SINGULARITIES OF THETA DIVISORS AND THE GEOMETRY OF $\mathcal{A}_5$

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Abstract. We study the codimension two locus H in $\mathcal{A}_g$ consisting of principally polarized abelian varieties whose theta divisor has a singularity that is not an ordinary double point. We compute the class $[H] \in CH^2(\mathcal{A}_g)$ for every g. For $g = 4$, this turns out to be the locus of Jacobians with a vanishing theta-null. For $g = 5$, via the Prym map we show that $H \subset \mathcal{A}_5$ has two components, both unirational, which we describe completely. We then determine the slope of the effective cone of $\overline{\mathcal{A}}_5$ and show that the component $\overline{N_0'}$ of the Andreotti-Mayer divisor has minimal slope and the Iitaka dimension $\kappa(\overline{\mathcal{A}}_5, \overline{N_0'})$ is equal to zero.

Introduction

The theta divisor Θ of a generic principally polarized abelian variety (ppav) is smooth. The ppav (A, Θ) with a singular theta divisor form the Andreotti-Mayer divisor N_0 in the moduli space $\mathcal{A}_g$, see [AM67] and [Bea77]. The divisor N_0 has two irreducible components, see [Mum83] and [Deb92], which are denoted θ_{null} and N_0': here θ_{null} denotes the locus of ppav for which the theta divisor has a singularity at a two-torsion point, and N_0' is the closure of the locus of ppav for which the theta divisor has a singularity not at a two-torsion point. The theta divisor Θ of a generic ppav $(A, \Theta) \in \theta_{\text{null}}$ has a unique singular point, which is a double point. Similarly, the theta divisor of a generic element of N_0' has two distinct double singular points x and $-x$. Using this fact, one can naturally assign multiplicities to both components of N_0 such the following equality of cycles holds, see [Mum83],[Deb92]

$$N_0 = \theta_{\text{null}} + 2N_0'. \tag{1}$$

As it could be expected, generically for both components the double point is an ordinary double point (that is, the quadratic tangent cone to the theta divisor at such a point has maximal rank g — equivalently, the Hessian matrix of the theta function at such a point is non-degenerate). Motivated by a conjecture of H. Farkas [HF06], in [GSM08] two of the authors considered the locus in θ_{null} in genus 4 where the double point is not ordinary. In [GSM07] this study was extended to arbitrary g, considering the sublocus $\theta_{\text{null}}^{g-1} \subset \theta_{\text{null}}$ parameterizing ppav (A, Θ) with a singularity at a two-torsion point, that is not an ordinary double point of Θ. In particular it has been proved that

$$\theta_{\rm null}^{g-1}\subset\theta_{\rm null}\cap N_0'. \tag{2}$$

In fact the approach yielded a more precise statement: Let $\phi:\mathcal{X}_g\to\mathcal{A}_g$ be the universal family of ppav over the orbifold $\mathcal{A}_g$ and $\mathcal{S}\subset\mathcal{X}_g$ be the locus of singular points of theta divisors. Note that $\mathcal{S}$ can be viewed as a subscheme of $\mathcal{X}_g$ given by the vanishing of the theta functions and all its partial derivatives, see Section 1. Then $\mathcal{S}$ decomposes into three equidimensional components [Deb92]: $\mathcal{S}_{\rm null}$, projecting to $\theta_{\rm null}$, $\mathcal{S}'$, projecting to N_0', and $\mathcal{S}_{\rm dec}$, projecting (with $(g-2)$-dimensional fibers) onto $\mathcal{A}_1\times\mathcal{A}_{g-1}$. It is proved in [GSM07] that set-theoretically, $\theta_{\rm null}^{g-1}$ is the image in $\mathcal{A}_g$ of the intersection $\mathcal{S}_{\rm null}\cap\mathcal{S}'$. An alternative proof of these results has been found by Smith and Varley [SV12a],[SV12b].

It is natural to investigate the non-ordinary double points on the other component N_0' of the Andreotti-Mayer divisor. Similarly to $\theta_{\rm null}^{g-1}$, we define $N_0'^{g-1}$, or, to simplify notation, H, to be the closure in N_0' of the locus of ppav whose theta divisor has a non-ordinary double point singularity. Note that H is the pushforward under ϕ of a subscheme $\mathcal{H}$ of $\mathcal{X}_g$ given by a Hessian condition on theta functions. In particular H can be viewed as a codimension 2 cycle (with multiplicities) on $\mathcal{A}_g$. Since an explicit modular form defining N_0' and the singular point is not known, we consider the cycle

$$N_0^{g-1}:=\theta_{\rm null}^{g-1}+2N_0'^{g-1}=\theta_{\rm null}^{g-1}+2H. \tag{3}$$

We first note that $\theta_{\rm null}^{g-1}$ is a subset of H. Then, after recalling that the Andreotti-Mayer loci N_i are defined as consisting of ppav $(A,\Theta)\in\mathcal{A}_g$ with dim $\mathrm{Sing}(\Theta)\geq i$, we establish the set-theoretical inclusion $N_i\subset H$, for $i\geq 1$. From this we deduce:

Proposition 0.1. *For $g\geq 5$ we have $\theta_{\rm null}^{g-1}\subsetneq H$.*

To further understand the situation, especially in low genus, we compute the class:

Theorem 0.2. *The class of the cycle H inside $\mathcal{A}_g$ is equal to*

$$[H]=\left(\frac{g!}{16}(g^3+7g^2+18g+24)-(g+4)2^{g-4}(2^g+1)\right)\lambda_1^2\in CH^2(\mathcal{A}_g).$$

As usual, $\lambda_1:=c_1(\mathbb{E})$ denotes the first Chern class of the Hodge bundle and CH^i denotes the $\mathbf{Q}$-vector space parameterizing algebraic cycles of codimension i with rational coefficients modulo rational equivalence. Comparing classes and considering the cycle-theoretic inclusion $3\theta_{\rm null}^3\subset H$, we get the following result, see Section 4 for details:

Theorem 0.3. *In genus 4 we have the set-theoretic equality $\theta_{\rm null}^3=H$.*

We then turn to genus 5 with the aim of obtaining a geometric description of $H\subset\mathcal{A}_5$ via the dominant Prym map $P:\mathcal{R}_6\to\mathcal{A}_5$. A key role in the study of the Prym map is played by its branch divisor, which in this case equals $N_0' \subset \mathcal{A}_5$, and its ramification divisor $\mathcal{Q} \subset \mathcal{R}_6$. We introduce the *antiramification* divisor $\mathcal{U} \subset \mathcal{R}_6$ defined cycle-theoretically by the equality

$$P^*(N_0') = 2\mathcal{Q} + \mathcal{U}.$$

Using the geometry of the Prym map, we describe both $\mathcal{Q}$ and $\mathcal{U}$ explicitly in terms of Prym-Brill-Noether theory. For a Prym curve $(C, \eta) \in \mathcal{R}_g$ and an integer $r \geq -1$, we recall that $V_r(C, \eta)$ denotes the Prym-Brill-Noether locus (see Section 5 for a precise definition). It is known [Wel85] that $V_r(C, \eta)$ is a Lagrangian determinantal variety of expected dimension $g - 1 - \binom{r+1}{2}$. We denote $\pi : \mathcal{R}_g \to \mathcal{M}_g$ the forgetful map. Our result is the following:

Theorem 0.4. *The ramification divisor $\mathcal{Q}$ of the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ equals the Prym-Brill-Noether divisor in $\mathcal{R}_6$, that is,*

$$\mathcal{Q} = \{(C, \eta) \in \mathcal{R}_6 : V_3(C, \eta) \neq 0\}.$$

The antiramification divisor is the pull-back of the Gieseker-Petri divisor from $\mathcal{M}_6$, that is, $\mathcal{U} = \pi^*(\mathcal{GP}^1_{6,4})$. The divisor $\mathcal{Q}$ is irreducible and reduced.

