On the Relation of the Bonan 4-form and Maximal Linear Subspaces of Strong Self-dual 2-forms

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Set-up and methods...

E: vector bundle over a manifold M,

∇: a connection on the vector bundle,

F: the curvature 2-form of the connection,

$\int_M \mathcal{L}(F) dvol$: Action integral involving F,

$\int_M P(p_i)$: Integrals of polynomials in Pontrjagin classes of E,

Aim: Find topological lower bounds for $\int_M \mathcal{L}(F) dvol$, expressed locally in terms of invariant polynomials of F.

Results: The curvatures that saturate the lower bounds have a property that we defined as "strong self-duality"

Methods: Matrix inequalities, Clifford algebra representations, computer algebra in lower dimensions..

Notation: We reserve Greek letters to forms and omit wedge product throughout the presentation.

- Definition of strong self-dual/anti self-dual 2 forms in even dimensions greater than four; Equivalence with previous definitions of self-duality [BDK, 1996, BDK 1997]

- The geometry of strong SD/ASD 2-forms; Maximal linear subspaces of strong SD 2-forms [BDK, 1997]

- Strong self-dual forms in 8-dimensions and the Bonan form [BDK,1997], [Ozdemir, B, 1999].

- Applications to actions realizing topological lower bounds; Linear self-duality in 8-dimensions, the CDNF equations; Nonlinear self-duality and the GKS solution. [Corrigan et.al, 1986, Acharya and O'Loughlin, 1997, BDK 1999, Grossman et.al, 1984]

Hodge duality of 2-forms in 4- dimensions...

- M: 4-dimensional manifold, e^i, $i = 1, \ldots, 4$: a local orthonormal basis for the cotangent bundle of M,

- ω: any 2-form $\omega = \sum_{i<j} \omega_{ij} e^i \wedge e^j$. Denote the corresponding skew-symmetric matrix by $\mu(\omega)$

$$\mu(\omega) = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix}$$

- Eigenvalues of $\mu(\omega)$: $\pm i\lambda_1$ and $\pm i\lambda_2$,

$$\lambda_1 \mp \lambda_2 = \sqrt{(\omega_{12} \mp \omega_{34})^2 + (\omega_{13} \pm \omega_{24})^2 + (\omega_{14} \mp \omega_{23})^2}$$

When ω is Hodge self-dual, or anti self-dual...

$$\omega_{12} \mp \omega_{34} = 0, \quad \omega_{13} \pm \omega_{24} = 0, \quad \omega_{14} \mp \omega_{23} = 0.$$

ω is self-dual/anti self-dual $\Leftrightarrow |\lambda_1| = |\lambda_2|$.

Two cases are distinguished by the sign of

$$\frac{1}{2} * \omega^2 = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}.$$

In higher dimensions..

Definition. Let ω be a real 2-form in $2n$ dimensions, and denote the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some local orthonormal basis by $\mu(\omega)$. Let $\pm i\lambda_1, ..., \pm i\lambda_n$ be the eigenvalues of $\mu(\omega)$. Then ω is said to be strong self-dual (respectively, strong anti-self-dual) if

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$$

and $*\omega^n \geq 0$ (respectively $*\omega^n \leq 0$).

Properties of self-dual/anti self-dual 2 forms in 4-dimensions..
Generalization to higher dimensions..

1. The eigenvalues of the corresponding skew-symmetric matrix are equal in absolute value (fact)

   The eigenvalues of the corresponding skew-symmetric matrix are equal in absolute value (definition),

2. ω is self-dual/anti self-dual

   In $4n$ dimensions, ω^n is self-dual,

3. $*\omega = k\omega$

   In $2n$ dimensions, $*\omega^{n-1} = k\omega$,

4. They form linear subspaces and they are eigenspaces of the Hodge duality map $T_1(\omega) = *(\omega) = k\omega$.

   The set of strong self-dual forms form a manifold and they contain linear spaces whose dimensions are given by the Radon-Hurwitz numbers; In 8-dimensions they are realized as eigenspaces of the $T_\phi(\omega) = *(\phi \wedge \omega) = k\omega$, where ϕ is the Bonan form.

