The enumerative geometry of the moduli space of curves

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What are curves?

A smooth projective curve C is a compact complex Riemann surface. As such, C has a unique topological invariant, its *genus*:

$$g := \dim_{\mathbb{C}} H^0(C, \Omega^1_C) = \frac{1}{2}\dim_{\mathbb{Z}} H^1(C, \mathbb{Z}).$$

Dimension=complex/algebraic dimension. In algebraic geometry Riemann surfaces are called *algebraic curves*.
To classify curves of genus g we define the set:

$$\mathcal{M}_g := \{[C] : C \text{ is a smooth projective curve of genus } g\}.$$

The idea to view $\mathcal{M}_g$ as a space goes back to Riemann (1857). The space $\mathcal{M}_g$ should satisfy the following property:
For any family $f : X \to S$ of smooth curves, the *moduli* map

$$m = m_f : S \to \mathcal{M}_g, \ S \ni s \mapsto [f^{-1}(s)],$$

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Theorem

For $g \geq 2$, the moduli space $\mathcal{M}_g$ exists as an irreducible algebraic variety of dimension $3g - 3$.

History: O. Teichmüller (1940) provided the first *analytic* proof of the existence of $\mathcal{M}_g$. Completed by L. Bers (1960).
Another analytic approach via theta functions: F. Schottky (1890s), carried out by Baily-Borel (1966).
Algebraic approach: D. Mumford (1965), D. Gieseker (1982)-Geometric invariant theory. Mumford-Fields Medal 1974.
Modern approaches: J. Kollár, E. Viehweg et al. (1990s)-stacks.
$g = 0 : \mathcal{M}_0 = \{\text{point}\}$.
$g = 1$: Up to isomorphism every curve of genus 1 (*elliptic curve*) can be described by a plane equation

$$y^2 = x(x-1)(x-\lambda), \text{ where } \lambda \in \mathbb{C} - \{0, 1\}.$$

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Question

For $g \geq 2$: What sort of space is $\mathcal{M}_g$?

Example

$g = 3$: The generic curve of genus 3 is a plane quartic, that is, it can be given by an equation:

$$X := \{[x, y, z] \in \mathbf{P}^2 : \sum_{i+j+k=4} a_{ijk} x^i y^j z^k = 0\}.$$

Varying the coefficients, one has a *dominant* map

$$\mathbf{P}^{14} = \{[a_{ijk}] : i + j + k = 4\} - - >> \mathcal{M}_3, \quad [a_{ijk}] \mapsto X.$$

Definition

We say that an n-dimensional algebraic variety X is *unirational* if there exists a dominant rational map $f : \mathbf{P}^n - - >> X$.

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There exist rational functions $\{F_i(z_0, \ldots, z_n)\}_{i=0,\ldots,N}$ such that

$$[F_0(z_0, \ldots, z_n), \ldots, F_N(z_0, \ldots, z_n)] \in \mathbf{P}^N,$$

with $[z_0, \ldots, z_n] \in \mathbf{P}^n$ varying freely, fill-up the whole X.

The unirationality of $\mathcal{M}_g$ is equivalent to being able to write down the general genus g curve in a family depending on *free* complex parameters.

$\mathcal{M}_g$ unirational $\Longleftrightarrow$ "I have seen every curve once" (D. Mumford)

Theorem

(F. Severi 1915) $\mathcal{M}_g$ is unirational for $g \leq 10$.

Conjecture

(Severi) $\mathcal{M}_g$ is unirational for all $g \geq 0$.

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Modern classification theory of varieties:

Definition

For an n-dimensional smooth projective variety X, we set $K_X := \Omega_X^n$ its *canonical sheaf*. The function

$$\mathbb{Z} \ni l \mapsto \dim_{\mathbb{C}} H^0(X, K_X^{\otimes l}) \sim l^{\kappa(X)} \text{ grows like a polynomial in } l.$$

$\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim(X)\}$ is called *the Kodaira dimension of* X.

