\mathbb{E}}(r)$, then the Hessian form $(\mathrm{Hess}\ \mathcal{E})_\sigma$ on $\mathcal{V}(\sigma)^\perp \cong T_\sigma \Gamma^\infty(S_{\mathbb{E}}(r))$ is given by

$$(\mathrm{Hess}\ \mathcal{E})_\sigma \varphi = \int_M (\|\nabla\varphi\|^2 - \|\varphi\|^2 \|\nabla\sigma\|^2) dv.$$

$$\tan \tau(\sigma) \quad = \quad \left((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp\right)^{hor} \circ \sigma + \left(\frac{1}{r^2}\langle \nabla^*\nabla\sigma, \sigma\rangle\sigma - \nabla^*\nabla\sigma\right)^{vert} \circ \sigma.$$

Proposition

Let $\pi : \mathbb{E} \rightarrow M$ be a vector bundle with a metric connection over a closed and oriented Riemannian manifold and $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$. Then, we have:

(i) the map $\sigma : (M, \langle\cdot,\cdot\rangle) \rightarrow (S_{\mathbb{E}}(r), \langle\cdot,\cdot\rangle^S)$ is harmonic if and only if $R_{(\sigma,\langle\cdot,\cdot\rangle)} = 0$ and $\nabla^*\nabla\sigma$ is collinear with σ.

(ii) σ is a critical point of $\mathcal{E}$ restricted to $\Gamma^\infty(S_{\mathbb{E}}(r))$ if and only if $\nabla^*\nabla\sigma$ is collinear with σ.

$\nabla^*\nabla\sigma = \frac{1}{r^2}\|\nabla\sigma\|^2\sigma,$ *σ harmonic section into a sphere bundle*

$$(\nabla\sigma)^{\mathsf{t}} : \Gamma^\infty(\mathbb{E}) \to \mathfrak{X}(M), \qquad \langle (\nabla\sigma)^{\mathsf{t}}\varphi, X\rangle = \langle \varphi, \nabla_X\sigma\rangle,$$

Lemma

Given a harmonic section σ of the sphere bundle $S_{\mathbb{E}}(r)$,

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \mathrm{div}\left((\nabla\sigma)^{\mathrm{t}}\nabla_X\sigma\right) + \langle \nabla_{[X,e_i]}\sigma, \nabla_{e_i}\sigma\rangle - \tfrac{1}{2}X\left(\|\nabla\sigma\|^2\right).$$

Moreover, if $\langle \nabla_X \sigma, \nabla_Y \sigma \rangle$ is locally expressed by

$$\langle \nabla_X \sigma, \nabla_Y \sigma \rangle = \sum_{i=1}^{n} k_i e_i^\flat \otimes e_i^\flat (X, Y),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame field and $k_1, \dots, k_n$ are smooth functions, then

$$R_{(\sigma, \langle \cdot, \cdot \rangle)} = \sum_{i=1}^{n} \{ e_i(k_i) + \sum_{j=1}^{n} (k_i - k_j) \langle \nabla_{e_j} e_i, e_j \rangle \} e_i^\flat - \tfrac{1}{2} d(\sum_{j=1}^{n} k_j).$$

In particular, if $k_1 = \cdots = k_n = \lambda$, where λ is a (non-negative) constant, then σ is a harmonic map into $(S_{\mathbb{E}}(r), \langle \cdot, \cdot \rangle^S)$.

$$\langle \Psi, \Phi \rangle = \Psi(e_{i_1}, \dots, e_{i_p}) \Phi(e_{i_1}, \dots, e_{i_p}),$$

Theorem

Let $(M, \langle \cdot, \cdot \rangle)$ be an n-dimensional Riemannian manifold and (Ψ, Φ) a pair of differential forms of constant length $\|\Psi\| = r_1$ and $\|\Phi\| = r_2$, $\Psi \in \Omega^p M$ and $\Phi \in \Omega^{p+1} M$. If $\nabla_X \Psi = \lambda X \lrcorner \Phi$ and $\nabla_X \Phi = \mu X^\flat \wedge \Psi$, where λ, μ are constants and $0 \leq p < n$, then Ψ and Φ are harmonic maps into the corresponding sphere bundles $S_{\Omega^p M}(r_1)$ and $S_{\Omega^{p+1} M}(r_2)$.