Matrices whose eigenvalues are equal in absolute value...

Proposition. *Let ω be a 2-form in $2n$-dimensions. If the eigenvalues of the corresponding matrix are equal in absolute value, then*

$$(\omega^k, \omega^k) = \frac{n!k!}{n^k(n-k)!}(\omega, \omega)^k.$$

Lemma [Marcus, M., Minc, H. A Survey of Matrix Theory and Matrix Inequalities, Ally and Bacon, 1964.] *Let s_k be the k^{th} elementary symmetric function of the numbers $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, with $\alpha_i \in R$ and let the weighted elementary symmetric polynomials, q_k's, be defined by $\binom{n}{k} q_k = s_k$. Then*

$$q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \cdots \geq q_n^{1/n},$$

$$q_{r-1}q_{r+1} \leq q_r^2, \qquad 1 \leq r < n$$

and the equalities hold if and only if all α_i's are equal.

Apply to even dimensional skew-symmetric matrices and 2-forms...

- If A is real, skew-symmetric, $2n \times 2n$, then its eigenvalues are $\pm i\lambda_k$, $k = 1, 2, .., n$:

$$\begin{aligned} s_2 = &\quad \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2, \\ \cdots & \\ s_{2n} = &\quad \lambda_1^2 \lambda_2^2 \ldots \lambda_n^2 \ . \end{aligned}$$

- If A is $\mu(\omega)$, then

$$\begin{aligned} s_2 &= (\omega, \omega) = n q_1, \\ s_{2k} &= \tfrac{1}{(k!)^2}(\omega^k, \omega^k) = \tfrac{n!}{k!(n-k)!} q_k, \\ &\vdots \\ s_{2n} &= \tfrac{1}{(n!)^2}(\omega^n, \omega^n) = \tfrac{1}{(n!)^2} \mid *\omega^n \mid^2 = q_n. \end{aligned}$$

Using the definitions $s_{2k} = \binom{n}{k} q_k$, $(\omega, \omega) = n q_1$, $\quad (\omega^k, \omega^k) = \frac{n!k!}{(n-k)!} q_k$, and in the case of equality of the eigenvalues,

$$(\omega^k, \omega^k) = \frac{n!k!}{n^k (n-k)!} (\omega, \omega)^k.$$

How this definition relates to other notions of self-duality?

- Trautman [Trautman A., Solutions of the Maxwell and Yang-Mills equations associated with Hopf fiberings, Int. J. Theor. Phys., 1977, V.16, 561-565.]: ω is self-dual if $*\omega = k\omega^{n-1}$,

- Grossman, Kephart, Statshef [Grossman B., Kephart T.W. and Statsheff J.D., Solutions to Yang-Mills field equations in eight dimensions and the last Hopf map, Comm. Math. Phys., 1984, V. 96, 431-437.]: Use an action with $F^2 = *F^2$.

- Corrigan,Devchand, Fairly, Nuyts [Corrigan E., Devchand C., Fairlie D. and Nuyts J., First order equations for gauge fields in spaces of dimensions greater than four, Nucl. Phys. B., 1983, V. 214, 452-464.]: ω belongs to an eigenspace of a certain linear map.

Relation to other notions of self-duality...

Proposition 1. *Let ω be a 2-form in in $4n$-dimensions. Then ω is strong self-dual (anti self-dual) if and only if ω^n is self-dual (anti self-dual) in the Hodge sense, that is $*\omega^n = \pm\omega^n$.*

Proposition 2. *Let ω be a 2-form in $2n$ dimensions. Then*

$$\omega^{n-1} = \pm k * \omega$$

where k is a constant, if and only if ω is strong SD/ASD and $k = \frac{n!}{n^{n/2}}(\omega,\omega)^{\frac{n}{2}-1}$.

Geometric structure of strong SD/ASD 2-forms...

