BMS Algebraic Geometry 2008, Problem Set Nr. 4

1. Let \( H_i \) and \( H_j \) be hyperplanes in \( \mathbb{P}^n \) defined by \( x_i = 0 \) and \( x_j = 0 \), with \( i \neq j \). Show that any regular function on \( \mathbb{P}^n - (H_i \cap H_j) \) is constant.

2. Let \( \mathbb{P}^n \) be a hyperplane of \( \mathbb{P}^{n+1} \) and we fix a point \( p \in \mathbb{P}^{n+1} - \mathbb{P}^n \). We define a map \( \phi : \mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n \) by \( \phi(x) := \) the point of intersection of the line containing \( p \) and \( x \) with the hyperplane \( \mathbb{P}^n \).
   - Show that \( \phi \) is a morphism of prevarieties.
   - Let \( Y \subset \mathbb{P}^3 \) be the twisted cubic curve given by points \( [x_0, x_1, x_2, x_3] = [s^3, s^2t, st^2, t^3] \), where \( [s, t] \in \mathbb{P}^1 \). Assume that \( p = [0, 0, 1, 0] \in \mathbb{P}^3 \) and let \( \mathbb{P}^2 \) be the hyperplane \( x_2 = 0 \). Find the equations in the plane of the curve \( \phi(Y) \).

3. Let \( X \) be any prevariety and \( p \in X \). Show that there is a 1 : 1 correspondence between the prime ideals of the local ring \( \mathcal{O}_{X,p} \) and the closed subvarieties of \( X \) containing \( p \).

4. We fix \( n, d > 0 \) and let \( M_0, M_1, \ldots, M_N \) be all monomials of degree \( d \) in the variables \( x_0, \ldots, x_d \), where \( N = \binom{n+d}{n} - 1 \). We define the map
   \[
   \rho_d : \mathbb{P}^n \to \mathbb{P}^N
   \]
   obtained by sending a point \( p = [a_0, \ldots, a_n] \) to the point \( \rho_d(p) = [M_0(p), \ldots, M_N(p)] \) obtained by evaluating all the monomials \( M_j \) at the point \( (a_0, \ldots, a_n) \). This is called the \( d \)-uple embedding of \( \mathbb{P}^n \) in \( \mathbb{P}^N \).
   - Describe this map in the case \( n = 1, d = 2 \). What is the image of \( \rho_2 \)?
   - Prove that the image \( \rho_d(\mathbb{P}^n) \) is always a projective subvariety of \( \mathbb{P}^N \) given by some homogeneous ideal \( I \subset k[X_0, \ldots, x_n] \).
   - Show that the 3-uple embedding of \( \mathbb{P}^1 \) into \( \mathbb{P}^3 \) has as image the twisted cubic curve in \( \mathbb{P}^3 \).

5. Let \( Y \subset \mathbb{P}^5 \) be the 2-uple embedding \( \rho_2; \mathbb{P}^2 \to \mathbb{P}^5 \). Describe the homogeneous ideal of \( Y \).