syzygy:

- shortest English word containing three y's.
- longest English word without vowels.
- ancient Greek: συζυγια : copulation, conjunction
- astronomy
- name of an alternate reality game
- psychology (C.G. Jung): archetypal pairing of contrasexual opposites.
- math: Sylvester

= Syzygies of varieties and Green's Conjecture =

what are syzygies ? History: $\begin{cases} \text{Sylvester 1850:} \\ \text{astronomy (17th century)} \end{cases}$

$X \subseteq \mathbb{P}^r$ variety. $S = k[x_0 \dots x_r]$. $I(X) \subseteq S$ ideal: describe $I(X)$ by generators and relations: $f_1 \dots f_t \in S$ generators of $I(X)$ $\deg(f_i) = e_i$

$$0 \to \underbrace{M_1}_{} \to \bigoplus_{i=1}^{t} S(-e_i) \xrightarrow{(f_1 \dots f_t)} I(X) \to 0.$$

1st module of syzygies of X (i.e. relations between generators of X).

$\{g^\alpha = (G_1^\alpha \dots G_t^\alpha) \in M_1\}_{\alpha=1,e}$ basis for M_1 : $G_1^\alpha f_1 + \dots + G_t^\alpha f_t = 0$

$\deg G_i^\alpha + e_i = f_\alpha \ \forall i.$

$$0 \to \underbrace{M_2}_{} \to \bigoplus_{\alpha=1}^{e} S(-f_\alpha) \xrightarrow{(G_i^\alpha)} \bigoplus_{i=1}^{t} S(-e_i) \xrightarrow{(f_i)} I(X) \to 0 \quad \text{exact}.$$

2nd module of syzygies of X ... : In this way we get a free S-resolution of X

$$0 \to F_{r-1} \to \dots \to F_2 \to F_1 \to I(X) \to 0$$

(Hilbert syzygy thm).

F_i = free S-modules $= \bigoplus_j S(-i-j)^{b_{ij}(X)}$ $\qquad$ $b_{ij}(X)$ – graded Betti #'s of X.

Betti diagram of X:

j \ i		i
		$\vdots$
j	$\dots$	$b_{ij}(X)$

what information is encoded in syzygies ? (or in the Betti #'s).

$h_X(m) = \dim_k S(X)_m$ (Hilbert polynomial of X) : if $F_i = \bigoplus S(-a_{ij})$

$$h_X(m) = \sum_{i \geq 0} (-1)^i \sum_j \binom{r+m-a_{ij}}{r}$$

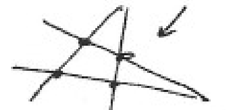
: in fact the Betti diagram encodes a lot more information than the Hilbert polynomial.

<u>Example</u>: 4 points in $\mathbb{P}^2$: p_1, p_2, p_3, p_4.

If $p_1 \dots p_4$ are general $\Rightarrow$ they are the intersection of two conics Q_1, Q_2.

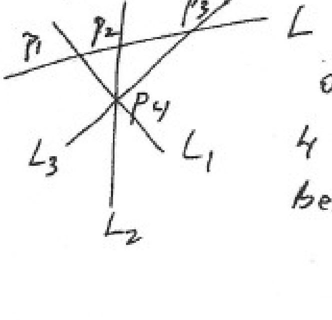
$$0 \to S(-4) \to S(-2)^2 \to I_X \to 0$$

1	0	0
0	2	0
0	0	1



syzygy : $Q_2 \cdot Q_1 - Q_1 \cdot Q_2 = 0$

If 3 points are collinear:



$Q_1 := L \cup L_1$, $Q_2 := L \cup L_2$.

$$0 \to S(-3) \oplus S(-4) \to S(-2)^2 \oplus S(-3) \to I_X \to 0 \ :$$

4 points in $\mathbb{P}^2$ always have the same Hilbert polynomial but the Betti #'s reflect their geometry !