Let A_{2n} be the set of anti-symmetric matrices in $2n$ dimensions. Define

$$\mathcal{S}_{2n} = \{A \in A_{2n} \mid A^2 + \lambda^2 I = 0, \lambda \in R, \lambda \neq 0\}.$$

In 4-dimensions $\mathcal{S}_{2n}$ is the union of two linear spaces. In $2n$-dimensions,

Proposition. *$\mathcal{S}_{2n}$ is diffeomorphic to the homogeneous manifold $\left(O(2n) \times R^+\right)/U(n) \times \{1\}$, and dim$\mathcal{S}_{2n} = n^2 - n + 1$.*

Linear subspaces of $\mathcal{S}_{2n}$

Proposition. *The dimension of the maximal linear subspaces of $\mathcal{S}_{2n}$ is equal to the number of linearly independent vector fields on S^{2n-1} (the Radon-Hurwitz number).*

Eigenspaces of linear maps

Assume there is a globally defined 4-form ϕ on M. Define a linear map on 2-forms by

$$T_\phi(\omega) = *(*\phi \wedge \omega).$$

This map is self adjoint and the equations restricting ω to some eigenspace will be called linear self-duality equations.

Example: In 8-dimensions, let ϕ be the Bonan form, locally expressed as

$$\begin{aligned}\phi \;=\; & e_1e_2e_3e_4 + e_1e_2e_5e_6 + e_1e_2e_7e_8 + e_1e_3e_5e_7 - e_1e_3e_6e_8 - e_1e_4e_5e_8 - e_1e_4e_6e_7 \\ & -e_2e_3e_5e_8 - e_2e_3e_6e_7 - e_2e_4e_5e_7 + e_2e_4e_6e_8 + e_3e_4e_5e_6 + e_3e_4e_7e_8 + e_5e_6e_7e_8\end{aligned}$$

Define a map on 2-forms by

$$T\phi(\omega) = *(*\phi \wedge \omega).$$

T_ϕ is self-adjoint, its eigenvalues are 3 and -1. The corresponding eigenspaces are 7 and 21 dimensional. Defining equations were first given by Corrigan et.al [CDFN,1984]

Self-duality equations of Corrigan et.al

a) 21-dimensional subspace:

Proposition. *Let* ω *be a 2-form,* ω_{ij} *be its coordinate components. If* ω *belongs to the* -1 *eigenspace of the* T_ϕ *map, then the equations below are satisfied.*

$$
\begin{aligned}
\omega_{12} + \omega_{34} + \omega_{56} + \omega_{78} = 0,\\
\omega_{13} - \omega_{24} + \omega_{57} - \omega_{68} = 0,\\
\omega_{14} + \omega_{23} - \omega_{67} - \omega_{58} = 0,\\
\omega_{15} - \omega_{26} - \omega_{37} + \omega_{48} = 0,\\
\omega_{16} + \omega_{25} + \omega_{38} + \omega_{47} = 0,\\
\omega_{17} - \omega_{28} + \omega_{35} - \omega_{46} = 0,\\
\omega_{18} + \omega_{27} - \omega_{36} - \omega_{45} = 0.
\end{aligned}
$$

b) 7-dimensional subspace:

Proposition. *Let ω be a 2-form, ω_{ij} be its coordinate components. If ω belongs to the $+3$ eigenspace of the T_ϕ map, then the equations below are satisfied.*

$$
\begin{array}{lll}
\omega_{12}-\omega_{34}=0 & \omega_{12}-\omega_{56}=0 & \omega_{12}-\omega_{78}=0 \\
\omega_{13}+\omega_{24}=0 & \omega_{13}-\omega_{57}=0 & \omega_{13}+\omega_{68}=0 \\
\omega_{14}-\omega_{23}=0 & \omega_{14}+\omega_{67}=0 & \omega_{14}+\omega_{58}=0 \\
\omega_{15}+\omega_{26}=0 & \omega_{15}+\omega_{37}=0 & \omega_{15}-\omega_{48}=0 \\
\omega_{16}-\omega_{25}=0 & \omega_{16}-\omega_{38}=0 & \omega_{16}-\omega_{47}=0 \\
\omega_{17}+\omega_{28}=0 & \omega_{17}-\omega_{35}=0 & \omega_{17}+\omega_{46}=0 \\
\omega_{18}-\omega_{27}=0 & \omega_{18}+\omega_{36}=0 & \omega_{18}+\omega_{45}=0.
\end{array}
$$