As the referee pointed out to us, the irreducibility of $\mathcal{Q}$ (as well as that of $\mathcal{U}$) also follows from Donagi's results [Don92] on the monodromy of the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$. Apart from the Brill-Noether characterization provided by Theorem 0.4, the divisor $\mathcal{Q}$ has yet a third (respectively a fourth!) geometric incarnation as the closure of the locus of points $(C, \eta) \in \mathcal{R}_6$ with a linear series $L \in W^2_6(C)$, such that the sextic model $\varphi_L(C) \subset \mathbf{P}^2$ has a totally tangent conic, see Theorem 8.1 (respectively as the locus of section $(C, \eta) \in \mathcal{R}_6$ of Nikulin surfaces [FV11]). The rich geometry of $\mathcal{Q}$ enables us to (i) compute the classes of the closures $\overline{\mathcal{Q}}$ and $\overline{\mathcal{U}}$ inside the Deligne-Mumford compactification $\overline{\mathcal{R}}_6$, then (ii) determine explicit codimension two cycles in $\mathcal{R}_6$ that dominate the irreducible components of H. In this way we find a complete geometric characterization of 5-dimensional ppav whose theta divisor has a non-ordinary double point. First we characterize θ^4_{null} as the image under P of a certain component of the intersection $\mathcal{Q} \cap P^*(\theta_{\mathrm{null}})$:

Theorem 0.5. *A ppav $(A, \Theta) \in \mathcal{A}_5$ belongs to θ^4_{null} if an only if it is lies in the closure of the locus of Prym varieties $P(C, \eta)$, where $(C, \eta) \in \mathcal{R}_6$ is a curve with two vanishing theta characteristics θ_1 and θ_2, such that*

$$\eta = \theta_1 \otimes \theta_2^\vee.$$

Furthermore, θ^4_{null} is unirational and $[\theta^4_{\mathrm{null}}] = 27 \cdot 44\, \lambda_1^2 \in CH^2(\mathcal{A}_5)$.

Denoting by $\mathcal{Q}_5 \subset \mathcal{R}_6$ the locus of Prym curves $(C, \eta = \theta_1 \otimes \theta_2^\vee)$ as above, we prove that $\mathcal{Q}_5$ (and hence θ^4_{null} which is the closure of $P(\mathcal{Q}_5)$ in $\mathcal{A}_5$) is unirational, by realizing its general element as a nodal curve

$$C \in \left| \mathcal{I}^2_{R_1 \cdot R_2 / \mathbf{P}^1 \times \mathbf{P}^1}(5, 5) \right|,$$

where $R_1 \in |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(3, 1)|$ and $R_2 \in |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 3)|$, with the vanishing theta-nulls θ_1 and θ_2 being induced by the projections on the two factors.

Observing that $[H] \neq [\theta^4_{\rm null}]$ in $CH^2(\mathcal{A}_5)$, the locus H must have extra irreducible components corresponding to ppav with a non-ordinary singularity that occurs generically not at a two-torsion point. We denote by $H_1 \subset \mathcal{A}_5$ the union of these components, so that at the level of cycles

$$H = \theta^4_{\rm null} + H_1,$$

where $[H_1] = 27 \cdot 49\, \lambda_1^2$. We have the following characterization of H_1:

Theorem 0.6. *The locus H_1 is unirational and its general point corresponds to a Prym variety $P(C, \eta)$, where $(C, \eta) \in \mathcal{R}_6$ is a Prym curve such that $\eta \in W_4(C) - W^1_4(C)$ and $K_C \otimes \eta$ is very ample.*

As an application of this circle of ideas, we determine the slope of $\overline{\mathcal{A}}_5$. Let $\overline{\mathcal{A}}_g$ be the perfect cone (first Voronoi) compactification of $\mathcal{A}_g$ — this is the toroidal compactification of $\mathcal{A}_g$ constructed using the first Voronoi (perfect) fan decomposition of the cone of semi-positive definite quadratic forms with rational nullspace (see e.g. [Vor1908] for the origins, and [SB06] for recent progress). The Picard group of $\overline{\mathcal{A}}_g$ with rational coefficients has rank 2 (for $g \geq 2$), and it is generated by the first Chern class λ_1 of the Hodge bundle and the class of the *irreducible* boundary divisor $D := \overline{\mathcal{A}}_g - \mathcal{A}_g$. The slope of an effective divisor $E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)$ is defined as the quantity

$$s(E) := \inf \left\{ \frac{a}{b} : a, b > 0,\ a\lambda_1 - b[D] - [E] = c[D],\ c > 0 \right\}.$$

If E is an effective divisor on $\overline{\mathcal{A}}_g$ with no component supported on the boundary and $[E] = a\lambda_1 - bD$, then $s(E) := \frac{a}{b} \geq 0$. One then defines the slope (of the effective cone) of the moduli space as $s(\overline{\mathcal{A}}_g) := \inf_{E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)} s(E)$. This important invariant governs to a large extent the birational geometry of $\overline{\mathcal{A}}_g$; for instance $\overline{\mathcal{A}}_g$ is of general type if $s(\overline{\mathcal{A}}_g) < g+1$, and $\overline{\mathcal{A}}_g$ is uniruled when $s(\overline{\mathcal{A}}_g) > g+1$. Any effective divisor class calculation on $\overline{\mathcal{A}}_g$ provides an upper bound for $s(\overline{\mathcal{A}}_g)$. It is known [SM92] that $s(\overline{\mathcal{A}}_4) = 8$ (and the minimal slope is computed by the divisor $\overline{\mathcal{J}}_4$ of Jacobians). In the next case $g = 5$, the class of the closure of the Andreotti-Mayer divisor is $[\overline{N_0'}] = 108\lambda_1 - 14D$, giving the upper bound $s(\overline{\mathcal{A}}_5) \leq \frac{54}{7}$.

Theorem 0.7. *The slope of $\overline{\mathcal{A}}_5$ is computed by $\overline{N_0'}$, that is, $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, $\kappa(\overline{\mathcal{A}}_5, \overline{N_0'}) = 0$, that is, the only effective divisors on $\overline{\mathcal{A}}_5$ having minimal slope are the multiples of $\overline{N_0'}$.*

To prove this result, we define a partial compactification $\widetilde{\mathcal{R}}_6$ of $\mathcal{R}_6$ and via the (rational) Prym map $P : \widetilde{\mathcal{R}}_6 \dashrightarrow \overline{\mathcal{A}}_5$ we investigate the pull-back

$$P^*(\overline{N_0'}) = 2\widetilde{\mathcal{Q}} + \widetilde{\mathcal{U}} + 20\delta_0'',$$

where $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{U}}$ denote the closure of $\mathcal{Q}$ and $\mathcal{U}$ respectively in $\widetilde{\mathcal{R}}_6$, and δ_0'' is the divisor of degenerate Wirtinger double covers (see Section 6 for precise definitions). Since each of the divisors appearing in this linear system admits a uniruled parametrization in terms of plane sextics having a totally tangent conic, we are ultimately able to establish the rigidity of any multiple of $\overline{N_0'}$.

A final application concerns the divisor in $\overline{\mathcal{A}}_5$ of Pryms obtained from branched covers. The Prym variety associated to a double cover $f:\tilde{C}\to C$ branched over two points is still a ppav. When $g(C)=5$ (and only in this case), the Prym varieties constructed in this way form an irreducible divisor $\mathcal{D}^{\mathrm{ram}}:=P_*(\Delta_0^{\mathrm{ram}})$ inside the moduli space. We have the following formula for the class of the closure of $\mathcal{D}^{\mathrm{ram}}$ in $\overline{\mathcal{A}}_5$:

Theorem 0.8. $[\overline{\mathcal{D}^{\mathrm{ram}}}]=4\,(153\lambda_1-19D)\in CH^1(\overline{\mathcal{A}}_5)$.

Since the classes $[P^*(\overline{\mathcal{D}^{\mathrm{ram}}})]$ and δ_0^{ram} are not proportional, one obtains that the general Prym variety $(A,\Theta)\in\mathcal{D}^{\mathrm{ram}}$ obtained from a ramified cover $\tilde{C}\to C$ (with $g(C)=5$ and $g(\tilde{C})=10$), is also the Prym variety induced by an étale cover $\tilde{C}_1\to C_1$ (with $g(C_1)=6$ and $g(\tilde{C}_1)=11$).

We summarize the structure of the paper. The cycle structure of H and $\theta_{\mathrm{null}}^{g-1}$ is described in Section 2, whereas the classes $[\theta_{\mathrm{null}}^{g-1}],[H]\in CH^2(\mathcal{A}_g)$ are computed in Section 3. The particular case $g=4$ is treated in Section 4. After some background on singularities of Prym theta divisors (Section 5), the different geometric realizations of the ramification and antiramification divisors of the Prym map $P:\mathcal{R}_6\to\mathcal{A}_5$, as well as the corresponding class calculations on $\widetilde{\mathcal{R}}_6$ are presented in Sections 6 and 7. A proof of Theorem 0.7, thus determining the slope of $\overline{\mathcal{A}}_5$ is given in Section 8. The final sections of the paper are devoted to a complete geometric description in terms of Pryms of the two components of the cycle H in genus 5, see Theorems 0.5 and 0.6. We close by expressing our thanks to the referee for the many pertinent comments which clearly improved the presentation of the paper.

