Infinitesimal deformations of nearly parallel G_2-structures

Bogdan Alexandrov
(joint with U. Semmelmann)

1 The group G_2

G_2 is the group of automorphisms of the Cayley numbers $\mathbb{O}$. Equivalently, G_2 is the stabilizer in $GL(7,\mathbb{R})$ of the 3-form

$$\begin{aligned}\varphi = {} & e^1\wedge e^2\wedge e^3 + e^1\wedge e^4\wedge e^5 \\ & -e^1\wedge e^6\wedge e^7 + e^2\wedge e^4\wedge e^6 + e^2\wedge e^5\wedge e^7 \\ & +e^3\wedge e^4\wedge e^7 - e^3\wedge e^5\wedge e^6.\end{aligned}$$

It holds $G_2 \subset SO(7)$.

Rem. The form φ is stable, i.e., its $GL(7,\mathbb{R})$-orbit is open in $\Lambda^3\left(\mathbb{R}^7\right)^*$.

We shall need the following representations of G_2:

$$
\begin{aligned}
V_{0,0} &\cong \mathbb{R}, \\
V_{1,0} &\cong \mathbb{R}^7, \\
V_{0,1} &\cong \mathfrak{g}_2, \quad \dim V_{0,1} = 14, \\
V_{2,0} &\cong S_0^2\left(\mathbb{R}^7\right), \quad \dim V_{2,0} = 27.
\end{aligned}
$$

The G_2-decomposition of the exterior algebra of $\mathbb{R}^7$ is then the following:

$$
\begin{aligned}
\Lambda^1\left(\mathbb{R}^7\right)^* &= \Lambda^1_7 \cong V_{1,0}, \\
\Lambda^2\left(\mathbb{R}^7\right)^* &= \Lambda^2_{14} \oplus \Lambda^2_7 \cong V_{0,1} \oplus V_{1,0}, \\
\Lambda^3\left(\mathbb{R}^7\right)^* &= \Lambda^3_{27} \oplus \Lambda^3_7 \oplus \Lambda^3_1 \cong V_{2,0} \oplus V_{1,0} \oplus \mathbb{R}, \\
\Lambda^4\left(\mathbb{R}^7\right)^* &= \Lambda^4_{27} \oplus \Lambda^4_7 \oplus \Lambda^4_1 \cong V_{2,0} \oplus V_{1,0} \oplus \mathbb{R}, \\
\Lambda^5\left(\mathbb{R}^7\right)^* &= \Lambda^5_{14} \oplus \Lambda^5_7 \cong V_{0,1} \oplus V_{1,0}, \\
\Lambda^6\left(\mathbb{R}^7\right)^* &= \Lambda^6_7 \cong V_{1,0}.
\end{aligned}
$$

The spaces Λ^3_1 and Λ^4_1 are spanned by φ and $*\varphi$ respectively.

2 G_2-structures

A G_2-structure on a 7-dimensional manifold M is a reduction of the structure group to G_2. Equivalently, a G_2-structure is a global 3-form σ on M such that at each point it can be written in the same form as φ.

Gray: There exists a G_2-structure on M iff M is orientable and spin (i.e., the first two Stiefel-Whitney classes vanish).

3 Types of G_2-structures (Fernandez-Gray)

The intrinsic torsion of a G_2-structure takes values in

$$\begin{aligned}\mathbb{R}^7 \otimes \mathfrak{g}_2^{\perp} = V_{1,0} \otimes V_{1,0} &\cong \mathbb{R} \oplus V_{0,1} \oplus V_{2,0} \oplus V_{1,0} \\ &= W_1 \oplus W_2 \oplus W_3 \oplus W_4.\end{aligned}$$

We say, for example, that a G_2-structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ if the intrinsic torsion lies in $W_1 \oplus W_2 \oplus W_3$ at each point of M.

Rem. The classification of Fernandez-Gray was made in terms of $\nabla\sigma$. This tensor can be identified with the intrinsic torsion.

There are potentially 16 classes of G_2-structures. The most interesting is the class $\mathcal{O}$ which consists of the torsion-free G_2-structures, i.e., the Riemannian 7-dimensional manifolds with holonomy contained in G_2. The existence of metrics with holonomy G_2 has been proved by Bryant in the non-compact case and by Joyce in the compact case. In fact, there exist compact G_2-manifolds in each of the 16 classes with only one exception: $\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_1 \bigcup \mathcal{W}_2$.

In this talk we are interested in the class $\mathcal{W}_1$.

4 Nearly parallel G_2-structures

Def. A G_2-structure of type $\mathcal{W}_1$ is called a *nearly parallel* G_2*-structure*. It is called *proper*, if it is not induced by an Einstein-Sasakian structure.

