Eigenvalue estimates for Dirac Operators in geometries with parallel torsion

Mario Kassuba[1 2] (HU-Berlin)

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[1]supervised by Dr. habil. I. Agricola and Prof. Dr. sc. T. Friedrich
[2]supported by the Berlin Mathematical School

Outline

1. Introduction – the operator $D^{1/3}$

2. Eigenvalue estimates on five dimensional Sasakian spaces

3. The almost Hermitian case –

   $D^{1/3}$ on spaces of pure type $W1, W3$ and $W4$

1. Introduction

Consider a compact Riemannian spin-mfd. (M^n, g) with fixed Spin-structure and corresponding spinor-bundle ΣM. Then the Schrödinger-Lichnerowicz formula states that

$$D^{g2} = \Delta^g + \frac{1}{4}\,\mathrm{Scal}^g.$$

Consequently: $D^{g2} \geq 1/4\ \mathrm{Scal}^g_{\mathrm{min}}$, but: this never gives an optimal positive lower bound for $\mathrm{spec}\ D^{g2}$! Solution [Friedrich 1980]: the operator $D^g - f$ satisfies

$$(D^g - f)^2 = \Delta^f + \frac{1}{4}\,\mathrm{Scal}^g + (1-n)f^2$$

where Δ^f is the Laplacian of a *new spinorial metric covariant derivative*. Integrating and choosing f to be a suitable constant yield

$$D^{g2} \geq \frac{1}{4}\,\frac{n}{n-1}\mathrm{Scal}^g_{\mathrm{min}}$$

and spinors realizing the limiting case correspond to Killing-spinors!

Now, let $(M^n, g, \mathcal{R} \subset \mathcal{F}(M^n))$ be a compact Riemannian spin-mfd. equipped with a non-integrable G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ such that a *characteristic connection* ∇^c with ∇^c-*parallel characteristic torsion* T^c exists, e.g. (and in particular)

• *Sasakian structures* in dim. 5 $\Rightarrow T^c = \eta \wedge \mathrm{d}\eta$ [Friedrich,Ivanov (2002)]

• *almost Hermitian structures* in dim. 6 $\Rightarrow T^c = -J((\nabla^g_{\cdot} J)(.))$ [Gray,Kirichenko,Bismut,Friedrich,Ivanov]

• *cocalibrated G_2-structures* in dim. 7 $\Rightarrow T^c = \frac{1}{6}(\mathrm{d}\varphi, *\varphi) \cdot \varphi - *\mathrm{d}\varphi$ [Friedrich,Ivanov (2002)]

Instead of ∇^c consider the affine metric connection $\nabla^{1/3}$ with torsion $1/3\ T^c$. Then, we introduce the elliptic first order dop.

$$D^{1/3} = \mu \circ \nabla^{1/3} : \Gamma(\Sigma M) \to \Gamma(\Sigma M).$$

Why $\nabla^{1/3}$?

Facts

•$D^{1/3}$ coincides with *Kostant's Dirac operator* in the naturally reductive case and satisfies a Parthasarathy-type formula [Agricola 2003].

•for almost Hermitian spaces, $D^{1/3}$ can be identified with the *Dolbeault operator*

•$(D^{1/3})^2$ satisfies the (universal) S.-L. formula [Agricola, Friedrich 2003]

$$(D^{1/3})^2 = \Delta^T + \tfrac{1}{4}|T|^2 + \tfrac{1}{8}T^2 + \tfrac{1}{4}\,\mathrm{Scal}^g,$$

where $\Delta^T = \nabla^{c*}\nabla^c$.

Consequence: $(D^{1/3})^2 \circ T^c = T^c \circ (D^{1/3})^2$, i.e. $(D^{1/3})^2$ can be restricted to sections of each eigensubbundle $\Sigma_\mu = \ker(T^c - \mu)$ of T^c and we will estimate the restricted operator $(D^{1/3})^2_{|\Sigma_\mu}$ for all (finitely many) $\mu \in \mathrm{spec}_p T^c$.

Remark: $D^{1/3}$ and T^c do not commute!