Remark:

- The first set of 7 equations form an elliptic system for connection (under the Coulomb gauge condition) hence they lead to solutions that depend on arbitrary functions, on generic manifolds: Linear self-duality

- The second set of 21 equations lead to an overdetermined system for the connection. If there is a solution, (under the Coulomb gauge) it can depend on arbitrary constants only. The existence of solutions seems to determine the base manifold (work in progress): Non-linear self-duality

Strong Self-duality in Eight-Dimensions

Dealing with a single SD/ASD 2-form ω...

In eight dimensions these basic inequalities reduce to

$$(\omega,\omega)^2 \geq \frac{2}{3}(\omega^2,\omega^2) \geq \frac{2}{3}|*\omega^4|.$$

When ω is strong self dual we have the equalities

$$(\omega,\omega)^2 = \frac{2}{3}(\omega^2,\omega^2) = \frac{2}{3}*\omega^4, \quad \omega^3 = \frac{3}{2}(\omega,\omega)*\omega,$$

while when ω is strong anti-self-dual

$$(\omega,\omega)^2 = \frac{2}{3}(\omega^2,\omega^2) = -\frac{2}{3}*\omega^4, \quad \omega^3 = -\frac{3}{2}(\omega,\omega)*\omega.$$

Dealing with two strong SD 2-forms ω, η...

Proposition 3. Let ω and η be orthogonal, strongly self-dual 2-forms lying in the same linear subspace. Then

i. $\omega\eta$ is Hodge self-dual,

ii. $\omega^2\eta = \frac{1}{2}(\omega, \omega) * \eta$.

Proposition 4. Let ω and α be orthogonal, strong self-dual 2-forms such that α lies in the linear subspace transversal to the tangent space of $\mathcal{S}_{2n}$ at ω. Then

i. $\omega\alpha$ is Hodge anti self-dual,

ii. $\omega^2\alpha = -\frac{1}{2}(\omega, \omega) * \alpha$.

Outline of proofs... When ω and η lie in the same linear subspace, matrix inequalities are sufficient to prove that $\omega \wedge \eta$ is Hodge self-dual: Apply the inequalities to $\omega \pm \eta$, :

$$\begin{aligned}
(\omega,\omega)^2 + (\eta,\eta)^2 + 2(\omega,\omega)(\eta,\eta) &\geq \frac{2}{3}\big[(\omega^2,\omega^2) + (\eta^2,\eta^2) \\
&+ 2(\omega^2,\eta^2) \pm 4(\omega^2+\eta^2,\omega\eta) + 4(\omega\eta,\omega\eta)\big] \\
&\geq \frac{2}{3} * \left[\omega^4 + \eta^4 + 6\omega^2\eta^2 \pm 4\omega^3\eta \pm 4\omega\eta^3\right].
\end{aligned}$$

When ω and η do not lie in the same linear subspace, we need to go to their relation with representations of the Clifford algebra $Cl(7)$. For such forms, the corresponding matrices commute and they are simultaneously diagonalizable. We prove our claims by passing to canonical forms.

Representations of $Cl(7)$ and strong self-dual 2-forms...

The Clifford algebra $Cl(7)$ with generators a_i has a representation on R^8. Let

$$\rho(a_i) = A_{1i}, \quad i = 1, \ldots 7, \quad A_{ij} = A_{1i}A_{1j}.$$

The 2-forms corresponding to the set $\{A_{1i}, A_{ij}\}$ form a basis of 2-forms consisting of strong SD forms. Fix $A = A_{12}$, and the ω be the corresponding 2-form.

- The 12 matrices $\{A_{1i}\}$, $\{A_{2i}\}$ anti-commute with $A = A_{12}$ and constitute L_A. We will show that for $\eta \in L_A$, $\omega\eta$ is Hodge self-dual (using form inequalities).