$$\nabla^* \nabla \Psi = -(n-p)\lambda\mu\Psi$$
$$\nabla^* \nabla \Phi = -(p+1)\lambda\mu\Phi$$

$$\nabla_X \phi = \tfrac{k}{4} X \lrcorner * \phi, \qquad \nabla_X * \phi = -\tfrac{k}{4} X^\flat \wedge \phi, \qquad \rho = \frac{k^2}{16}$$

$4\|\phi\|^2 = \| * \phi\|^2 = 7.4!$

$$\nabla^* \nabla \phi = \tfrac{k^2}{4} \phi = 4\rho\phi, \quad \nabla^* \nabla * \phi = \tfrac{k^2}{4} * \phi = 4\rho * \phi, \quad R_{(\phi, \langle \cdot, \cdot \rangle)} = R_{(*\phi, \langle \cdot, \cdot \rangle)} = 0.$$

Theorem

For a nearly parallel G_2-manifold, the differential forms ϕ and $*\phi$ are harmonic maps into their respective sphere bundles.

$3w_1^+\Psi_+ := d\omega$, where $5\left(w_1^+\right)^2 = \rho$, ($\rho$ is the Einstein constant)

$$\Psi_- := -\Psi_+(J\cdot,\cdot,\cdot)$$

$\Psi_+ + i\Psi_-$ complex volume form

$$T^*M \otimes \mathfrak{su}(3)^\perp = \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3^{SU(3)} + \mathcal{W}_4^{SU(3)} + \mathcal{W}_5^{SU(3)},$$

general nearly Kähler 6-manifolds: $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5^{SU(3)}$

Our case is: $\mathcal{W}_1^+$ (because of the fixed complex volume form $\Psi_+ + i\Psi_-$)

If the $SU(3)$-structure is of type $\mathcal{W}_1^+ + \mathcal{W}_5^{SU(3)}$, then it is of type $\mathcal{W}_1^+$ or of type $\mathcal{W}_5^{SU(3)}$.

$$\nabla_X \omega = w_1^+ X \lrcorner \Psi_+, \qquad \nabla_X \Psi_+ = -w_1^+ X^\flat \wedge \omega$$

$$\nabla^* \nabla \omega = 4 \left(w_1^+\right)^2 \omega, \quad \nabla^* \nabla \Psi_+ = 3 \left(w_1^+\right)^2 \Psi_+, \qquad R_{(\omega, \langle\cdot,\cdot\rangle)} = R_{(\Psi_+, \langle\cdot,\cdot\rangle)} = 0.$$

$$\nabla_X \Psi_- = \tfrac{1}{2} w_1^+ X \lrcorner (\omega \wedge \omega), \quad \nabla_X (\omega \wedge \omega) = -2 w_1^+ X^\flat \wedge \Psi_-$$

$$\nabla^* \nabla \Psi_- = 3 \left(w_1^+\right)^2 \Psi_-, \quad \nabla^* \nabla (\omega \wedge \omega) = 4 \left(w_1^+\right)^2 \omega \wedge \omega,$$
$$R_{(\Psi_-, \langle\cdot,\cdot\rangle)} = R_{(\omega\wedge\omega, \langle\cdot,\cdot\rangle)} = 0.$$

Theorem

For a nearly Kähler 6-manifold, the differential forms ω, Ψ_+, Ψ_- and $\omega \wedge \omega$ are harmonic maps into their respective sphere bundles.

An almost contact metric manifold
$(M^{2n+1}, \langle\cdot,\cdot\rangle, \varphi, \eta)$

$$\langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)$$
$$\varphi^2 = -I + \eta \otimes \zeta, \qquad \zeta^\flat = \eta$$

$$U(n) \times 1 \subseteq SO(2n+1), \qquad T^*_m M = \mathbb{R}\eta + \eta^\perp$$

$$\mathfrak{so}(2n+1) \cong \Lambda^2 T^* M \cong \Lambda^2 \eta^\perp + \eta^\perp \wedge \mathbb{R}\eta = \mathfrak{u}(n) + \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \wedge \mathbb{R}\eta$$

$$\mathfrak{u}(n)^\perp = \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \wedge \mathbb{R}\eta$$