Theorem ((S-def.) S.-L. formula). If $S \in \mathrm{End}(\Sigma M)$ is a symmetric and ∇^c-parallel endomorphism, then

$$\langle (D^{1/3}+S)^2\psi,\psi\rangle_{L^2} = ||\nabla^S\psi||^2_{L^2} - \tfrac{1}{4}\textstyle\sum_{i=1}^n ||(e_i\cdot S + Se_i)\psi||^2_{L^2} -$$

$$\tfrac{1}{4}||T^c\psi||^2_{L^2} + \tfrac{1}{8}|T^c|^2||\psi||^2_{L^2} + \tfrac{1}{4}\textstyle\int_M \mathrm{Scal}^g|\psi|^2 + ||S\psi||^2_{L^2} - \langle T^c\cdot S\psi,\psi\rangle_{L^2},$$

where

$$\nabla^S_X\psi := \nabla^c_X\psi - \frac{1}{2}(SX\cdot\psi + X\cdot S\psi).$$

Two different Ansätze:

•For $(D^{1/3})^2_{|\Sigma_\mu}, \mu \neq 0$ we choose S to be a polynomial in the torsion form T^c !

•For $(D^{1/3})^2_{|\Sigma_0}$ we refine the general S.-L. formula (under an additional technical and sensible assumption) and obtain as corresponding covariant derivative on $\Gamma(\Sigma_0)$

$$\nabla^f_X\psi = \nabla^c_X\psi + f\,\mathrm{pr}_{\Sigma_0}(X\cdot\psi).$$

2. $D^{1/3}$ on 5-dim. Sasakian spaces

Let $(M^5, g, \xi, \eta, \phi)$ be a 5-dim. compact Sasakian spin-mfd. with fixed Spin-structure.

The charact. torsion is given by $T^c = \eta \wedge \mathrm{d}\eta$ and we can set $|T^c|^2 = 8$.

The spinor-bundle ΣM splits into

$$\Sigma M = \Sigma_4 \oplus \Sigma_0 \oplus \Sigma_{-4}.$$

If $-4 < \mathrm{Scal}_{\mathrm{min}}$ we obtain the

Theorem.

$$\lambda_{\mathrm{min}}((D^{1/3})^2_{|\Sigma_0}) \geq \lambda_{\mathrm{min}}((D^{1/3})^2_{|\Sigma_{\pm 4}}).$$

Proof. We apply the general method to the bundle Σ_0 and obtain for every eigenvalue λ of the operator $(D^{1/3})^2_{|\Sigma_0}$ with eigenspinor ψ the estimate

$$\lambda \geq \frac{\langle D^{1/3}\psi, \psi\rangle^2_{L^2}}{||\psi||^4_{L^2}} + 1 + \frac{1}{4}\,\mathrm{Scal}_{\mathsf{min}}. \tag{1}$$

In general, ψ is not an eigenspinor of the operator $D^{1/3}$!

- Trivial case: if there is a $\psi \in \mathsf{Eig}_\lambda((D^{1/3})^2)$ with $\mathsf{pr}_{\Sigma_4} D^{1/3}\psi \neq 0$ or $\mathsf{pr}_{\Sigma_{-4}} D^{1/3}\psi \neq 0$, then $\mathsf{pr}_{\Sigma_4} D^{1/3}\psi \in \mathsf{Eig}_\lambda((D^{1/3})^2_{|\Sigma_4})$ or $\mathsf{pr}_{\Sigma_{-4}} D^{1/3}\psi \in \mathsf{Eig}_\lambda((D^{1/3})^2_{|\Sigma_{-4}})$ resp.

- Interesting case: $\mathsf{pr}_{\Sigma_4} D^{1/3}\psi = 0, \mathsf{pr}_{\Sigma_{-4}} D^{1/3}\psi = 0$ for all $\psi \in \mathsf{Eig}_\lambda(D^{1/3})^2_{|\Sigma_0}$

  $\Rightarrow$ exists a Spinor $\psi \in \mathsf{Eig}_\lambda((D^{1/3})^2)$ which appears to be an eigenspinor of $D^{1/3}$ and $(D^{1/3})^2$ simultaneously! Plugging ψ into (1) yields the result.

The bundle Σ_4

- The universal S.-L. formula for $(D^{1/3})^2$ yields the simple estimate

$$\lambda_{\min}((D^{1/3})^2_{|\Sigma_4}) \geq -3 + \tfrac{1}{4}\mathrm{Scal}_{\min}.$$

- As an application of the $P(T^c)$-deformed S.-L. formula we get

$$\lambda_{\min}((D^{1/3})^2_{|\Sigma_4}) \geq y_\psi^2 + 4y_\psi + 1 + \tfrac{1}{4}\mathrm{Scal}_{\min},$$

where $y_\psi := \langle D^{1/3}\psi, \psi\rangle_{L^2}/||\psi||^2_{L^2}$ and $(D^{1/3})^2\psi = \lambda_{\min}\psi$.