- The 15 matrices $\{A_{jk}\}$, j, k different from 1 and 2 commute with $A = A_{12}$ and constitute P_A. We will show that for $\alpha \in P_A$, $\omega\alpha$ is Hodge anti self-dual (using matrices)

Construction of the Bonan form.

Let ω be a strongly self-dual 2-form with unit norm. Then ω^2 is a self-dual 4-form and for each such ω we have a self adjoint linear map on 2-forms,

$$T_\omega(\eta) = *(\omega^2\eta).$$

T_ω has 1, 12 and 15 dimensional eigenspaces, corresponding to the eigenvalues, 3/2, 1/2 and $-1/2$. They consist respectively of the span of ω, strong SD 2-forms in the same linear subspace as ω and the ones in the transversal subspace.

Propositon. *Let ω_{1j}, $j = 2, \ldots 8$ be orthogonal strong self-dual 2-forms belonging to the same linear subspace. Then*

$$\phi = \left[\omega_{12}^2 + \cdots + \omega_{18}^2\right]$$

is self-dual, and T_ϕ has eigenvalues $9/2$ and $-3/2$, respectively with 7 and 21 dimensional eigenspaces (or with eigenvalues 3 and -1 after scaling ϕ).

Proof.

$$T_\phi(\omega_{1j}) = T_{\omega_{1j}}(\omega_{1j}) + \sum_{k \neq j} T_{\omega_{1k}}(\omega_{1j}) = \frac{3}{2}\omega_{1j} + 6 \times \frac{1}{2}\omega_{1j}.$$

$$T_\phi(\omega_{jk}) = T_{\omega_{1j}}(\omega_{jk}) + T_{\omega_{1k}}(\omega_{jk}) + \sum_{l \neq j,k} T_{\omega_{1l}}(\omega_{jk}) = 2 \times \frac{1}{2}\omega_{jk} - 5 \times \frac{1}{2}\omega_{jk}.$$

Propositon. *Let ω_{ij}, $i,j \neq 1$ be strong self-dual orthogonal 2-forms that complement the ω_{1i}'s to a basis. Then*

$$\phi' = \left[\omega_{23}^2 + \cdots + \omega_{78}^2\right]$$

is self-dual, and $T_{\phi'}$ has eigenvalues $-9/2$ and $3/2$, respectively with 7 and 21 dimensional eigenspaces (or with eigenvalues -3 and 1 after scaling ϕ).

$$T_{\phi'}(\omega_{1j}) = T_{\omega_{2j}}(\omega_{1j}) + \sum_{k\neq j} T_{\omega_{jk}}(\omega_{1j}) = 6 \times \frac{1}{2}\omega_{1j} - 15 \times \frac{1}{2}\omega_{1j}.$$

$$T_{\phi'}(\omega_{jk}) = \left[\frac{3}{2} + (5+5) \times \frac{1}{2} - (5+5) \times \frac{1}{2}\right]\omega_{jk} = \frac{3}{2}$$

$$T_{\phi}(\omega_{jk}) = T_{\omega_{jk}}(\omega_{jk}) + T_{\omega_{1k}}(\omega_{jk}) + \sum_{l\neq j,k} T_{\omega_{1l}}(\omega_{jk}) = 2 \times \frac{1}{2}\omega_{jk} - 5 \times \frac{1}{2}\omega_{jk}.$$

$$[(23+24+25+26+27+28)+(34+35+\cdots+67+68+78)](12)$$

$$[(23)+(24+25+26+27+28)+(34+35+36+37+38)+(45+46+\cdots+68+78)](23)$$

If we choose

$$
\begin{aligned}
\omega_{12} &= e_{12} + e_{34} + e_{56} + e_{78}, \\
\omega_{13} &= e_{13} - e_{24} + e_{57} - e_{68}, \\
\omega_{14} &= e_{14} + e_{23} - e_{67} - e_{58}, \\
\omega_{15} &= e_{15} - e_{26} - e_{37} + e_{48}, \\
\omega_{16} &= e_{16} + e_{25} + e_{38} + e_{47}, \\
\omega_{17} &= e_{17} - e_{28} + e_{35} - e_{46}, \\
\omega_{18} &= e_{18} + e_{27} - e_{36} - e_{45},
\end{aligned}
$$