Example: Twisted cubic: $R \subseteq \mathbb{P}^3$. $R : \det_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0$

$$0 \to S(-3)^2 \to S(-2)^3 \to \mathcal{I}_X \to 0$$

$\begin{array}{|ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{array}$

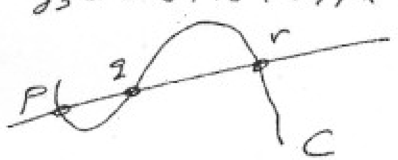
NB: $\exists$ similar description for $R \subseteq \mathbb{P}^r$ (rnc). $\det_2 \begin{pmatrix} x_0 & x_1 & \dots & x_{r-1} \\ x_1 & x_2 & \dots & x_r \end{pmatrix} = 0$.

Example: $C \xrightarrow{|K_C|} \mathbb{P}^4$ $g=5$, $d=8$. : C no g^1_2 : $\underbrace{\mathrm{Sym}^2 H^0(K_C)}_{15} \to \underbrace{H^0(K_C^2)}_{12} \to 0$

$\Rightarrow \exists$ 3 quadrics containing C.

$$0 \to S(-5) \to S(-4)^3 \to S(-2)^3 \to \mathcal{I}_C \to 0 \quad (\text{if } \nexists\, g^1_3).$$

$\exists\, g^1_3 = L \in \mathrm{Pic}^3(C)$, $h^0(L) = 2$. $L = \mathcal{O}_C(p+q+r) \Rightarrow p, q, r \in \mathbb{P}^4$ collinear



$\Rightarrow$ C cannot be an intersection of quadrics in this case.

$$\dots \to S(-2)^3 \oplus S(-3)^3 \to \mathcal{I}_C \to 0$$

$\exists$ 2 classical theorems: M. Noether: C non-hyperelliptic $\Rightarrow C \subseteq \mathbb{P}^{g-1}$ proj. normal.

C has no g^1_3 or g^2_5 $(g=6)$: $\mathcal{I}_C$ generated by quadrics (K. Petri)

Q: Since these classical theorems govern the first steps of the resolution can we extend them in such a way to take into account the whole resolution?

The property (N_p): A way to encode that the resolution is "nice". $X \subseteq \mathbb{P}^r$

(N_0) : X proj. normal : $0 \to F_{r-1} \to \quad \to F_2 \to F_1 \to \mathcal{I}_X \to 0$

(N_1): $F_1 = \oplus S(-2)$

$\vdots$

(N_p): N_{p-1} + $F_p = \oplus S(-p-1)$ i.e. all syzygies of X of order $\leq p$ are linear

NB: $(N_p) \Leftrightarrow b_{i,2}(X) = 0$ $i \leq p$.

Noether: C satisfies $(N_0) \Leftrightarrow C$ has no g^1_2

Petri: C satisfies $(N_1) \Leftrightarrow C$ has no g^1_3.

$\begin{array}{c|l} j=0 & 1 \ 0 \dots 0 \\ j=1 & \underline{\quad\quad\quad} \\ & \boxed{p+1 \text{ zeros}} \end{array}$

Green's Conjecture (1983). $C \subseteq \mathbb{P}^{g-1}$ canonical. Then C satisfies (N_p) $\Leftrightarrow p < \mathrm{Cliff}(C)$ (Clifford index of C).

Moral: intrinsic geometry of C can be recovered from the equations of its canonical embedding.

Clifford index: $C \xrightarrow{|L|} \mathbb{P}^r$: $r = h^0(L) - 1$. $\mathrm{Cliff}(C) = \min\{d(L) - 2r(L)\}$

$\mathrm{Cliff}(L) = \deg(L) - 2r(L) \geq 0$ for $\deg(L) \leq g-1$. $\quad 0 \leq d(L) \leq g-1$

(Clifford's theorem)

Ex: $\mathrm{Cliff}(C) = 0 \Leftrightarrow C$ has g^1_2 i.e. C is hyperelliptic.

$\mathrm{Cliff}(C) = 1 \Leftrightarrow C$ has g^1_3 or g^2_5 ...

Q: what is $\mathrm{Cliff}(C)$ when $[C] \in \mathcal{M}_g$ is general?

Brill-Noether thm: C - general then $\exists\, L \in \mathrm{Pic}^d(C)$, $h^0(L) \geq r+1 \Leftrightarrow g - (r+1)(g-d+r) \geq 0$.