Key observation: $y_\psi^2 \leq \lambda \Rightarrow$

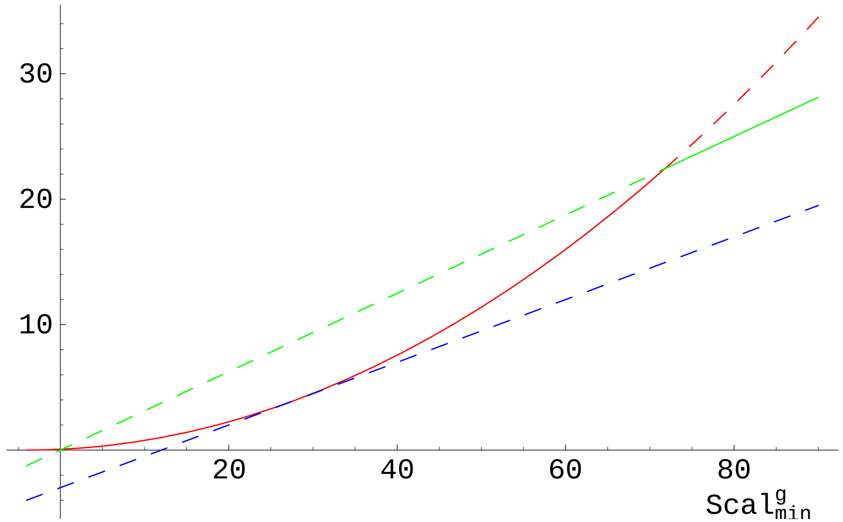
$$\lambda_{\min}((D^{1/3})^2_{|\Sigma_4}) \geq \inf_{y\in[-\sqrt{\lambda},\sqrt{\lambda}]} y_\psi^2 + 4y_\psi + 1 + \frac{1}{4}\mathrm{Scal}_{\min}$$

Furthermore, using $(D^{1/3})^2_{|\Sigma_0} \geq (D^g)^2 = 5/16\ \mathrm{Scal}^g_{\min}$ gives

Theorem. $(M^5, g, \xi, \eta, \phi)$ compact Sasakian spin-mfd. with $-4 < \mathrm{Scal}^g_{\mathsf{min}} \Rightarrow$

$$\lambda_{\mathsf{min}}((D^{1/3})^2) \geq \begin{cases} 1/16\ (1 + 1/4\ \mathrm{Scal}^g_{\mathsf{min}})^2; -4 < \mathrm{Scal}^g_{\mathsf{min}} \leq 4(9+4\sqrt{5}) \\ 5/16\ \mathrm{Scal}^g_{\mathsf{min}}; 4(9+4\sqrt{5}) \leq \mathrm{Scal}^g_{\mathsf{min}} \end{cases}.$$

Furthermore, if $\lambda_{\mathsf{min}}((D^{1/3})^2) < \frac{1}{16}(1 + \frac{1}{4}\mathrm{Scal}^g_{\mathsf{min}})^2$ then necessarily $\lambda_{\mathsf{min}}((D^{1/3})^2_{|\Sigma_0}) = \lambda_{\mathsf{min}}((D^{1/3})^2_{|\Sigma_{\pm 4}})$.



Remarks

- quadric dependence on the scalar curvature
- positive lower bound even for negative scalar curvature
- for $\mathrm{Scal}_{\mathsf{min}} = 28$ both curves coincide $\Rightarrow$ spaces with ∇^c-parallel spinors

[Friedrich,Ivanov (2002)]

The limiting case

Definition. A Sasakian mfd. (M, g, ξ, η, ϕ) is called η-Einstein, if $\mathrm{Ric} = \lambda g + \mu \eta \otimes \eta$.

Theorem. 1) Let (M, g, ξ, η, ϕ) be a compact Sasakian spin-mfd. with $-4 < \mathrm{Scal}_{\mathsf{min}} \leq 4(9 + 4\sqrt{5})$. If $\psi \in \Gamma(\Sigma_{\pm 4})$ is an eigenspinor to the eigenvalue $\lambda = 1/16(1 + 1/4\mathrm{Scal}_{\mathsf{min}})^2$, then (M, g, ξ, η, ϕ) is an η-Einstein space.