then,

$$
\begin{aligned}
\phi \;=\; & e_1e_2e_3e_4 + e_1e_2e_5e_6 + e_1e_2e_7e_8 + e_1e_3e_5e_7 - e_1e_3e_6e_8 - e_1e_4e_5e_8 - e_1e_4e_6e_7 \\
& -e_2e_3e_5e_8 - e_2e_3e_6e_7 - e_2e_4e_5e_7 + e_2e_4e_6e_8 + e_3e_4e_5e_6 + e_3e_4e_7e_8 + e_5e_6e_7e_8
\end{aligned}
$$

Independence of ϕ of the basis ω_{ij}

We have constructed a Bonan form starting from a basis of a 7-dimensional subspace.

Proposition. Let L_7 be a 7-dimensional linear subspace of strong self-dual 2-forms. If $\{\omega_{ij}\}$ and $\{\tilde{\omega}_{ij}\}$ are two orthonormal bases for L_7 then

$$\sum \omega_{ij}^2 = \sum \tilde{\omega}_{ij}^2.$$

This proves that the Bonan form ϕ depends on the linear subspace only and Bonan forms are in 1-1 correspondence with 7-dimensional linear subspaces of strong self-dual 2-forms.

$SO(8)$ action on the Bonan form...

We have thus constructed a Bonan form ϕ, corresponding to a linear subspace of strongly self-dual 2-forms. The form ϕ is invariant under the corresponding realization of $Spin(7)$ The orbit of ϕ under the action of the full $SO(8)$ is 7-dimensional. We will obtain it by using linear subspaces of strong self-dual 2-forms.

1-parameter subgroups of $SO(8)$:

Let A_{1i}'s be representations of the generators of $Cl(7)$ and $A_{ij} = A_{1i}A_{1j}$. The one parameter subgroup generated by any matrix A with $A^2 = -I$ is

$$exp(tA) = \cos(t)I + \sin(t)A.$$

The A_{ij}'s belong to the Lie algebra of $Spin(7)$. By computer algebra, we can see that their 1-parameter groups leave ϕ invariant and the orbit of ϕ under $SO(8)$ is given by the actions of A_{1i}'s. *But, practically it is impossible to obtain the full orbit.*

Construction of all 7-dimensional linear subspaces of strong self-dual 2-forms

Let ω be a strong self-dual 2-form. Then the set of 2-forms that lie in the same linear subspace as ω can be obtained by computer algebra. Start with

$$\omega_{12} = e_{12} + e_{34} + e_{56} + e_{78}.$$

The strong self-dual 2-forms in the same linear subspace as ω are

$$\begin{aligned}\eta \quad &= r_1[\cos(\alpha_1)(e_{13} - e_{24}) + \sin(\alpha_1)(e_{14} + e_{23}) + \cos(\beta_1)(e_{57} - e_{68}) + \sin(\beta_1)(e_{58} + e_{67})] \\ &+ r_2[\cos(\alpha_2)(e_{15} - e_{26}) + \sin(\alpha_2)(e_{16} + e_{25}) + \cos(\beta_2)(e_{37} - e_{48}) + \sin(\beta_2)(e_{38} + e_{47})] \\ &+ r_3[\cos(\alpha_3)(e_{17} - e_{28}) + \sin(\alpha_3)(e_{18} + e_{27}) + \cos(\beta_3)(e_{35} - e_{46}) + \sin(\beta_3)(e_{36} + e_{45})]\end{aligned}$$

subject to the relations

$$\beta_3 = \beta_1 + \alpha_1 - \alpha_3, \quad \beta_2 = \beta_1 + \alpha_1 - \alpha_2 - \pi.$$

Introduce an angular parameter u by,

$$\alpha_1 + \beta_1 = u.$$

Then

$$\beta_1 = u - \alpha_1, \quad \beta_2 = u - \alpha_2 - \pi, \quad \beta_3 = u - \alpha_3.$$