$\Rightarrow \exists\, g^1_{\lceil \frac{g+3}{2} \rceil} = L$ s.t. $\mathrm{Cliff}(C) = \mathrm{Cliff}(g^1_{\lceil \frac{g+3}{2} \rceil}) = [\frac{g-1}{2}]$

GC is wide open for arbitrary curves. However we have the following:

Theorem: (C. Voisin 2002–2003): G.C. holds for the general curve of every genus.
$\begin{cases} g = 2k+1 \Rightarrow C \text{ satisfies } (N_{k-1}) \\ g = 2k+2. \end{cases}$

(* say something about why is it difficult to prove such a thing).

How about (N_p) for other situations? $X \xrightarrow{|L|} \mathbb{P}^r$.

- curves: $\deg(L) \geq 2g+1+p \Rightarrow X$ satisfies (N_p). (Green – Lazarsfeld).
- X – ab. variety. $L = \mathcal{O}_X(\Theta)$ ample: $L^{\otimes(p+3)}$ satisfies (N_p). (Pareschi)
- $\mathbb{P}^n \xrightarrow{|\mathcal{O}(d)|} \mathbb{P}^{\binom{n+d}{n}-1}$: should have (N_p) for $p \leq 3d-3$. ($p = 3d-2$ does not work).
- Fujita conjecture: L–ample $\Rightarrow k_X + (n+2)L$ – very ample; $k_X + (n+2+p)L$ satisfies (N_p).

<u>koszul cohomology:</u> $X \subseteq \mathbb{P}^r$: $b_{ij}(X) = \dim_k \operatorname{Tor}^i_S(S(X), k)_{i+j}$. $V = H^0(X, L)$.

koszul resolution of k as a graded S-module:

$$0 \to \Lambda^{r+1} V(-r-1) \to \cdots \to \Lambda^2 V(-2) \to V \otimes \mathcal{O}(-1) \to k \to 0. \quad \Big| \otimes_k S(X).$$

$$(*)$$

$$\to \Lambda^{i+1} H^0(L) \otimes H^0(L^{j-1}) \xrightarrow{d_{i+1,j-1}} \Lambda^i H^0(L) \otimes H^0(L^j) \xrightarrow{d_{i,j}} \Lambda^{i-1} H^0(L) \otimes H^0(L^{j+1}) \to \cdots$$

$$d_{i,j}(f_1 \wedge \ldots \wedge f_i \otimes u) = \sum_{l=1}^{i} (-1)^l f_1 \wedge \ldots \wedge \hat{f_l} \wedge \ldots \wedge f_i \otimes (f_l u)$$

koszul differential.

$$k_{i,j}(X, L) = \frac{\ker\{d_{i,j}\}}{\operatorname{Im}\{d_{i+1,j-1}\}}$$: koszul cohomology group

$\boxed{b_{i,j}(X, L) = \dim_k k_{i,j}(X, L)}$.

So $b_{i,j}(X, L) = 0 \iff (*)$ exact on global sections. To handle $(*)$ one introduces the following vector bundle:

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \xrightarrow{eval} L \to 0$$

$$0 \to \Lambda^{i+1} M_L \to \Lambda^{i+1} H^0(L) \otimes \mathcal{O}_X \to \Lambda^i M_L \otimes L \to 0 \quad \forall i. \ \Big| \otimes L^{j-1}$$

$$\boxed{k_{i,j}(X, L) = \frac{H^0(\Lambda^i M_L \otimes L^j)}{\operatorname{Im}(\Lambda^{i+1} H^0(L) \otimes H^0(L^{j-1}))}}$$

koszul cohomology groups enjoy other properties:

(1) <u>duality</u>: $k_{i,j}(X, k_X) = k_{g-i-2,3-j}(X, k_X)^\vee$

(2). <u>Lefschetz</u>: $Y \xrightarrow{|L|} \mathbb{P}^r$, $X \subset Y$ hyperplane section: ($X = \{l = 0\}$, $l \in H^0(Y, L)$) : $\boxed{k_{i,j}(Y, L) \simeq k_{i,j}(X, L|_X)}$.