2) If (M, g, ξ, η, ϕ) is a simply-connected η-Einstein space with $-4 < \mathrm{Scal}^g_{\mathsf{min}}$, then $\lambda = 1/16(1 + 1/4\,\mathrm{Scal}^g_{\mathsf{min}})^2$ is eigenvalue of the operator $(D^{1/3})^2_{|\Sigma_{\pm 4}}$ and realizes the smallest eigenvalue for $\mathrm{Scal}^g_{\mathsf{min}} \leq 4(9 + 4\sqrt{5})$.

Examples. 5-dim. η-Einstein Sasakian spaces can be obtained as the total spaces of S^1-bundles over 4-dim. Kähler-Einstein spaces

Examples. There are many non-regular examples.
[Boyer,Galicki,Matzen 2006].

3. $D^{1/3}$ on 6-dim. almost Hermitian spaces

1. Spaces of pure type $W1$ – the *nearly Kähler* case

Let (M^6, g, J) be a compact (and simply connected) *nearly Kähler*, non Kähler mfd..

•(M^6, g, J) is a complete Einstein space of positive scalar curvature Scal^g satisfying $\mathrm{Scal}^g = 15/2\,|T^c|^2$.

•For $M^6 \neq S^6$ there exist (according to the unique) spin-structure two Riemannian and ∇^c-parallel Killing-spinors φ_1, φ_2 [Friedrich,Ivanov].

•The spinor-bundle decomposes under the action of T^c into

$$\Sigma M = \underbrace{\Sigma_0}_{6-\mathrm{dim}} + \underbrace{\Sigma_{2|T^c|}}_{1-\mathrm{dim}} + \underbrace{\Sigma_{-2|T^c|}}_{1-\mathrm{dim}}.$$

Proposition. φ_1 and φ_2 trivialize the bundles $\Sigma_{\pm 2|T^c|}$ and realize the lower bound for $\mathrm{spec}\,(D^{1/3})^2_{|\pm 2|T^c|}$ given by the universal S.-L. formula for $(D^{1/3})^2_{|\pm 2|T^c|}$, $(D^{1/3})^2\varphi_{1/2} = 2/15\ \mathrm{Scal}^g\,\varphi_{1/2}$.

The bundle Σ_0

• The simple lower bound for $\mathrm{spec}\ (D^{1/3})^2_{|\Sigma_0}$ is given by $(D^{1/3})^2_{|\Sigma_0} \geq 4/15\ \mathrm{Scal}^g = 8/9\ (1/4 \cdot 6/5\ \mathrm{Scal}^g)$ – can never be optimal since $(D^{1/3})^2_{|\Sigma_0} \geq (D^g)^2 \geq 1/4 \cdot 6/5\ \mathrm{Scal}^g$!

• Taking $\langle \psi, (D^{1/3})^2\psi\rangle_{L^2} = \langle \psi, (D^g)^2\psi\rangle_{L^2}, \psi \in \Gamma(\Sigma_0)$ into account we observe

$$\langle (D^{1/3})^2\psi, \psi\rangle_{L^2}/||\psi||^2_{L^2} \geq \inf_{\psi\in\Gamma(\Sigma_0)} \langle (D^g)^2\psi, \psi\rangle_{L^2}/||\psi||^2_{L^2}$$

But: (for $M^6 \neq S^6$) $\ker((D^g)^2 - \lambda_1^g) = \langle \varphi_1, \varphi_2\rangle \perp \Gamma(\Sigma_0) \Rightarrow$

$$\langle (D^{1/3})^2\psi, \psi\rangle_{L^2}/||\psi||^2_{L^2} \geq \inf_{\psi\in\mathrm{Eig}^{\perp}_{\lambda_{\mathrm{min}}}(D^g)^2} \langle (D^g)^2\psi, \psi\rangle_{L^2}/||\psi||^2_{L^2} = \lambda_2^g$$

Theorem. (M^6, g, J) simply connected and compact nearly Kähler space and not isometric to the sphere S^6, then

$$\lambda_1((D^{1/3})^2_{|\Sigma_0}) \geq \lambda_2((D^g)^2).$$

Natural questions:

- What is the optimal lower bound for $\lambda_2^g((D^g)^2)$? – remains open
- necessary conditions for $\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2((D^g)^2)$
- which nearly Kähler spaces satisfy $\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2((D^g)^2)$?