Using trigonometric identities, we get

$$\begin{aligned}
\eta \quad &= r_1 \cos(\alpha_1)[(e_{13} - e_{24}) + \cos(u)(e_{57} - e_{68}) + \sin(u)(e_{58} + e_{67})] \\
&+ r_1 \sin(\alpha_1)[(e_{14} + e_{23}) + \sin(u)(e_{57} - e_{68}) - \cos(u)(e_{58} + e_{67})] \\
&+ r_2 \cos(\alpha_2)[(e_{15} - e_{26}) - \cos(u)(e_{37} - e_{48}) - \sin(u)(e_{38} + e_{47})] \\
&+ r_2 \sin(\alpha_2)[(e_{16} + e_{25}) - \sin(u)(e_{37} - e_{48}) + \cos(u)(e_{38} + e_{47})] \\
&+ r_3 \cos(\alpha_3)[(e_{17} - e_{28}) + \cos(u)(e_{35} - e_{46}) + \sin(u)(e_{36} + e_{45})] \\
&+ r_3 \sin(\alpha_3)[(e_{18} + e_{27}) + \cos(u)(e_{35} - e_{46}) - \cos(u)(e_{36} + e_{45})]
\end{aligned}$$

Note that there is only one parameter that appears nonlinearly!

Orbit of the basis of L_7 onder $SO(8)$:

$$\begin{aligned}
\omega_{12} &= e_{12} + e_{34} + e_{56} + e_{78} \\
\omega_{13} &= a_{13}[(e_{13} - e_{24}) + \cos(u)(e_{57} - e_{68}) + \sin(u)(e_{58} + e_{67})] \\
\omega_{14} &= a_{14}[(e_{14} + e_{23}) + \sin(u)(e_{57} - e_{68}) - \cos(u)(e_{58} + e_{67})] \\
\omega_{15} &= a_{15}[(e_{15} - e_{26}) - \cos(u)(e_{37} - e_{48}) - \sin(u)(e_{38} + e_{47})] \\
\omega_{16} &= a_{16}[(e_{16} + e_{25}) - \sin(u)(e_{37} - e_{48}) + \cos(u)(e_{38} + e_{47})] \\
\omega_{17} &= a_{17}[(e_{17} - e_{28}) + \cos(u)(e_{35} - e_{46}) + \sin(u)(e_{36} + e_{45})] \\
\omega_{18} &= a_{18}[(e_{18} + e_{27}) + \cos(u)(e_{35} - e_{46}) - \cos(u)(e_{36} + e_{45})]
\end{aligned}$$

Substituting these in

$$\phi = \omega_{12}^2 + \omega_{13}^2 + \omega_{14}^2 + \omega_{15}^2 + \omega_{16}^2 + \omega_{17}^2 + \omega_{18}^2$$

we obtain the orbit of the Bonan form we started with...

The Bonan form...

$$
\begin{aligned}
\frac{1}{2}\phi &= \left(1 + a_{13}^2 + a_{14}^2\right)\left(e_{1234} + e_{5678}\right) \\
&+ \left(1 + a_{15}^2 + a_{16}^2\right)\left(e_{1256} + e_{3478}\right) \\
&+ \left(1 + a_{17}^2 + a_{18}^2\right)\left(e_{1278} + e_{3456}\right) \\
&+ \cos(u)\big[\left(a_{13}^2 + a_{15}^2 + a_{17}^2\right)\left(e_{1357} + e_{2468}\right) \\
&\quad - \left(a_{13}^2 + a_{16}^2 + a_{18}^2\right)\left(e_{1368} + e_{2457}\right) \\
&\quad - \left(a_{14}^2 + a_{15}^2 + a_{18}^2\right)\left(e_{1458} + e_{2367}\right) \\
&\quad - \left(a_{14}^2 + a_{16}^2 + a_{17}^2\right)\left(e_{1467} + e_{2358}\right)\big] \\
&+ \sin(u)\big[\left(a_{13}^2 + a_{15}^2 + a_{18}^2\right)\left(e_{1358} - e_{2467}\right) \\
&\quad + \left(a_{13}^2 + a_{16}^2 + a_{17}^2\right)\left(e_{1367} - e_{2458}\right) \\
&\quad + \left(a_{14}^2 + a_{15}^2 + a_{17}^2\right)\left(e_{1457} - e_{2368}\right) \\
&\quad - \left(a_{14}^2 + a_{16}^2 + a_{18}^2\right)\left(e_{1468} - e_{2357}\right)\big]
\end{aligned}
$$

This completes the construction of the orbit of a fixed Bonan 4-form under $SO(8)$, using linear subspaces of strong self-dual 2-forms.