Friedrich, Kath, Moroianu, Semmelmann: A G_2-structure is nearly parallel iff there exists a constant $\tau \neq 0$, so that $d\sigma = \tau * \sigma$.

Baum, Friedrich, Grunewald, Kath: The nearly parallel G_2-manifolds are exactly the 7-dimensional manifolds which admit a real Killing spinor. Every nearly parallel G_2-manifold is Einstein with scalar curvature $s = \frac{21}{8}\tau^2$.

Bär: The metric cone over a 7-dimensional manifold $M \neq S^7$ has holonomy contained in $Spin(7)$ iff M is a nearly parallel G_2-manifold.

Def. The *canonical connection* ∇^0 of a G_2-structure is the unique G_2-connection whose torsion is the intrinsic torsion of the G_2-structure.

Friedrich, Ivanov: The canonical connection of a nearly parallel G_2-structure is given by

$$\nabla^0 = \nabla + \frac{\tau}{12}\sigma, \quad \text{i.e.,}$$

$$g(\nabla^0_X Y, Z) = g(\nabla_X Y, Z) + \frac{\tau}{12}\sigma(X, Y, Z).$$

In particular, ∇^0 has parallel and totally skew-symmetric torsion.

Examples:

1. S^7 with its canonical metric.
2. Squashed 3-Sasakian metrics.
3. Allof-Wallach spaces. These are homogeneous spaces of the form $SU(3)/U(1)$.

Friedrich, Kath, Moroianu, Semmelmann: List of all homogeneous nearly parallel G_2-manifolds.

5 Infinitesimal deformations

Let σ_t be a smooth family of G_2-structures on M. Then $\dot{\sigma}_t \in \Omega^3_0 \oplus \Omega^3_7 \oplus \Omega^3_{27}$ and therefore there exist differential forms $f^0_t \in \Omega^0$, $f^0_t \in \Omega^1$, $f^3_t \in \Omega^3_{27}$ so that

$$\begin{aligned}
&\dot{\sigma} = 3f^0\sigma + *_\sigma\left(f^1 \wedge \sigma\right) + f^3,\\
&\dot{g} = 2f^0 g + \frac{1}{2}j\left(f^3\right),\\
&(*_\sigma\sigma)^\cdot = 4f^0 *_\sigma \sigma + f^1 \wedge \sigma - *_\sigma f^3,\\
&(*_\sigma 1)^\cdot = 7f^0 *_\sigma 1,
\end{aligned}$$

where $j : \Lambda^3_{27} \longrightarrow S^2_0$ is certain G_2-invariant map identifying $\Lambda^3_{27} \cong V_{2,0}$ and $S^2_0 \cong V_{2,0}$.

We are interested in the case when all G_2-structures are nearly parallel and M is compact.

Thm 1 (**Friedrich**) *On S^7 there exists only one (up to isometry) nearly parallel G_2-structure which induces the standard metric.*

So we can assume that the initial nearly parallel G_2-structure (M, σ_0) is different from the standard one on S^7. Furthermore, since every nearly parallel G_2-structure is Einstein, we have an Einstein deformation and according to the theorem of Berger-Ebin we can assume

$$\operatorname{tr} \dot{g} = 0, \quad \delta \dot{g} = 0.$$

As $\operatorname{tr} \dot{g} = 14 f^0$, it follows $f^0 = 0$ and the above equations simplify to

$$\begin{aligned}
&\dot{\sigma} = *_\sigma \left(f^1 \wedge \sigma \right) + f^3, \\
&\dot{g} = \frac{1}{2} j \left(f^3 \right), \\
&(*_\sigma \sigma)^{\cdot} = f^1 \wedge \sigma - *_\sigma f^3, \\
&(*_\sigma 1)^{\cdot} = 0.
\end{aligned}$$

From

$$\dot{s} = \Delta(\operatorname{tr} \dot{g}) + d^*(\delta \dot{g}) - g(Ric, \dot{g}) = -\frac{s}{7} \operatorname{tr} \dot{g} = 0$$

and $s = \frac{21}{8} \tau^2$ follows $\dot{\tau} = 0$. So we can assume that $\tau = const$ and by rescaling the 3-Form σ we can ensure that $\tau = 1$.

Now $d\sigma = *_\sigma \sigma$ implies $d\dot{\sigma} = (*_\sigma \sigma)^{\cdot}$ and therefore

(1) $d\left(*_\sigma\left(f^1 \wedge \sigma\right)\right) + df^3 = f^1 \wedge \sigma - *_\sigma f^3.$

Let us consider the condition $\delta \dot{g} = 0$, i.e., $\delta\left(j\left(f^3\right)\right) = 0$.