1. Theta divisors and their singularities

In this section we recall notation, definitions, as well as some results from [GSM08]. We denote $\mathbb{H}_g$ the *Siegel upper half-space*, i.e. the set of symmetric complex $g\times g$ matrices τ with positive definite imaginary part. If $\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{Sp}(2g,\mathbf{Z})$ is a symplectic matrix in $g\times g$ block form, then its action on $\tau\in\mathbb{H}_g$ is defined by $\sigma\cdot\tau:=(a\tau+b)(c\tau+d)^{-1}$, and the moduli space of complex principally polarized abelian variety (ppav for short) is the quotient $\mathcal{A}_g=\mathbb{H}_g/\mathrm{Sp}(2g,\mathbf{Z})$, parameterizing pairs $(A_\tau,\,\Theta_\tau)$ with $A_\tau=\mathbf{C}^g/\mathbf{Z}^g\tau+\mathbf{Z}^g$, an abelian variety and Θ_τ the (symmetric) polarization bundle. We denote by $A_\tau[2]$ the group of two-torsion points of A_τ. Let $\varepsilon,\delta\in(\mathbf{Z}/2\mathbf{Z})^g$, thought of as vectors of zeros and ones; then $x=\tau\varepsilon/2+\delta/2\in A_\tau[2]$, and the shifted bundle $t_x^*\Theta$ is still a symmetric line bundle. Up to a multiplicative constant the unique section of the above bundle is given by the *theta function with* *characteristic* $[\varepsilon, \delta]$ defined by

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) := \sum_{m \in \mathbf{Z}^g} \exp \pi i \left[{}^t(m + \frac{\varepsilon}{2})\tau(m + \frac{\varepsilon}{2}) + 2\, {}^t(m + \frac{\varepsilon}{2})(z + \frac{\delta}{2}) \right].$$

We shall write $\theta(\tau, z)$ for the theta function with characteristic $[0, 0]$. The zero scheme of $\theta(\tau, z)$, as a function of $z \in A_\tau$, defines the principal polarization Θ_τ on A_τ.

Theta functions satisfy the heat equation

$$\frac{\partial^2 \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)}{\partial z_j \partial z_k} = 2\pi i (1 + \delta_{j,k}) \frac{\partial \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)}{\partial \tau_{jk}},$$

(where $\delta_{j,k}$ is Kronecker's symbol).

The *characteristic* $[\varepsilon, \delta]$ is called *even* or *odd* corresponding to whether the scalar product ${}^t\varepsilon\delta \in \mathbf{Z}/2\mathbf{Z}$ is zero or one. Consequently, depending on the characteristic, $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)$ is even or odd as a function of z. A *theta constant* is the evaluation at $z = 0$ of a theta function. All odd theta constants of course vanish identically in τ.

A holomorphic function $f : \mathbb{H}_g \to \mathbf{C}$ is called a *modular form* of weight k with respect to a finite index subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbf{Z})$ if

$$f(\sigma \cdot \tau) = \det(c\tau + d)^k f(\tau) \qquad \forall \tau \in \mathbb{H}_g, \forall \sigma \in \Gamma,$$

and if additionally f is holomorphic at all cusps of $\mathbb{H}_g/\Gamma$. Theta constants with characteristics are modular forms of weight $\frac{1}{2}$ with respect to a finite index subgroup $\Gamma_g(4, 8) \subset \mathrm{Sp}(2g, \mathbf{Z})$. We refer to [Igu72] for a detailed study of theta functions.

We denote by

$$\phi : \mathcal{X}_g = \mathbb{H}_g \times \mathbf{C}^g / (\mathrm{Sp}(2g, \mathbf{Z}) \rtimes \mathbf{Z})^{2g}) \to \mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbf{Z})$$

the universal family of ppav, and let $\Theta_g \subset \mathcal{X}_g$ be the universal theta divisor — the zero locus of $\theta(\tau, z)$. Following Mumford [Mum83], we denote by $\mathcal{S} := \mathrm{Sing}_{\mathrm{vert}}\, \Theta_g$ the locus of singular points of theta divisors of ppav:

(4)

$$\mathcal{S} = \bigcup_{\tau \in \mathcal{A}_g} \mathrm{Sing}\, \Theta_\tau = \left\{ (\tau, z) \in \mathbb{H}_g \times \mathbf{C}^g : \theta(\tau, z) = \frac{\partial \theta}{\partial z_i}(\tau, z) = 0,\ i = 1, \ldots, g \right\}$$

(computationally, by an abuse of notation, we will often work locally on $\mathcal{S}$, thinking of it as a locus inside the cover $\mathbb{H}_g \times \mathbf{C}^g$ of $\mathcal{X}_g$). It is known that $\mathcal{S} \subset \mathcal{X}_g$ is of pure codimension $g + 1$, and has three irreducible components [CvdG00], denoted $\mathcal{S}_{\mathrm{null}}$, $\mathcal{S}_{\mathrm{dec}}$, and $\mathcal{S}'$. Here $\mathcal{S}_{\mathrm{null}}$ denotes the locus of even two-torsion points that lie on the theta divisor, given locally by $g + 1$ equations

(5)

$$\mathcal{S}_{\mathrm{null}} := \left\{ (\tau, z) \in \mathcal{X}_g : \theta(\tau, z) = 0,\ z = (\tau\varepsilon + \delta)/2 \text{ for some } [\varepsilon, \delta] \in (\mathbf{Z}/2\mathbf{Z})^{2g}_{\mathrm{even}} \right\}.$$

To define $\mathcal{S}_{\rm dec}$, recall that a ppav is called decomposable if it is isomorphic to a product of lower-dimensional ppav. We denote then

$$\mathcal{S}_{\rm dec} := \mathcal{S} \cap \phi^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}). \tag{6}$$

Since the theta divisor of a product $(A_1, \Theta_1) \times (A_2, \Theta_2)$ is given by the union $(\Theta_1 \times A_2) \cup (A_1 \times \Theta_2)$, its singular locus contains $\Theta_1 \times \Theta_2$ and is of codimension 2 (see the work [EL97] of Ein and Lazarsfeld for a proof of a conjecture [ADC84] of Arbarello and De Concini that N_{g-2} is in fact equal to the decomposable locus). Thus the fibers of $\mathcal{S}_{\rm dec} \to \mathcal{A}_{1,g-1}$ are all of dimension $g-2$, and the codimension of $\mathcal{S}_{\rm dec} \subset \mathcal{X}_g$ is equal to $g+1$. (We note that any other locus of products $\mathcal{A}_h \times \mathcal{A}_{g-h}$ has codimension $h(g-h)$, and contributes no irreducible component of $\mathcal{S}$).

Finally, $\mathcal{S}'$ is the closure of the locus of singular points of theta divisors of indecomposable ppav that are not two-torsion points. Observe that $\mathcal{S}_{\rm null}, \mathcal{S}'$, and $\mathcal{S}_{\rm dec}$ all come equipped with an induced structure as determinantal subschemes of $\mathbb{H}_g \times \mathbf{C}^g$.

The Andreotti-Mayer divisor is then defined (as a cycle) by

$$N_0 := \phi_*(\mathcal{S}) = \{\tau \in \mathcal{A}_g : \operatorname{Sing}\Theta_\tau \neq \emptyset\}.$$

It can be shown that N_0 is a divisor in $\mathcal{A}_g$, which has at most two irreducible components, see [Deb92],[Mum83].

The *theta-null divisor* $\theta_{\rm null} \subset \mathcal{A}_g$ is the zero locus of the modular form

$$F_g(\tau) := \prod_{m \,\rm even} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, 0).$$

Geometrically, it is the locus of ppav for which an even two-torsion point lies on the theta divisor, and it can be shown that $\theta_{\rm null} = \phi_*(\mathcal{S}_{\rm null})$, viewed as an equality of cycles. Similarly for the other component we have $N_0' = \frac{1}{2}\phi_*(\mathcal{S}')$ (the one half appears because a generic ppav in N_0' has two singular points $\pm x$ on the theta divisor).