Remark

If X is unirational then $\kappa(X) = -\infty$ (because $K_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-n-1) < 0)$.
If $\kappa(X) = dim(X)$, then X is said to be of general type.

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Theorem

(Harris, Mumford, Eisenbud 1982-87) $\mathcal{M}_g$ *is of general type for* $g \geq 24$.

So Severi was not only wrong: He was maximally wrong!

The (lack of) unirationality of $\mathcal{M}_g$ has dramatic consequences:

Corollary

For $g \geq 24$*, if a general* $[C] \in \mathcal{M}_g$ *is the hyperplane section of a surface* S*, then* S *must be birationally equivalent to* $C \times \mathbf{P}^1$*. That is,* C *does not move in any non-trivial way.*

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The compactification of $\mathcal{M}_g$

Curves naturally degenerate in families:

$$y^2 = x(x-1)(x-\lambda)$$

When λ approaches 0, we get a *singular nodal curve*. So $\mathcal{M}_g$ cannot be compact!

Algebraic geometry is designed for compact (projective) varieties: numerical invariants, enumerative geometry, degeneration techniques.

Definition

A *stable curve* of genus g, is a connected 1-dimensional variety such that:
(1) All the singularities of C are nodes.
(2) $\#(Aut(C)) < \infty$.

Theorem

(Deligne-Mumford 1969) There exists a compact (projective) irreducible moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g. The space $\overline{\mathcal{M}}_g$ contains $\mathcal{M}_g$ as an open dense subset.

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$$\overline{\mathcal{M}}_g - \mathcal{M}_g = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{[g/2]},$$

where Δ_i are irreducible *boundary divisors* (hypersurfaces) in $\overline{\mathcal{M}}_g$. Precisely,

$$\Delta_0 := \{[\frac{C}{p \sim q}] : [C] \in \mathcal{M}_{g-1}\}^-, \text{ and}$$

$$\Delta_i := \{[C_1 \cup C_2] : [C_1] \in \mathcal{M}_i, [C_2] \in \mathcal{M}_{g-i}\}^-, \text{ for } 1 \leq i \leq [g/2].$$

The Picard group of $\overline{\mathcal{M}}_g$: Determining the Kodaira dimension of a space is a question about codimension 1 subvarieties of that space!
For a projective variety X we define

$$\mathrm{Pic}(X) := \frac{\{D = \sum_{H \subset X} a_H \cdot H, \quad a_H \in \mathbb{Z}, H \subset X \text{ hypersurface}\}}{\{\mathrm{div}(f) : f \in \mathbb{C}(X)\}} =$$

$$= \{\text{isomorphism classes of line bundles on } X\}.$$

For $0 \leq i \leq [g/2]$ we set $\delta_i := [\Delta_i] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$.

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The *Hodge class* $\lambda := [\mathbb{E}] \in \text{Pic}(\overline{\mathcal{M}}_g)$, where $\mathbb{E} \to \overline{\mathcal{M}}_g$ is a line bundle with fibres

$$\mathbb{E}[C] = \wedge^g H^0(C, \omega_C).$$

Theorem

(Harer, Arbarello-Cornalba) For $g \geq 3$, the group $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$ is freely generated by the classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$.

What is the expression of $K_{\overline{\mathcal{M}}_g}$? Answer given by Mumford:

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}.$$

Strategy to prove that $K_{\overline{\mathcal{M}}_g}$ is positive:

Find a geometric divisor (hypersurface) $D \subset \overline{\mathcal{M}}_g$ such that $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$, with $a, b_i \geq 0$ and

$$\text{slope}(D) := \frac{a}{\min_{i \geq 0} b_i} < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}.$$

Then for $\alpha, \beta > 0$ we can write that

$$K_{\overline{\mathcal{M}}_g} = \alpha \cdot \lambda + \beta \cdot D + \{ \text{ non-negative combination of } \delta_i \}.$$

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$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}.$$

Strategy to prove that $K_{\overline{\mathcal{M}}_g}$ is positive:

Find a geometric divisor (hypersurface) $D \subset \overline{\mathcal{M}}_g$ such that $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$, with $a, b_i \geq 0$ and

$$\mathrm{slope}(D) := \frac{a}{\min_{i \geq 0} b_i} < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}.$$

Then for $\alpha, \beta > 0$ we can write that

$$K_{\overline{\mathcal{M}}_g} = \alpha \cdot \lambda + \beta \cdot D + \{ \text{ non-negative combination of } \delta_i \}.$$

The *Hodge class* $\lambda := [\mathbb{E}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$, where $\mathbb{E} \to \overline{\mathcal{M}}_g$ is a line bundle with fibres

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Since $\lambda >> 0$ (Siegel modular forms), it follows that $K_{\overline{\mathcal{M}}_g} >> 0$, that is, $\overline{\mathcal{M}}_g$ is of general type.

$\overline{\mathcal{M}}_g$ of general type $\Leftrightarrow$ there exists $D \subset \overline{\mathcal{M}}_g$ with $\mathrm{slope}(D) < \frac{13}{2}$.

Geometric divisors on $\overline{\mathcal{M}}_g$

For a curve C, a *linear system* $\mathfrak{g}^r_d$ is a line bundle $L \in \mathrm{Pic}^d(C)$ with $\dim_{\mathbb{C}} H^0(C, L) \geq r + 1$.

$$L = \mathfrak{g}^r_d, \{s_i\}_{i=0}^r \subset H^0(C, L) \Leftrightarrow \phi_L : C \to \mathbf{P}^r, \phi(p) := [s_0(p), \ldots, s_r(p)].$$

$\mathfrak{g}^r_d$ = group of d points on C moving with r degrees of freedom.

Theorem

(Brill-Noether 1874; Griffiths-Harris 1980) A generic curve $[C] \in \mathcal{M}_g$ has a $\mathfrak{g}^r_d$ if and only if $\rho(g, r, d) := g - (r + 1)(g - d + r) \geq 0$.

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Fix integers $r, d \geq 1$ such that $\rho(g, r, d) = -1$. Set

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : C \text{ has a } \mathfrak{g}^r_d\}.$$

Then we expect $\mathcal{M}^r_{g,d}$ to be a hypersurface inside $\mathcal{M}_g$.

Theorem

(Eisenbud-Harris 1987). When $\rho(g, r, d) = -1$,the locus $\mathcal{M}^r_{g,d}$ is an irreducible divisor in $\mathcal{M}_g$. Moreover, the class of its closure in $\overline{\mathcal{M}}_g$ is:

$$\overline{\mathcal{M}}^r_{g,d} = c\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big).$$

Thus slope$(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1} < \frac{13}{2}$ for $g \geq 24$. This proves the Harris-Mumford-Eisenbud theorem (at least when $g+1$ is composite).

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$\mathrm{slope}(D) \geq 6 + \frac{12}{g+1}$ *for any effective divisor* $D \subset \overline{\mathcal{M}}_g$.

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(of the Slope Conjecture) $\kappa(\overline{\mathcal{M}}_g) = -\infty$ *for all* $g < 23$.

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(Sernesi 1982, Chang-Ran 1987, Verra 2005).
For $11 \leq g \leq 14$ *the space* $\mathcal{M}_g$ *is unirational. For* $g = 15, 16$ *the Kodaira dimension of* $\overline{\mathcal{M}}_g$ *is* $-\infty$.

Note: Severi's proof for $g \leq 10$ is relatively straightforward. Each of the cases $g \geq 11$ is much more difficult.

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What is the Kodaira dimension of $\overline{\mathcal{M}}_g$ *for* $17 \leq g \leq 23$?

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The Slope Conjecture and $K3$ surfaces

Definition

A $K3$ surface is a complex projective surface S which is simply-connected ($\pi_1(S) = 0$) and carries a nowhere vanishing 2-form $\omega \in H^0(S, \Omega_S^2)$ (equivalently, $K_S = 0$).

$K3=$ Kummer, Kodaira, Kähler.