Going back to Green's Conjecture: $X = C \hookrightarrow \mathbb{P}^{g-1}$

$GC \iff K_{p,2}(C, K_C) = 0 \quad \forall p < \mathrm{Cliff}(C) \underset{\text{duality}}{\iff} K_{a,1}(C, K_C) = 0,\ a = g-1-\mathrm{Cliff}(C)$.

Suppose $g = 2a+1$: $\exists\, g^1_{a+2} : C \xrightarrow{a+2} \mathbb{P}^1$, $\nexists\, g^1_{a+1}$ $\qquad \mathrm{Cliff}(C) = a$.

Thm: (Voisin): If $[C] \in \mathcal{M}_g$ is general then C satisfies (N_{a-1}) and its resolution in the canonical embedding is

$$0 \to S(-g-1) \to S(-g+1)^{b_{11}} \to S(-a-2)^{b_{a1}} \to \underset{F_{a-1}}{S(-a)^{b_{a1}}} \to \cdots \to \underset{F_1}{S(-2)^{b_{11}}} \to I_C \to 0$$

"minimal resolution"

– (a similar statement for even genus).

Thm: (Hirschowitz–Ramanan): If C is an arbitrary curve of genus $g = 2a+1$ then C does not have minimal resolution iff C has a covering of degree $a+1$ of $\mathbb{P}^1$ ($C \xrightarrow{g^1_{a+1}} \mathbb{P}^1$).

The proof of this is a spectacular application of $\mathcal{M}_g$: although this is a statement about an arbitrary curve we work with all genus g curves at once and define two loci in $\mathcal{M}_g$:

$D_1 = \{[C] \in \mathcal{M}_g : C$ has non-minimal resolution $\iff K_{a,1}(C, K_C) \neq 0\}$.

$D_2 = \{[C] \in \mathcal{M}_g : \exists\, C \xrightarrow{g^1_{a+1}} \mathbb{P}^1\}$.

$D_2 \subseteq D_1$ (e.g. $g = 5$: if C has g^1_3 it can't be cut out by quadrics).

$D_1 \subsetneq \mathcal{M}_g$ (Voisin)

D_2 – divisor in $\mathcal{M}_g$: Independently compute the class of D_1 and D_2

in $\mathrm{Pic}(\mathcal{M}_g)$: $D_1 \equiv D_2$ in $\mathrm{Pic}(\mathcal{M}_g)$ $\left(\equiv 6a(a+2)\binom{2a-2}{a}\lambda\right)$. $\implies D_1 = D_2$.

$\exists\, \mathcal{M}_g \subseteq \widetilde{\mathcal{M}_g}$ with $\mathrm{codim}(\widetilde{\mathcal{M}_g} - \mathcal{M}_g, \widetilde{\mathcal{M}_g}) \geq 2$ $\qquad \boxtimes$.

<u>About Voisin's proof:</u> $\qquad a = g-1-\mathrm{Cliff}(C)$.

One has to find a single curve C of genus g for which $K_{a,1}(C, K_C) = 0$.

Voisin chooses $C \subseteq S \hookrightarrow$ K3 surface, $\mathrm{Pic}(S) = \mathbb{Z}[C]$.

$S \xrightarrow{L = \mathcal{O}_S(C)} \mathbb{P}^g$, $C \subseteq S$: hyperplane section. From the above mentioned Lefschetz principle $K_{a,1}(C, K_C) = 0 \iff K_{a,1}(S, L) = 0$.

Then reinterpret $K_{a,1}(S, L)$ in terms of $S^{[a+1]} = \mathrm{Hilb}^{a+1}(S)$.

$$\Sigma = \{(x, Z) \in S \times S^{[a+1]} : x \in Z\} \quad : \quad \mathcal{L}_d := \det(E) : E \to S^{[a+1]}$$

$$\begin{array}{ccc} & \Sigma & \\ \swarrow & & \searrow \\ S & & S^{[a+1]} \end{array} \qquad E(Z) = H^0(\mathcal{L}_Z).$$

$$K_{a,1}(S, L) = \mathrm{Coker}\{H^0(S^{[a+1]}, \mathcal{L}_Z) \to H^0(\Sigma, \pi^*\mathcal{L}_Z)\}$$