Remark 1.1. The two components of N_0 are zero loci of modular forms (with some character χ in genus 1 and 2): $\theta_{\rm null}$ is the zero locus of the modular form F_g of weight $2^{g-2}(2^g+1)$, while N_0' must be the zero locus of some modular form I_g of weight $g!(g+3)/4 - 2^{g-3}(2^g+1)$ (the class, and thus the weight, was computed by Mumford [Mum83]). Unlike the explicit formula for F_g, the modular form I_g is only known explicitly for $g = 4$, in which case it is the so called Schottky form [Igu81a],[Igu81b]. Various approaches to constructing I_g explicitly were developed in [Yos99],[KSM02].

2. Double points on theta divisors that are not ordinary double points

We shall now concentrate on studying the local structure of a theta divisor near its singular point. For this, we look at the tangent space to $\mathcal{S}$ and the map between the tangent spaces.

Proposition 2.1. *Let $x_0 = (\tau_0, z_0)$ be a smooth point of $\mathcal{S}$. Then the map $(d\phi)_{x_0} : T_{x_0}(\mathcal{S}) \to T_{\tau_0}(N_0)$ is injective if and only if the Hessian matrix*

$$H(x_0) := \begin{pmatrix} \frac{\partial^2\theta}{\partial z_1 \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_1 \partial z_g}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\theta}{\partial z_g \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_g \partial z_g}(x_0) \end{pmatrix}$$

has rank g.

Proof. Since the subvariety $\mathcal{S} \subset \mathcal{X}_g$ is defined by the $g+1$ equations (4), the point x_0 is smooth if and only if the $(\frac{g(g+1)}{2} + g) \times (g+1)$ matrix

$$M(\tau_0, z_0) := \begin{pmatrix} \frac{\partial\theta}{\partial \tau_{11}} & \dots & \frac{\partial\theta}{\partial \tau_{gg}} & \frac{\partial\theta}{\partial z_1} & \dots & \frac{\partial\theta}{\partial z_g} \\ \frac{\partial^2\theta}{\partial z_1 \partial \tau_{11}} & \dots & \frac{\partial^2\theta}{\partial z_1 \partial \tau_{gg}} & \frac{\partial^2\theta}{\partial z_1 \partial z_1} & \dots & \frac{\partial^2\theta}{\partial z_1 \partial z_g} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\theta}{\partial z_1 \partial \tau_{11}} & \dots & \frac{\partial^2\theta}{\partial z_1 \partial \tau_{gg}} & \frac{\partial^2\theta}{\partial z_g \partial z_1} & \dots & \frac{\partial^2\theta}{\partial z_g \partial z_g} \end{pmatrix}$$

evaluated at $x_0 = (\tau_0, z_0)$ has rank $g+1$. We compute

$$M(\tau_0, z_0) = \begin{pmatrix} \frac{\partial\theta}{\partial \tau_{11}}(x_0) & \dots & \frac{\partial\theta}{\partial \tau_{gg}}(x_0) & 0 & \dots & 0 \\ \frac{\partial^2\theta}{\partial z_1 \partial \tau_{11}}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_1 \partial \tau_{gg}}(x_0) & \frac{\partial^2\theta}{\partial z_1 \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_1 \partial z_g}(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\theta}{\partial z_1 \partial \tau_{11}}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_1 \partial \tau_{gg}}(x_0) & \frac{\partial^2\theta}{\partial z_g \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_g \partial z_g}(x_0) \end{pmatrix}$$

Since the map ϕ is the projection on the first $\frac{g(g+1)}{2}$ coordinates, the proposition follows. $\square$

Remark 2.2. From the heat equation for the theta function it follows that if the Hessian matrix $H(x_0)$ has rank g, then x_0 is a smooth point of $\mathcal{S}$.

We also note that from the product rule for differentiation and the heat equation it follows that the second derivative

$$\frac{\partial^2\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, z)}{\partial z_j \partial z_k}|_{z=0}$$

restricted to the locus $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau, 0) = 0$ is also a modular form for $\Gamma_g(4, 8)$.

Since we have different, easier to handle, local defining equations (5) for $\mathcal{S}_{\text{null}}$, we can obtain better results in this case.

Proposition 2.3. *A point $x_0 \in \mathcal{S}_{\text{null}}$ is a smooth point of $\mathcal{S}_{\text{null}}$ unless $\frac{\partial\theta}{\partial\tau_{ij}}(x_0) = 0$ for all $1 \le i, j \le g$. The map $(d\phi)_{x_0}$ is injective if and only if the Hessian matrix $H(x_0)$ has rank g.*

Remark 2.4. If $x_0 = (\tau_0, z_0)$ is a smooth point of $\mathcal{S}_{\text{null}}$, while τ_0 is singular in θ_{null}, this implies that at least two different theta constants vanish at τ_0.

Using the above framework, we get a complete description of the intersection $\mathcal{S}_{\text{null}} \cap \mathcal{S}'$, obtaining thus an easier proof of one of the main results of [GSM07].

Proposition 2.5. *For $x_0 \in \mathcal{S}_{\text{null}}$, the point x_0 lies in $\mathcal{S}'$ if and only if the rank of $H(x_0)$ is less than g.*

Proof. If $x_0 \in \mathcal{S}' \cap \mathcal{S}_{\text{null}}$, then it is a singular point in $\mathcal{S}$, hence the rank of $H(x_0)$ is less than g by the above proposition. To obtain a proof in the other direction, since z_0 is a two-torsion point, the matrix $M(\tau_0, z_0)$ appearing in the proof of the proposition above has the form

$$M(x_0) = \begin{pmatrix} \frac{\partial\theta}{\partial\tau_{11}}(x_0) & \dots & \frac{\partial\theta}{\partial\tau_{gg}}(x_0) & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\partial^2\theta}{\partial z_1 \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_1 \partial z_g}(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial^2\theta}{\partial z_g \partial z_1}(x_0) & \dots & \frac{\partial^2\theta}{\partial z_g \partial z_g}(x_0) \end{pmatrix}$$

Hence if the rank of $H(x_0)$ is less than g, x_0 is a singular point of $\mathcal{S}$; thus either it is a singular point of $\mathcal{S}_{\text{null}}$, or it lies in the intersection $\mathcal{S}_{\text{null}} \cap \mathcal{S}'$. The first case cannot happen for dimensional reasons (the singular locus of $\mathcal{S}_{\text{null}}$ is codimension at least 2 within $\mathcal{S}_{\text{null}}$, see also [CvdG00]), and thus we must have $x_0 \in \mathcal{S}_{\text{null}} \cap \mathcal{S}'$. ☐

Corollary 2.6. *Set theoretically we have*

$$\phi(\mathcal{S}_{\text{null}} \cap \mathcal{S}') = \theta_{\text{null}}^{g-1}.$$

Remark 2.7. From the previous proof it also follows that $\operatorname{Sing} \mathcal{S}_{\text{null}} \subset \mathcal{S}_{\text{null}} \cap \mathcal{S}'$.

Our further investigation will consider the subvariety

$$\mathcal{H} := \mathcal{S}'^{g-1} := \{x_0 = (\tau_0, z_0) \in \mathcal{S}' : \operatorname{rk} H(x_0) < g\} \subset \mathcal{X}_g$$

(notice that since the derivative of a section of a line bundle is a section of the same bundle when restricted to the zero locus of the section, this is an algebraic subvariety of $\mathcal{X}_g$). Note that $\mathcal{H}$, being defined by explicit equations in the (derivatives of) theta functions, comes equipped with a scheme structure. Then we define the pushforward cycle

$$2H := 2N_0'^{g-1} := \phi_*(\mathcal{H}) \subset \mathcal{A}_g$$

Unlike in the case of the theta-null, $\mathcal{H} \not\subset \operatorname{Sing} \mathcal{S}$. Indeed, if z_0 is not a two-torsion point, the condition $\operatorname{rk} H(x_0) < g$ does not imply that $\operatorname{rk} M(\tau_0, z_0) < g+1$, as the matrix M at z_0 does not have as many zero entries as in the theta-null case. Still, we have the set-theoretic inclusions

$$\mathcal{S}_{\text{null}} \cap \mathcal{S}' \subset \mathcal{H} \quad \text{and} \quad \theta_{\text{null}}^{g-1} \subset H.$$

The locus $\mathcal{H}$ is given locally by $(g+2)$ equations (the $g+1$ equations for $\mathcal{S}'$ together with the vanishing of the Hessian determinant), and thus each irreducible component of $\mathcal{H}$ has codimension at most $g+2$ in $\mathcal{X}_g$. However, we note that $\mathcal{S}_{\rm dec} \subset \mathcal{H} \subset \mathcal{S}$ is an irreducible component of codimension $g+1$. We now check that all other irreducible components of $\mathcal{H}$ are indeed of expected codimension $g+2$. Indeed, we first note that by the results of Ciliberto and van der Geer [CvdG08] the Andreotti-Mayer locus N_k (parameterizing ppav whose theta divisor has singular locus of dimension at least k) with $1 \le k \le g-3$ has codimension at least $k+2$ in $\mathcal{A}_g$, and thus its preimage in $\mathcal{S}$ cannot be an irreducible component of $\mathcal{S}$ for dimension reasons. Now for both $\mathcal{S}'$ and $\mathcal{S}_{\rm null}$ it is known that generically the singular points of the theta divisors are ordinary double points, and thus $\mathcal{H}$ cannot be equal to either of these loci. Finally, by the results of Ein and Lazarsfeld [EL97] the locus N_{g-2} is equal to the locus of indecomposable ppav, and each component $\mathcal{A}_h \times \mathcal{A}_{g-h}$ of it has codimension too high, except for $h=1$.