Theorem

(F-2003) If $D \subset \overline{\mathcal{M}}_g$ is an effective divisor with $\mathrm{slope}(D) < 6 + \frac{12}{g+1}$, then

$$D \supset \mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \textit{ lies on a } K3 \textit{ surface}\}.$$

The locus $\mathcal{K}_g$ appears as an obstruction for a divisor on $\overline{\mathcal{M}}_g$ to have exceptionally small slope.

Theorem

(F-Popa 2004) The Slope Conjecture is false on $\overline{\mathcal{M}}_{10}$! The locus $\mathcal{K}_{10}$ is a divisor on $\mathcal{M}_{10}$ and its class is

$$\overline{\mathcal{K}}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5.$$

Thus $\text{slope}(\overline{\mathcal{K}}_{10}) = 7 < 6 + \frac{12}{11}$. Is this an isolated example or the first in a series?

Theorem

There are Koszul divisors D on $\overline{\mathcal{M}}_g$ with $s(D) < 6 + \frac{12}{g+1}$ for every $g = s(2s + si + i + 1)$, where $s \geq 2, i \geq 0$.

Question

What is $s_g := \inf_{D \subset \overline{\mathcal{M}}_g} \text{slope}(D)$?

Known: $s_g \geq O(1/g)$ (Harris 90). Conjecturally $s_g \geq 4$ for any g.

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(F 2006) The moduli space $\overline{\mathcal{M}}_{22}$ is of general type.

Strategy of proof: Construct an effective divisor $\mathcal{D}_{22} \subset \mathcal{M}_{22}$ such that slope$(\overline{\mathcal{D}}_{22}) < 6.5$.

The construction: Define the auxiliary moduli space
$\mathcal{W}^6_{25} = \{[C, L] : [C] \in \mathcal{M}_{22}, L = \mathfrak{g}^6_{25}\}$.
Since $\rho(22, 6, 25) = 1$, each curve of genus 22 has a 1-dimensional family of embeddings $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^6$ of degree 25, that is,

$$\sigma : \mathcal{W}^6_{25} \to \mathcal{M}_{22},\ \sigma[C, L] := [C]$$

has 1-dimensional fibres. On $\mathcal{W}^6_{25}$ we define two vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks 28 and 29 respectively, and a map $\phi : \mathcal{E} \to \mathcal{F}$, such that

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We define the degeneracy locus

$$\mathcal{D}_{22} := \sigma_*\{[C, L] : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ not injective}\}.$$

In other words: $\mathcal{D}_{22}$ consists of those curves $[C] \in \mathcal{M}_{22}$ which possess an embedding $C \overset{\mathfrak{g}^6_{25}}{\hookrightarrow} \mathbf{P}^6$, in which C lies on a quadric hypersurface.

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$$\overline{\mathcal{D}}_{22} = 132822768\Big(\frac{17121}{2636}\lambda - \delta_0 - \sum_{j=1}^{11} b_j\delta_j\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_{22}), \ \textit{where } b_j > 1.$$

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We define the degeneracy locus

$$\mathcal{D}_{22} := \sigma_*\{[C, L] : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ not injective}\}.$$

In other words: $\mathcal{D}_{22}$ consists of those curves $[C] \in \mathcal{M}_{22}$ which possess an embedding $C \overset{\mathfrak{g}^6_{25}}{\hookrightarrow} \mathbf{P}^6$, in which C lies on a quadric hypersurface.

Theorem

(F 2006). The locus $\mathcal{D}_{22}$ is a divisor on $\mathcal{M}_{22}$. The class of its compactification in $\overline{\mathcal{M}}_{22}$ is

$$\overline{\mathcal{D}}_{22} = 132822768\Big(\frac{17121}{2636}\lambda - \delta_0 - \sum_{j=1}^{11} b_j\delta_j\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_{22}), \ \textit{where } b_j > 1.$$

Thus

$$\mathrm{slope}(\overline{\mathcal{D}}_{22}) = \frac{17121}{2636} = 6.49506... < 6.5,$$

so $\overline{\mathcal{M}}_{22}$ is of general type!

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