$$T^*M \otimes \mathfrak{u}(n)^\perp = \eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \otimes \eta^\perp \wedge \eta + \eta \otimes \eta^\perp \wedge \eta$$

$$T^*M \otimes \mathfrak{u}(n)^\perp = \eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \otimes \eta^\perp \wedge \eta + \eta \otimes \eta^\perp \wedge \eta$$

$$\begin{aligned}
\eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \\
\eta^\perp \otimes \eta^\perp \wedge \eta &= \mathcal{C}_5 + \mathcal{C}_8 + \mathcal{C}_9 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_{10} \\
\eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_{11} \\
\eta \otimes \eta^\perp \wedge \eta &= \mathcal{C}_{12}
\end{aligned}$$

D. Chinea and J. C. González-Dávila, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.* (4) 156 (1990), 15–36.

$F = \langle \cdot, \varphi \cdot \rangle$ fundamental two-form

a-Sasakian manifolds ($\mathcal{C}_6$), $\nabla_X F = -aX^\flat \wedge \eta$

$\nabla_X \eta = aX \lrcorner F$

Theorem

For an a-Sasakian $(2n+1)$-manifold, the differential forms $\eta \wedge F^r$ and F^{r+1}, $0 \leq r \leq n$, are harmonic maps into their respective sphere bundles.

There are also examples on 3-a-Sasakian manifolds b-Kenmotsu manifolds ($\mathcal{C}_5$), $\nabla_X F = b\eta \wedge (X \lrcorner F)$

$\nabla_X \eta = -bX^\flat + b\eta(X)\eta, \qquad db = f\eta$

Proposition

For b-Kenmotsu manifolds, we have:

$$\begin{aligned}
\nabla^*\nabla(F^r) &= 2rb^2 F^r, \\
R_{(F^r,\langle\cdot,\cdot\rangle)} &= 2rb\|F^r\|^2(rb^2 - f)\eta, \\
\nabla^*\nabla(\eta \wedge F^r) &= 2(n-r)b^2\eta \wedge F^r, \\
R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)} &= 2(2r+1)(n-r)\|F^r\|^2 b(b^2 - f)\eta,
\end{aligned}$$

where $0 \leq r \leq n$.

$$\langle \nabla_X F^r, \nabla_Y F^r \rangle = \frac{r^2}{n} b^2 \|F^r\|^2 (\langle X, Y \rangle - \eta(X)\eta(Y))$$

$$\langle \nabla_X(\eta \wedge F^r), \nabla_Y(\eta \wedge F^r) \rangle = \frac{(2r+1)(n-r)}{n} b^2 \|F^r\|^2 (\langle X, Y \rangle - \eta(X)\eta(Y))$$

Moreover, we also have:

(a) $R_{(F^r,\langle\cdot,\cdot\rangle)} = 0$ if and only if one of the following equivalent conditions is satisfied:

(i) $db(\zeta) = rb^2$,
(ii) $2n\,\mathrm{grad}\,(\mathrm{div}\,(\zeta)) = -r\mathrm{div}^2\,(\zeta)\zeta$,
(iii) $2n\,\Delta\eta = r(d^*\eta)^2\eta$;

(b) $R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)} = 0$ if and only if one of the following equivalent conditions is satisfied:

(i) $db(\zeta) = b^2$,
(ii) $2n\,\mathrm{grad}\,(\mathrm{div}\,(\zeta)) = -\mathrm{div}^2\,(\zeta)\zeta$,
(iii) $2n\,\Delta\eta = (d^*\eta)^2\eta$;

where Δ denotes the Hodge Laplacian, $\Delta = dd^* + d^*d$.

Ψ is a p-form of constant length.

Definition

A p-form Ψ on a Riemannian n-manifold M is said to be *locally conformal parallel*, if there exists a closed one-form θ on M such that

$$\nabla_X \Psi = X^\flat \wedge (\theta^\sharp \lrcorner \Psi) - \theta \wedge (X \lrcorner \Psi),$$

for all $X \in \mathfrak{X}(M)$. We will refer to the one-form θ as the Lee form of Ψ.