The above discussion leads to the following result:

Proposition 2.8. *The Andreotti-Mayer locus N_1 is contained in H.*

Proof. Indeed, for $\tau_0 \in N_1$ we let $z(t) \subset \operatorname{Sing}\Theta_\tau$ be a curve of singular points such that $z(0) = z_0$ is a smooth point of the curve. Differentiating (4) with respect to t, we get g non-zero equations (the derivative of the first one will vanish):

$$\sum_{j=1}^{g} \frac{\partial^2\theta(\tau_0, z(t))}{\partial z_i \partial z_j} \frac{\partial z_j(t)}{\partial t} = 0.$$

Denoting

$$v := \left(\frac{\partial z_1}{\partial t}, \ldots, \frac{\partial z_g}{\partial t}\right)|_{t=0}$$

this means that $H(x_0) \cdot v = 0$, and since by our assumption z_0 is a smooth point of the curve and thus $v \ne 0$, the matrix $H(x_0)$ has a kernel, and in particular is not of maximal rank. $\square$

Corollary 2.9. *For $g \ge 5$ the locus of Jacobians $\mathcal{J}_g$ is contained in H. Hence for $g \ge 5$ set-theoretically $\theta_{\rm null}^{g-1} \subsetneq H$.*

Proof. Indeed, we have $\mathcal{J}_g \subset N_1 \subset H$ for $g \ge 5$. However, since for all g the divisor $\theta_{\rm null}$ does not contain $\mathcal{J}_g$, we must have $\mathcal{J}_g \subset \overline{H \setminus \theta_{\rm null}^{g-1}}$. $\square$

3. Class computations in cohomology

In this section we compute the class of the components of the expected dimension of the loci $\mathcal{H}$ and H in Chow and cohomology rings (our computation works in both, as we only use Chern classes of vector bundles) of $\mathcal{X}_g$ and $\mathcal{A}_g$, respectively.

Recall that Mumford [Mum83] computed the class of N_0' in the Picard group of the partial toroidal compactification of the moduli space $\mathcal{A}_g$ (the class of $\theta_{\rm null}$ is easier, and was computed previously by Freitag [Fre83]). We shall compute the classes of the codimension 2 cycles H and $\theta_{\rm null}^{g-1}$ on $\mathcal{A}_g$.

As a consequence we will obtain a complete description of H in genus 4, rederive some result of [GSM08], and reprove that for $g \geq 5$ the locus H has other components besides $\theta_{\text{null}}^{g-1}$. Debarre [Deb92, Section 4] computed the class of the intersection $\theta_{\text{null}} \cap N_0'$ and used this to show that this intersection is not irreducible. In spirit our computation is similar, though much more involved.

For the universal family $\phi : \mathcal{X}_g \to \mathcal{A}_g$ we denote by $\Omega_{\mathcal{X}_g/\mathcal{A}_g}$ the relative cotangent bundle, by $\mathbb{E} := \phi_* \Omega_{\mathcal{X}_g/\mathcal{A}_g}$ we denote its pushforward — the rank g vector bundle that is called the Hodge bundle. Then the Hodge class $\lambda_1 := c_1(\mathbb{E})$ is the Chern class of the line bundle of modular forms of weight one on $\mathcal{A}_g$.

The basic tool for our computation of pushforwards is the following:

Lemma 3.1. *The pushforward under ϕ of powers of the universal theta divisor $\Theta \subset \mathcal{X}_g$ can be computed as follows:*

$$\phi_*([\Theta^k]) = \begin{cases} 0 & \text{if } k < g \\ g! & \text{if } k = g \\ \frac{(g+1)!}{2}\lambda_1 & \text{if } k = g+1 \\ \frac{(g+2)!}{8}\lambda_1^2 & \text{if } k = g+2 \end{cases}$$

Proof. The first three cases are consequence of the computation in [Mum83, page 373]. The last case is the next step of the same computation, recalling that $c_2(\mathbb{E}) = \lambda_1^2/2$. In full generality the pushforwards of the universal theta divisor were computed and studied in [vdG99] (note that the universal theta divisor trivialized along the zero section, that is, the class $[\Theta] - \lambda_1/2$, is used there, and it is shown that $\phi_* \left(([\Theta] - \lambda_1/2)^k\right) = 0$ unless $k = g$). $\square$

Note that the locus $\mathcal{S}$ is given as the scheme of zeroes of theta function and its derivatives, i.e. given by zeroes of a section of $\Omega_{\mathcal{X}_g/\mathcal{A}_g}(\Theta) \otimes \mathcal{O}_{\mathcal{X}_g}(\Theta)$ (see [Mum83]). Hence

$$[N_0] = \phi_* \left(c_g \left(\Omega_{\mathcal{X}_g/\mathcal{A}_g}(\Theta) \otimes_{\mathcal{O}_{\mathcal{X}_g}} \mathcal{O}_\Theta \right) \right).$$

Recall now that $\mathcal{S}^{g-1} \subset \mathcal{S}$ is defined by the equation $\det H(x_0) = 0$. On $\mathcal{S}$, each second derivative of the theta function is a section of Θ, and the determinant of the Hessian matrix is known (see [GSM07],[dJ10]) to be a section of

$$\mathcal{O}_{\mathcal{X}_g}(g\Theta) \otimes \phi^*(\det \mathbb{E})^{\otimes 2} \otimes \mathcal{O}_{\mathcal{S}}.$$

Using the above formula for the class of $\mathcal{S}$, to get H we will need to compute the pushforward

$$\phi_* \left(c_g \left(\Omega_{\mathcal{X}_g/\mathcal{A}_g}(\Theta)_{|\Theta} \right) \cdot (g\Theta + 2\phi^*\lambda_1) \right)$$

The computation becomes rather delicate since $\mathcal{S}^{g-1}$ is not equidimensional. We set

$$\mathcal{S}_{\text{indec}} := \overline{\mathcal{S} \setminus \mathcal{S}_{\text{dec}}} = \mathcal{S}' \cup \mathcal{S}_{\text{null}},$$

which is then purely of codimension $g+2$ in $\mathcal{X}_g$, and thus we have

$$[\mathcal{S}^{g-1}_{\text{indec}}] = [\mathcal{S}_{\text{indec}}] \cdot (g[\Theta] + 2\lambda_1) \in CH^{g+2}(\mathcal{X}_g).$$

However, for dimension reasons it turns out that we often do not need to deal with the class of $\mathcal{S}_{\text{dec}}$:

Proposition 3.2. *For $g \geq 4$ we have the equality of codimension 2 classes on $\mathcal{A}_g$:*

$$[N_0^{g-1}] = [N^{g-1}_{0,\,\text{indec}}] := [\phi_*(\mathcal{S}^{g-1}_{\text{indec}})]$$

Moreover this class can be computed as

$$[N_0^{g-1}] = \frac{g!}{8}(g^3 + 7g^2 + 18g + 24)\lambda_1^2 \in CH^2(\mathcal{A}_g).$$