More on strong self-dual 2-forms...
Dealing with three strong SD/ASD 2-forms ω, η, α...

Proposition 5. *Let* ω, η *and* α *be mutually orthogonal strongly self-dual 2-forms such that* $\omega + \eta + \alpha$ *is also strongly self-dual. Then*

$$\omega\eta\alpha \quad = 0.$$

If ω and η belong to the same linear subspace, and α is any form orthogonal to η, then

$$\omega^2\eta\alpha = 0.$$

Non zero products of strong self-dual 2-forms in eight dimensions... Let ω_{ij}'s be the orthogonal basis of 2-forms constructed above with $(\omega_{ij}, \omega_{ij}) = 4$. One can prove that quadruple products are nonzero only for the following cases:

$$\begin{aligned} *\omega_{ij}^4 &= 24, \\ *(\omega_{ij}^2 \omega_{ik}^2) &= 8, \\ *(\omega_{ij}^2 \omega_{kl}^2) &= -8, \\ *(\omega_{ij}\omega_{ik}\omega_{kl}\omega_{jl}) &= 8, \\ *(\omega_{ij}\omega_{kl}\omega_{mn}\omega_{pq}) &= -8. \end{aligned} \tag{1}$$

The proofs for the first four cases use matrix inequalities. For the last proof, i.e, when ω, η, α, β are orthogonal strongly self-dual 2-forms such that every pair belongs to transversal subspaces, use the fact that the corresponding matrices are simultaneously diagonalizable: Without loss of generality, choose

$$\omega = e_{12} + e_{34} + e_{56} + e_{78}, \quad \eta = e_{12} + e_{34} - e_{56} - e_{78},$$

$$\alpha = e_{12} - e_{34} - e_{56} + e_{78}, \quad \beta = e_{12} - e_{34} + e_{56} - e_{78}$$

and check the value of the quadruple product

Comment: Gauge Theories in Eight Dimensions

Example 1.

Let F be an iR valued curvature 2-form. Then $\int *(F,F)$ is minimized when F belongs to the 21-dimensional subspace of T_ϕ where $\Phi$is the Bonan form and the topological lower bound is $\int p_1^2$.

Example 2.

Let F be the curvature 2-form of an $SO(8)$ bundle. Let

$$F = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} & \omega_{16} & \omega_{17} & \omega_{18} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} & \omega_{25} & \omega_{26} & \omega_{27} & \omega_{28} \\ -\omega_{13} & \omega_{23} & 0 & \omega_{34} & \omega_{35} & \omega_{36} & \omega_{37} & \omega_{38} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 & \omega_{45} & \omega_{46} & \omega_{47} & \omega_{48} \\ -\omega_{15} & -\omega_{25} & -\omega_{35} & -\omega_{45} & 0 & \omega_{56} & \omega_{57} & \omega_{58} \\ -\omega_{16} & -\omega_{26} & -\omega_{36} & -\omega_{46} & -\omega_{56} & 0 & \omega_{67} & \omega_{68} \\ -\omega_{17} & -\omega_{27} & -\omega_{37} & -\omega_{47} & -\omega_{57} & -\omega_{67} & 0 & \omega_{78} \\ -\omega_{18} & -\omega_{28} & -\omega_{38} & -\omega_{48} & -\omega_{58} & -\omega_{68} & -\omega_{78} & 0 \end{pmatrix}$$

Let ω_{ij}'s be strong self-dual 2-forms labeled as before. Then

F^2 is Hodge self dual,

$F^3 = k * F$

$\int *(F^2, F^2)$ reaches its topological lower bound given by $\int p_2$.

Thank you...

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