We have $\delta = -c \circ \nabla$, where c is a contraction. As $\nabla^0 - \nabla = \frac{1}{12}\sigma$ is G_2-invariant, it follows

$$(\nabla^0 - \nabla)\left(j\left(f^3\right)\right) \in V_{2,0} \cong S_0^2 \cong \Lambda_{27}^3.$$

On the other hand, the contraction

$$c : T^* \otimes S_0^2 \cong V_{1,0} \otimes V_{2,0} \longrightarrow V_{1,0} \cong T^*$$

is also G_2-invariant. Therefore

$$-c \circ (\nabla^0 - \nabla)\left(j\left(f^3\right)\right) = 0 \quad \text{and}$$

$$0 = \delta\left(j\left(f^3\right)\right) = -c \circ \nabla^0\left(j\left(f^3\right)\right)$$
$$= -c \circ (1 \otimes j)\left(\nabla^0 f^3\right)$$

(the last equality holds because j is also G_2-invariant and ∇^0 is a G_2-connection). As

$$\nabla^0 f^3 \in T^* \otimes \Lambda^3_{27} \cong V_{1,0} \otimes V_{2,0}$$
$$= V_{1,0} \oplus V_{2,0} \oplus V_{0,1} \oplus V_{1,1} \oplus V_{3,0}$$

and

$$c \circ (1 \otimes j) : V_{1,0} \otimes V_{2,0} \longrightarrow V_{1,0}$$

is not zero, it follows

$$\left(\nabla^0 f^3\right)_{V_{1,0}} = 0.$$

Now consider

$$df^3 \in \Lambda^4_{27} \oplus \Lambda^4_7 \oplus \Lambda^4_1 \cong V_{0,1} \oplus V_{1,0} \oplus \mathbb{R}.$$

We have $d = \varepsilon \circ \nabla$, where ε is the exterior multiplication. Let $\pi : \Lambda^4 \longrightarrow \Lambda^4_7 \cong V_{1,0}$ be the projection. Then

$$\pi \circ \varepsilon \circ (\nabla^0 - \nabla)\left(f^3\right) = 0$$

because $\pi \circ \varepsilon \circ (\nabla^0 - \nabla)$ is G_2-invariant. Therefore

$$\pi \circ df^3 = \pi \circ \varepsilon \left(\nabla^0 f^3\right) = \pi \circ \varepsilon \left(\left(\nabla^0 f^3\right)_{V_{1,0}}\right) = 0.$$

Hence

$$\left(df^3\right)_{\Lambda^4_7} = 0$$

and from (1) also

$$\left(d\left(*_\sigma\left(f^1 \wedge \sigma\right)\right)\right)_{\Lambda^4_7} = f^1 \wedge \sigma.$$

In a similar manner (and using also integration) we obtain

(2) $\nabla f^1 \in \Lambda^2_7 \subset \Lambda^2$ i.e., f^1 is a Killing form,

(3) $df^1 = \frac{1}{2}\iota_\xi \sigma, \quad d^* f^1 = 0,$

(4) $df^3 = - *_\sigma f^3, \quad d^* f^3 = 0$

(here $\xi = (f^1)^\sharp$ and ι is the interior multiplication).

Now we want to show that $f^1 = 0$.

Consider the cone (CM, h), where

$$CM = \mathbb{R}_+ \times M, \quad h = dt^2 + t^2 g.$$

If the G_2-structure on M is proper, then $Hol(CM) = Spin(7)$, where the $Spin(7)$-structure is defined by the 4-form

$$\omega = 4t^3 dt \wedge \sigma + t^4 *_\sigma \sigma.$$

Since $L_\xi g = 0$ easily implies $L_\xi h = 0$, ξ is a Killing vector field for h too. Hence

$$0 = L_\xi \omega = 4t^3 dt \wedge (L_\xi \sigma) + t^4 L_\xi(*_\sigma \sigma),$$

i.e., $L_\xi \sigma = 0$, $L_\xi(*_\sigma\sigma) = 0$. But

$$L_\xi \sigma = d(\iota_\xi \sigma) + \iota_\xi d\sigma = 2d^2 f^1 + \iota_\xi(*_\sigma \sigma) = \iota_\xi(*_\sigma\sigma)$$

and therefore $\xi = 0$, i.e., $f^1 = 0$. Thus we obtain

Thm 2 *An infinitesimal deformation of a proper nearly parallel G_2-structure is uniquely determined by a 3-form $f^3 \in \Omega^3_{27}$, satisfying*

$$df^3 = - *_\sigma f^3, \quad d^* f^3 = 0$$

and hence also $(dd^* + d^*d) f^3 = f^3$. (Compare with

$$d\sigma = *_\sigma \sigma, \quad d^*\sigma = 0, \quad (dd^* + d^*d)\sigma = \sigma).$$

Rem. It follows from (2) and (3) that f^1 defines an Einstein-Sasakian structure on M (if $f^1 \neq 0$) and this yields another proof of the Theorem.