Proof. The first statement is a consequence of the fact that the map ϕ has $(g-2)$-dimensional fiber along $\mathcal{S}_{\text{dec}}$, and generically 0-dimensional fibers over $\mathcal{S}_{\text{indec}}$. (Note that this is the place in the argument where we are using the assumption $g \geq 4$ to ensure that $\mathcal{S}^{g-1}_{\text{indec}}$ is in fact non-empty, and that the codimension of its image under ϕ is lower than the codimension of $\mathcal{A}_1 \times \mathcal{A}_{g-1}$.) We now compute

$$\begin{aligned}
[N_0^{g-1}] &= \phi_*\left(c_g\left(\Omega_{\mathcal{X}_g/\mathcal{A}_g}(\Theta) \cdot \Theta\right) \cdot (g\Theta + 2\phi^*\lambda_1)\right) \\
&= \phi_*\left((\Theta^g + \Theta^{g-1}\phi^*\lambda_1 + \Theta^{g-2}\phi^*\lambda_2 + \dots) \cdot (g\Theta^2 + 2\Theta\phi^*\lambda_1)\right) \\
&= \phi_*\left(g\left(\Theta^{g+2} + \Theta^{g+1}\phi^*\lambda_1 + \Theta^g\frac{\phi^*\lambda_1^2}{2}\right) + (2\Theta^{g+1}\phi^*\lambda_1 + 2\Theta^g\phi^*\lambda_1^2)\right) \\
&= \left(\frac{g(g+2)!}{8} + \frac{g(g+1)!}{2} + \frac{g(g)!}{2} + (g+1)! + 2g!\right)\lambda_1^2 \\
&= \frac{g!}{8}(g^3 + 7g^2 + 18g + 24)\lambda_1^2.
\end{aligned}$$

□

We now compute the class of the locus $\theta^{g-1}_{\text{null}}$: recall that a theta constant is a modular form of weight $\frac{1}{2}$, and that the determinant of the Hessian matrix of $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z)$ evaluated at $z=0$ is a modular form of weight $\frac{g+4}{2}$ along the zero locus of $\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)$ (see [GSM08],[dJ10]). We thus get:

Proposition 3.3. *For $g \geq 2$ we have*

$$[\theta^{g-1}_{\text{null}}] = (g+4)2^{g-3}(2^g+1)\lambda_1^2.$$

Proof. Indeed, we have

$$\theta^{g-1}_{\text{null}} = \left\{ \tau \in \mathbb{H}_g : \exists[\varepsilon,\delta] \text{ even }, \theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau) = \det\left(\frac{\partial^2\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,z)}{\partial z_j \partial z_k}\right)|_{z=0} = 0 \right\},$$

Since there are $2^{g-1}(2^g+1)$ even characteristics, for each of them we get a contribution of $\lambda_1/2$ (for the zero locus of the corresponding theta constant) times $(g+4)\lambda_1/2$ (for the Hessian). $\square$

The proof of Theorem 0.2 comes by subtraction using the class formulas established in Propositions 3.2 and 3.3, while taking into account the relation given in formula (3).

4. The case $g = 4$

In this section we will work out the situation for genus 4 in detail, eventually proving Theorem 0.3. By the above formulas for $g = 4$ we have

$$[\theta_{\rm null}^3] = 272\lambda_1^2; \qquad [N_0^3] = 3 \cdot 272\lambda_1^2.$$

Moreover, going back from N_0^{g-1} to $H = N_0'^{g-1}$, we recall that for arbitrary genus by definition we have $N_0^{g-1} = \theta_{\rm null}^{g-1} + 2H$, and since at the intersection of the two components $\theta_{\rm null}$ and N_0' the singular points lie on both, we also have that set-theoretically

$$\theta_{\rm null}^{g-1} \subset H$$

As an immediate consequence we obtain:

Proposition 4.1. *The following identity holds at the level of codimension two cycles on $\mathcal{A}_4$:*

$$N_0^3 = 3\theta_{\rm null}^3.$$

Proof. From the formulas above we see that the cycle $2\theta_{\rm null}^3$ appears inside $2N_0'^3$, and thus that $3\theta_{\rm null}^3$ is a subcycle of N_0^3. Since the Chern classes are equal and $\theta_{\rm null}^3$ is equidimensional, we need to rule out the possibility of N_0^3 having an extra lower dimensional component. However, for genus 4 we know geometrically that N_0' is the locus of Jacobians. Using Riemann's Singularity Theorem for genus 4 curves we then see that the period matrix of a Jacobian is in $N_0'^3$ if and only if its theta divisor is singular at a two-torsion point, i.e. if this Jacobian lies in $\theta_{\rm null}^3$ (notice that this reproves a result of [GSM08]). $\square$

The proof of Theorem 0.3 is an immediate consequence of the above facts. We can prove something more: let I_4 be the Schottky modular form of weight 8 defining the Jacobian locus. Let then

$$\det \mathcal{D}(I_4) := \det \begin{pmatrix} \frac{\partial I_4}{\partial \tau_{11}} & \frac{1}{2}\frac{\partial I_4}{\partial \tau_{12}} & \cdots & \frac{1}{2}\frac{\partial I_4}{\partial \tau_{14}} \\ \frac{1}{2}\frac{\partial I_4}{\partial \tau_{21}} & \frac{\partial I_4}{\partial \tau_{22}} & \cdots & \frac{1}{2}\frac{\partial I_4}{\partial \tau_{24}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}\frac{\partial I_4}{\partial \tau_{41}} & \cdots & \cdots & \frac{\partial I_4}{\partial \tau_{44}} \end{pmatrix}$$

The restriction of this determinant to the zero locus of I_4 is a modular form of weight $34 = 8 \cdot 4 + 2$. By Proposition 2.1 we know that for a point in

$N_0' \setminus H$ the matrix $\mathcal{D}(I_4)$ is proportional to the Hessian matrix $H(x_0)$, hence it vanishes exactly along $\theta^3_{\rm null}$. The class of the cycle

$$\{I_4 = \det \mathcal{D}(I_4) = 0\}$$

is $8 \cdot 34\,\lambda_1^2 = 272\,\lambda_1^2$. Thus we obtain

Proposition 4.2. *The locus $\theta^3_{\rm null} \subset \mathcal{A}_4$ is a complete intersection given by*

$$I_4 = \det \mathcal{D}(I_4) = 0.$$

We observe that, by Riemann's Theta Singularity theorem, this is the locus of Jacobians with theta divisor singular at a two-torsion point. Moreover, the form $\sqrt{F_4}$ (recall that F_g is the product of all even theta constants) is well-defined along the Jacobian locus and it has the same weight, hence we get a different proof of the following result recently obtained by Matone and Volpato [MV11]:

Corollary 4.3. *On $\mathcal{J}_4$ we have the equality $\sqrt{F_4} = c \det \mathcal{D}(I_4)$ for some constant c. The locus $\theta^3_{\rm null}$ can also be given by equations*

$$I_4 = \sqrt{F_4} = 0.$$

Contrary to the situation in genus 4, for higher genera we know that we have other components, see Corollary 2.9. This fact can also be deduced from our class computation as follows.

Proof of Proposition 0.1. Recall that the statement we are proving is that at the level of effective cycles $\theta^{g-1}_{\rm null} \subsetneq H$ for any $g \geq 5$. We first note that the above discussion for the genus 4 case shows that the cycle-theoretic inclusion holds. Secondly, since we have computed both classes, we see that for $g \geq 5$ the class of N_0^{g-1} is not equal to 3 times the class of $\theta^{g-1}_{\rm null}$. In fact the growth orders of the degrees of these two classes are respectively

$$\deg \theta^{g-1}_{\rm null} \sim 4^{g-4} \deg \theta^3_{\rm null}; \qquad \deg N_0^{g-1} \sim \frac{g!}{4!} \deg N_0^3,$$

and one would thus expect many additional components. $\square$

The rest of the paper is devoted to studying the geometry for $g = 5$ in detail; in this case we will be able to describe all components explicitly, and will also obtain many results describing the classical Prym geometry of the situation.

5. Prym theta divisors and their singularities

While for higher g the geometry of the locus $H \subset \mathcal{A}_g$ appears quite intricate, for $g = 5$ one can use the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$. We begin by setting the notation and reviewing the basic facts about Prym varieties and their moduli, which will be used throughout the rest of the paper.

Let $\mathcal{R}_g$ be the moduli space of pairs (C, η) with $[C] \in \mathcal{M}_g$, and η a nonzero two-torsion point of the Jacobian $\mathrm{Pic}^0(C)$. We denote by $f : \tilde{C} \to C$ the étale double cover induced by η (so the genus of $\tilde{C}$ is equal to $2g-1$), by $i : \tilde{C} \to \tilde{C}$ the involution exchanging the sheets of f, and by $\varphi_{K_C \otimes \eta} : C \to \mathbf{P}H^0(C, K_C \otimes \eta)^\vee$ the Prym-canonical map. The map $\varphi_{K_C\otimes\eta}$ is an embedding if and only if $\eta \notin C_2 - C_2$ (where we denote $C_k := \mathrm{Sym}^k(C)$).