First we apply our general method to the nearly Kähler case:

Theorem 1. (M^6, g, J) compact nearly Kähler space. Then every eigenvalue with eigenspinor $(D^{1/3})^2_{|\Sigma_0}\psi = \lambda\psi$ is bounded from below by

$$\lambda \geq \frac{1}{4}\frac{\langle D^{1/3}\psi, \psi\rangle_{L^2}}{||\psi||^2_{L^2}} + \frac{4}{15}\,\mathrm{Scal}^g.$$

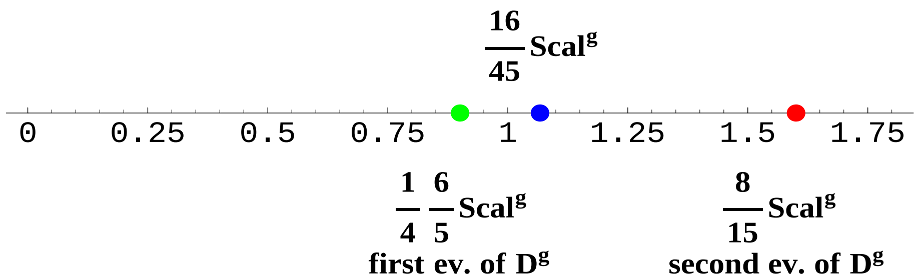
Theorem 2. Equality can never occur in theorem 1!

A consequence of theorem 2 is

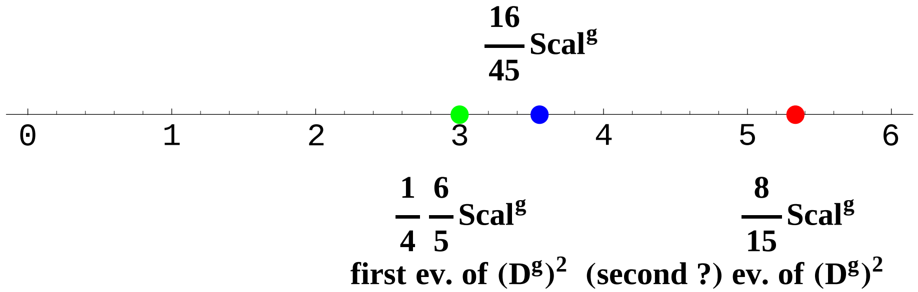
Theorem. If (M^6, g, J) is a simply connected and compact nearly Kähler space with $\lambda_1((D^{1/3})^2_{|\Sigma_0}) = \lambda_2^g((D^g))^2$, then λ_2^g satisfies the strict inequality

$$\frac{16}{45}\,\mathrm{Scal}^g < \lambda_2^g((D^g)^2).$$

Nearly Kähler structure on S^6, $\mathrm{Scal}^g = 3$



Nearly Kähler structure on $S^3 \times S^3$, $\mathrm{Scal}^g = 10$



2. Spaces of pure type $W3$

We consider (M^6, g, J) of pure type $W3$, compact und with fixed Spin-structure.

•In general $\mathrm{Iso}_o T^c = \mathrm{U}(2), \mathrm{SO}(3)$ or T^2 [Alexandrov,Friedrich,Schoemann]

•For $\mathrm{Iso}_o T^c = \mathrm{U}(2) \Rightarrow$ only the twistor-spaces over S^4 and $\mathbf{C}^4$ are possible

•For $\mathrm{Iso}_o T^c = \mathrm{SO}(3) \Rightarrow (M^6, g, J)$ is (locally) isomorphic to $(\mathrm{SL}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2) = \mathrm{SL}(2, \mathbf{C})$

$\Rightarrow$

We only treat the case $\mathrm{Iso}_o T^c = T^2$.

We scale $|T^c|^2 = 2$. Then the spinor-bundle splits into

$$\Sigma M = \underbrace{\Sigma_2}_{2-dim.} \oplus \underbrace{\Sigma_0}_{4-dim.} \oplus \underbrace{\Sigma_{-2}}_{2-dim.} .$$

The bundle Σ_2

•The universal estimate is given by

$$(D^{1/3})^2_{|\Sigma_2} \geq -\frac{3}{4} + \frac{1}{4}\,\mathrm{Scal}^g_{\mathsf{min}}$$

•Using the P-def. connection, we obtain

$$(D^{1/3})^2_{|\Sigma_2} \geq \frac{\langle D^{1/3}\psi, \psi\rangle^2_{L^2}}{||\psi||^4_{L^2}} + 2\frac{\langle D^{1/3}\psi, \psi\rangle_{L^2}}{||\psi||^2_{L^2}} + \frac{1}{4} + \frac{1}{4}\,\mathrm{Scal}^g_{\mathsf{min}}$$

Again, we control $\langle D^{1/3}\psi, \psi\rangle_{L^2}$ by the Cauchy-Schwarz inequality to get:

Theorem. (M^6, g, J) compact and of pure type $W3, \mathrm{Iso}_o T^c = T^2$. Then

1)
$$(D^{1/3})^2_{|\Sigma_2} \geq \begin{cases} 1/4\,(1/4 + 1/4\,\mathrm{Scal}^g_{\mathsf{min}})^2; \ -1 \leq \mathrm{Scal}^g_{\mathsf{min}} \leq 7 \\ -3/4 + 1/4\,\mathrm{Scal}^g_{\mathsf{min}}; \ 7 \leq \mathrm{Scal}^g_{\mathsf{min}} \end{cases}$$

2) (M^6, g, J) limiting mfd. $\Rightarrow$ (M^6, g, J) (locally) is naturally reductive.

Examples can be constructed out of a 3-form $T^c \in \Lambda^3_{12}(\mathbf{R}^6)$ and a curvature tensor $\mathcal{R}^c : \Lambda^2(\mathbf{R}^6) \rightarrow \mathfrak{t}^2$ such that the pair (T^c, R^c) defines an infinitesimal model.

The bundle Σ_0

The S.-L. formula for $(D^{1/3})^2$ gives the simple estimate

$$(D^{1/3})^2_{|\Sigma_0} \geq \frac{1}{8}\,|T^c|^2 + \frac{1}{4}\,\mathrm{Scal}^g_{\mathsf{min}}$$

– can never be an optimal postive lower bound.

We apply our general method and obtain the

Theorem 2.1. Every eigenvalue λ of the operator $(D^{1/3})^2_{|\Sigma_0}$ with eigenspinor $(D^{1/3})^2_{|\Sigma_0}\psi = \lambda\psi$ satisfies

$$\lambda \geq \frac{1}{2}\,\frac{\langle D^{1/3}\psi,\psi\rangle^2_{L^2}}{||\psi||^4_{L^2}} + \frac{1}{8}\,|T^c|^2 + \frac{1}{4}\,\mathrm{Scal}^2_{\mathsf{min}}.$$

A discussion of the integrability conditions yields

Theorem 2.2.. In theorem 2.1 equality never holds!

3. Spaces of pure type $W4$ – – the *locally conformal Kähler* case

(M^6, g, J) compact manifold of pure type $W4$ with fixed Spin-structure.

• there are two ∇^c-parallel and commuting Killing fields X, JX, where $\delta\Omega = 2\,X$

• the spinor bundle decomposes under the action of T^c into

$$\Sigma M = \underbrace{\Sigma_2}_{2-dim.} \oplus \underbrace{\Sigma_0}_{4-dim.} \oplus \underbrace{\Sigma_{-2}}_{2-dim.}$$

Theorem. (M^6, g, J) compact manifold of pure type $W4$ with fixed Spin-structure. Every eigenvalue $(D^{1/3})^2_{|\Sigma_2}\psi = \lambda\psi$ satisfies

$$(D^{1/3})^2_{|\Sigma_2} \geq \begin{cases} 1/4\,(1/4 + 1/4\,\mathrm{Scal}^g_{\mathsf{min}})^2; & -1 \leq \mathrm{Scal}^g_{\mathsf{min}} \leq 7 \\ -3/4 + 1/4\,\mathrm{Scal}^g_{\mathsf{min}}; & 7 \leq \mathrm{Scal}^g_{\mathsf{min}} \end{cases}$$

For the **limiting case** we suppose that

• JX induces a regular group action on M^6, s.t. $X^5 = M^6/JX$ becomes a smooth orbit space $\Rightarrow$ X^5 carries the structure of a Sasakian manifold $(X^5, \hat{g}, \xi, \eta, \varphi)$ [Vaisman]

• the Spin-structure on (M^6, g, J) can be projected to a Spin-structure on X^5

Theorem. (M^6, g, J) limiting space with limiting spinor $\psi \in \Gamma(\Sigma_2)$ $\Rightarrow$

1) the orbit space X^5 is a compact η-Einstein Sasakian manifold

2) ψ induces a spinor $\hat{\psi} \in \Gamma(\Sigma X^5)$ which realizes the limiting case for Sasakian Spin-manifolds.

Example. $(X^5, \hat{g}, \xi, \eta, \varphi)$ compact, simply connected η-Einstein Sasakian space with $-1 \leq \mathrm{Scal}^{\mathrm{g}} \leq 7$ and unique Spin-structure $\Rightarrow$ $X^5 \times S^1 \to X^5$ is a compact almost Hermitian space of pure type $W4$ realizing the limiting case.