Proposition

If Ψ is a locally conformal parallel p-form on a Riemannian n-manifold M with Lee form θ, then its coderivative $d^*\Psi$ and its rough Laplacian $\nabla^*\nabla\Psi$ are respectively given by

$$d^*\Psi = (p-n)\theta^\sharp \lrcorner \Psi,$$
$$\nabla^*\nabla\Psi = p\|\theta\|^2\Psi + (n-2p)\theta \wedge (\theta^\sharp \lrcorner \Psi).$$

In particular, if $2p = n$, then Ψ is a harmonic section of its corresponding sphere bundle.

Locally conformal Kähler $2n$-manifolds, $\nabla_X \omega = X^\flat \wedge (\theta^\sharp \lrcorner \omega) - \theta \wedge (X \lrcorner \omega)$

Proposition

If the Lee form θ of ω is not zero somewhere and $r < n$, then ω^r is a harmonic section of its corresponding sphere bundle if and only if $2r = n$.

$2r = n$

$$R_{(\omega^r,\langle\cdot,\cdot\rangle)} = \frac{1}{2}(n-1)(n!)^2\left(-(n-2)d(\|\theta\|^2) + d^*\theta\,\theta + d^*(J\theta)\,J\theta + \nabla_{J\theta^\sharp}J\theta\right).$$

Theorem

For a locally conformal Kähler $4n$-manifold, ω^n is a harmonic map into its sphere bundle if and only if

$$2(n-1)d(\|\theta\|^2) = (d^*\theta)\,\theta + d^*(J\theta)\,J\theta + \nabla_{J\theta^\sharp}J\theta,$$

where θ is the Lee form of ω. In particular, if θ is parallel, then ω^n is a harmonic map into its sphere bundle.

$S^{4n-1}\times S^1$

Locally conformal parallel $Spin(7)$-manifolds

$$4\nabla_X \Phi = X^\flat \wedge (\theta^\sharp \lrcorner \Phi) - \theta \wedge (X \lrcorner \Phi)$$

$$R_{(\Phi, \langle \cdot, \cdot \rangle)} = 12 \left((d^* \theta)\, \theta - 3d(\|\theta\|^2) \right)$$

Theorem

The fundamental form Φ of a locally conformal parallel $Spin(7)$-structure is a harmonic section of its corresponding sphere bundle. Furthermore, if θ denotes the Lee form of the $Spin(7)$-structure, then Φ is a harmonic map into its sphere bundle if and only if $(d^* \theta)\, \theta = 3d\left(\|\theta\|^2\right)$.

$S^7 \times S^1$

$$\Omega = \sum_{A=I,J,K} \omega_A \wedge \omega_A$$

Locally conformal quaternion Kähler manifolds,

$$\nabla_X \Omega = X^\flat \wedge (\theta^\sharp \lrcorner \Omega) - \theta \wedge (X \lrcorner \Omega)$$

Theorem

For a locally conformal quaternion-Kähler $8n$-manifold, Ω^n is a harmonic section of its corresponding sphere bundle.

H. Urakawa, *Calculus of variations and harmonic maps*, Transl. of Math. Monographs 132, Amer. Math. Soc., Providence, Rhode Island, 1993.

C. M. Wood, *Harmonic sections of homogeneous fibre bundles*, Differential Geom. Appl. 19 (2003), 193-210.

J. C. González Dávila, —-, *Harmonic G-structures*, `math.DG/0706.0116`

$(M, \langle\cdot,\cdot\rangle)$ compact and oriented, $G \subseteq SO(n)$, G closed and connected
The presence of a G-structure is equivalent to the presence of a section $\sigma : M \to \mathcal{SO}(M)/G$

$$\langle A, B\rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B\rangle + \langle \phi A, \phi B\rangle.$$

$\phi : T\mathcal{SO}(M)/G \to \pi^*\mathfrak{so}(M), \qquad \phi\,\sigma_* = -\xi^G$
The energy of a G-structure

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \|\sigma_*\|^2 dv = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\xi^G\|^2 dv$$

Harmonic G-structures

$$d^*\xi^G(X) = -(\nabla_{e_i}\xi^G)_{e_i}X$$

σ is a critical point for the energy functional on $\Gamma^\infty(\mathcal{SO}(M)/G)$ if and only if $d^*\xi^G = 0$

σ is a harmonic map if and only if $d^*\xi^G = 0$ and $\langle \xi^G_{e_i}, R(e_i, X)\rangle = 0$