We recall the definition of the Prym map $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Consider the norm map $\mathrm{Nm}_f : \mathrm{Pic}^{2g-2}(\tilde{C}) \to \mathrm{Pic}^{2g-2}(C)$ induced by the double cover f. The even component of the preimage

$$\mathrm{Nm}_f^{-1}(K_C)^+ := \Big\{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) : \mathrm{Nm}_f(L) = K_C,\ h^0(\tilde{C}, L) \equiv 0 \mod 2\Big\}$$

is then an abelian variety of dimension $g-1$. Denoting by $\Theta_{\tilde{C}} \subset \mathrm{Pic}^{2g-2}(\tilde{C})$ the Riemann theta divisor, scheme-theoretically we have the following equality $\Theta_{\tilde{C}}|_{\mathrm{Nm}_f^{-1}(K_C)^+} = 2\Xi$, where Ξ is a principal polarization. The *Prym variety* is defined to be the ppav

$$P(C, \eta) := \Big(\mathrm{Nm}_f^{-1}(K_C)^+, \Xi\Big) \in \mathcal{A}_{g-1}.$$

The polarization divisor can be described explicitly following [Mum74]:

$$\Xi(C, \eta) := \{L \in \mathrm{Nm}_f^{-1}(K_C)^+ : h^0(\tilde{C}, L) > 0\}.$$

A key role in what follows is played by the *Prym-Petri* map

$$\mu_L^- : \wedge^2 H^0(\tilde{C}, L) \to H^0(C, K_C \otimes \eta), \quad u \wedge v \mapsto u \cdot i^*(v) - v \cdot i^*(u),$$

where one makes the usual identification $H^0(C, K_C \otimes \eta) = H^0(\tilde{C}, K_{\tilde{C}})^-$ with the (-1) eigenspace under the involution i. Following [Wel85], for $(C, \eta) \in \mathcal{R}_g$ and $r \geq -1$, we define the determinantal locus

$$V_r(C, \eta) := \{L \in \mathrm{Nm}_f^{-1}(K_C) : h^0(L) \geq r+1,\ h^0(L) \equiv r+1 \mod 2\}.$$

For a general Prym curve $(C, \eta) \in \mathcal{R}_g$, the map μ_L^- is injective for every $L \in \mathrm{Nm}_f^{-1}(K_C)$, and $\dim V_r(C, \eta) = g - 1 - \binom{r+1}{2}$, see [Wel85].

For a point $L \in \Xi$, we recall the description of the tangent cone $TC_L(\Xi)$. Suppose $h^0(\tilde{C}, L) = 2m \geq 2$ and we fix a basis $\{s_1 \dots, s_{2m}\}$ of $H^0(\tilde{C}, L)$. Consider the skew-symmetric matrix

$$M_L := \big(\mu_L^-(s_k \wedge s_j)\big)_{1 \leq k, j \leq 2m}$$

and the pfaffian $\mathrm{Pf}(L) := \sqrt{\det(M_L)} \in \mathrm{Sym}^m H^0(C, K_C \otimes \eta)$. Via the identification $T_L(P(C, \eta)) = H^0(C, K_C \otimes \eta)^\vee$ we have the following result of [Mum74]:

Theorem 5.1. *If $h^0(\tilde{C}, L) = 2$ then $\mathrm{Pf}(L) = 0$ is the equation of the projectivized tangent space $\boldsymbol{P}T_L(\Xi)$. If $m \geq 2$ then $L \in \mathrm{Sing}(\Xi)$ and either $\mathrm{Pf}(L) \equiv 0$, in which case $\mathrm{mult}_L(\Xi) \geq m+1$, or else, $\mathrm{Pf}(L) = 0$ is the equation of the tangent cone $\boldsymbol{P}TC_L(\Xi)$.*

Note that one can have $L \in \mathrm{Sing}(\Xi)$ even when $m = 1$ and $\mathrm{Pf}(L)$ is identically zero, so that the Prym theta divisor Ξ can have two types of singularities, as follows:

Definition 5.2. For a point $L \in \mathrm{Sing}(\Xi)$, one says that
(1) L is a *stable* singularity if $h^0(\tilde{C}, L) = 2m \geq 4$,
(2) L is an *exceptional* singularity if $L = f^*(M) \otimes \mathcal{O}_{\tilde{C}}(B)$, where $M \in \mathrm{Pic}(C)$ is a line bundle with $h^0(C, M) \geq 2$ and B is an effective divisor on C.

Let $\mathrm{Sing}^{\mathrm{st}}_f(\Xi) = V_3(C, \eta)$ be the locus of stable singularities and $\mathrm{Sing}^{\mathrm{ex}}_f(\Xi)$ the locus of exceptional singularities. Clearly $\mathrm{Sing}(\Xi) = \mathrm{Sing}^{\mathrm{st}}_f(\Xi) \cup \mathrm{Sing}^{\mathrm{ex}}_f(\Xi)$. Both these notions depend on the étale double cover $f : \tilde{C} \to C$ and are not intrinsic to Ξ. Furthermore, there can be singularities that are simultaneously stable and exceptional. Every singularity of a 4-dimensional theta divisor Ξ can in fact be realized as both a stable and an exceptional singularity in different incarnations of $(A, \Xi) \in \mathcal{A}_5$ as a Prym variety.

For a decomposable vector $0 \neq u \wedge v \in \wedge^2 H^0(\tilde{C}, L)$, we set

$$\mathrm{div}(u) := D_u + B, \qquad \mathrm{div}(v) := D_v + B,$$

where D_u, D_v have no common components and $B \geq 0$ is an effective divisor on C. The next lemma is well known, see [ACGH85, Appendix C]:

Lemma 5.3. *For $0 \neq u \wedge v \in \wedge^2 H^0(\tilde{C}, L)$ the following are equivalent.*
(1) $\mu^-_L(u \wedge v) = 0$.
(2) $D_u, D_v \in |f^*M|$ *where* $M \in \mathrm{Pic}(C)$ *with* $h^0(C, M) \geq 2$.

In such a case we write $L = f^*(M) \otimes \mathcal{O}_{\tilde{C}}(B)$, hence $K_C = M^{\otimes 2} \otimes \mathcal{O}_C(f_*(B))$, in particular $h^0(C, K_C \otimes M^{\otimes(-2)}) \geq 1$, and the Petri map $\mu_0(M)$ is not injective. In particular, $\mathrm{Sing}^{\mathrm{ex}}_f(\Xi) = \emptyset$ if C satisfied the Petri theorem.

Suppose $L \in V_3(C, \eta)$ is a *quadratic stable singularity*, hence $h^0(\tilde{C}, L) = 4$ and $\mathrm{Pf}(L) \neq 0$. Setting $\mathbf{P}^5 := \mathbf{P}(\wedge^2 H^0(L)^\vee)$ and $\mathbf{P}^{g-2} := \mathbf{P}(H^0(K_C \otimes \eta)^\vee)$, we consider the projectivized dual of the Prym-Petri map

$$\delta := \mathbf{P}\left((\mu^-_L)^\vee\right) : \mathbf{P}^{g-2} \to \mathbf{P}^5.$$

The Plücker embedding of the Grassmannian $\mathbf{G}^* := G(2, H^0(L)^\vee) \subset \mathbf{P}^5$ is a rank 6 quadric whose preimage $Q_L := \delta^{-1}(\mathbf{G}^*)$ is defined precisely by the Pfaffian $\mathrm{Pf}(L)$. Note also that $\mathrm{rk}(Q_L) \leq \mathrm{rk}(\mu^-_L)$. On the other hand let $\mathbf{G} := G(2, H^0(L)) \subset \mathbf{P}(\wedge^2 H^0(L))$ be the dual Grassmannian. It is again a standard exercise in linear algebra to show the equivalence

$$\mathrm{rk}(Q_L) \leq 4 \Longleftrightarrow \mathbf{G} \cap \mathbf{P}\left(\mathrm{Ker}(\mu^-_L)\right) \neq \emptyset.$$

For a point $L \in \mathrm{Sing}^{\mathrm{st}}_f(\Xi)$ one has $\mathrm{mult}_L(\Xi) = 2$ if and only if $h^0(C, M) \leq 2$ for any line bundle M on C such that $h^0(\tilde{C}, L \otimes f^*M^\vee) \geq 1$, see [SV04]. We summarize this discussion as follows:

Proposition 5.4. *For a quadratic singularity $L \in \mathrm{Sing}^{\mathrm{st}}_f(\Xi)$ the following conditions are equivalent:*
(1) $\mathrm{rk}(Q_L) \leq 4$.
(2) $\boldsymbol{G} \cap \boldsymbol{P}\left(\mathrm{Ker}(\mu^-_L)\right) \neq \emptyset$.
(3) $L \in \mathrm{Sing}^{\mathrm{st}}_f(\Xi) \cap \mathrm{Sing}^{\mathrm{ex}}_f(\Xi)$.

6. Petri divisors and the Prym map in genus 6

This section is devoted to the study of singularities of Prym theta divisors of dimension 4 via the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$.

We review a few facts about the Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ of $\mathcal{R}_g$, and refer to [Don92] and [FL10] for details. The space $\overline{\mathcal{R}}_g$ is the coarse moduli space associated to the Deligne-Mumford stack $\overline{\mathbf{R}}_g$ of stable Prym curves of genus g. The geometric points of $\overline{\mathcal{R}}_g$ correspond to triples (X, η, β), where X is a quasi-stable curve with $p_a(X) = g$, $\eta \in \mathrm{Pic}(X)$ is a line bundle of total degree 0 on X such that $\eta_E = \mathcal{O}_E(1)$ for each smooth rational component $E \subset X$ with $|E \cap \overline{X - E}| = 2$ (such a component is called exceptional), and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. Denoting $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ the forgetful map, one has the formula [FL10, Example 1.4]

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \in CH^1(\overline{\mathcal{R}}_g), \tag{7}$$

where $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$, and $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ are boundary divisor classes on $\overline{\mathcal{R}}_g$ whose meaning we recall. Let us fix a general point $[C_{xy}] \in \Delta_0$ corresponding to a smooth 2-pointed curve (C, x, y) of genus $g - 1$ and the normalization map $\nu : C \to C_{xy}$, where $\nu(x) = \nu(y)$. A general point of $\Delta_0^{'}$ (respectively of $\Delta_0^{''}$) corresponds to a stable Prym curve $[C_{xy}, \eta]$, where $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_C$). A general point of Δ_0^{ram} is of the form (X, η), where $X := C \cup_{\{x,y\}} \mathbf{P}^1$ is a quasi-stable curve with $p_a(X) = g$, whereas $\eta \in \mathrm{Pic}^0(X)$ is a line bundle characterized by $\eta_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} = \mathcal{O}_C(-x - y)$.

For $1 \le i \le [\frac{g}{2}]$ we have a splitting of the pull-back of the boundary

$$\pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i} \in CH^1(\overline{\mathcal{R}}_g), \tag{8}$$

where the boundary classes $\delta_i := [\Delta_i], \delta_{g-i} := [\Delta_{g-i}]$ and $\delta_{i:g-i} := [\Delta_{i:g-i}]$ correspond to the possibilities of choosing a pair of two-torsion line bundles on a smooth curve of genus i and one of genus $g-i$, such that the first one, the second one, or neither of the corresponding bundles is trivial, respectively, see [FL10].

Often we content ourselves with working on the partial compactification $\widetilde{\mathcal{R}}_g := \pi^{-1}(\mathcal{M}_g \cup \Delta_0)$ of $\mathcal{R}_g$. When there is no danger of confusion, we still denote by $\delta_0^{''}, \delta_0^{''}$ and δ_0^{ram} the restrictions of the corresponding boundary classes to $\widetilde{\mathcal{R}}_g$. Note that $CH^1(\widetilde{\mathcal{R}}_g) = \mathbf{Q}\langle \lambda, \delta_0^{'}, \delta_0^{''}, \delta_0^{\mathrm{ram}} \rangle$.

The extension of the (rational) Prym map $P : \overline{\mathcal{R}}_g \dashrightarrow \overline{\mathcal{A}}_{g-1}$ over the general point of each of the boundary divisors of $\overline{\mathcal{R}}_g$ is well-understood, see e.g. [Don92]. The Prym map contracts $\Delta_0^{''}$ and all boundary divisors $\pi^*(\Delta_i)$ for $1 \le i \le \lfloor \frac{g}{2} \rfloor$. The Prym variety corresponding to a general point $[C_{xy}, \eta] \in \Delta_0^{''}$ as above is the Jacobian $\mathrm{Jac}(C)$ of the normalization. Thus $P(\Delta_0^{''}) = \mathcal{J}_{g-1}$. The pullback map P^* on divisors has recently been described in [GSM11]: one has

$$P^*(\lambda_1) = \lambda - \frac{\delta_0^{\mathrm{ram}}}{4}, \quad P^*(D) = \delta_0^{'}. \tag{9}$$

Remark 6.1. We sketch an alternative way of deriving the first formula in (9). For each $(C,\eta) \in \mathcal{R}_g$, there is a canonical identification of vector bundles $T^{\vee}_{P(C,\eta)} = H^0(C, K_C \otimes \eta) \otimes \mathcal{O}_{P(C,\eta)}$. The pull-back $P^*(\mathbb{E})$ of the Hodge bundle can be identified with the vector bundle $\mathcal{N}_1$ on $\overline{\mathcal{R}}_g$ with fiber $\mathcal{N}_1(C,\eta) = H^0(C, \omega_C \otimes \eta)$, over each point $(C,\eta) \in \overline{\mathcal{R}}_g$ (we skip details showing that this description carries over the boundary as well). Therefore $P^*(\lambda_1) = c_1(\mathcal{N}_1) = \lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}$, where we refer to [FL10] for the last formula.

We have seen that for $[\tilde{C} \xrightarrow{f} C] \in \mathcal{R}_g$ with $\mathrm{Sing}_f^{\mathrm{ex}}(\Xi) \neq \emptyset$, the curve C fails the Petri theorem. Let $\mathcal{GP}^1_{g,k} \subset \mathcal{M}_g$ denote the *Gieseker-Petri* locus whose general element is a curve C carrying a globally generated pencil $M \in W^1_k(C)$ with $h^0(C,M) = 2$, such that the multiplication map

$$\mu_0(M) : H^0(C,M) \otimes H^0(C, K_C \otimes M^{\vee}) \to H^0(C, K_C)$$

is not injective. It is proved in [Far05] that for $\frac{g+2}{2} \leq k \leq g-1$, the locus $\mathcal{GP}^1_{g,k}$ has a divisorial component. As usual, we denote by $\mathcal{M}^r_{g,d}$ the locus of curves $[C] \in \mathcal{M}_g$ such that $W^r_d(C) \neq \emptyset$.

In the case of $\mathcal{M}_6$ there are two Gieseker-Petri loci, both irreducible of pure codimension 1, described as follows:

- The locus $\mathcal{GP}^1_{6,4}$ of curves $[C] \in \mathcal{M}_6$ having a pencil $M \in W^1_4(C)$ with $h^0(C, K_C \otimes M^{\otimes(-2)}) \geq 1$. We have the following formula for the class of its closure in $\overline{\mathcal{M}}_6$, see [EH87]:

$$[\overline{\mathcal{GP}}^1_{6,4}] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3 \in CH^1(\overline{\mathcal{M}}_6).$$

- The locus $\mathcal{GP}^1_{6,5}$ of curves with a vanishing theta characteristic; then

$$[\overline{\mathcal{GP}}^1_{6,5}] = 8\,(65\lambda - 8\delta_0 - 31\delta_1 - 45\delta_2 - 49\delta_3) \in CH^1(\overline{\mathcal{M}}_6).$$

The Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is dominant of degree 27 and its Galois group equals the Weyl group of E_6, see [DS81],[Don92]. The differential of the Prym map at the level of stacks

$$(dP)_{(C,\eta)} : H^0(C, K_C^{\otimes 2})^{\vee} \to \left(\mathrm{Sym}^2 H^0(C, K_C \otimes \eta)\right)^{\vee}$$

is the dual of the multiplication map at the level of global sections for the Prym-canonical map $\varphi_{K_C \otimes \eta}$. Thus the ramification divisor of P is a Cartier divisor on $\mathcal{R}_6$ supported on the locus

$$\mathcal{Q} := \left\{ (C,\eta) \in \mathcal{R}_6 : \mathrm{Sym}^2 H^0(C, K_C \otimes \eta) \stackrel{\ncong}{\longrightarrow} H^0(C, K_C^{\otimes 2}) \right\}.$$

The closure of $P(\mathcal{Q})$ inside $\mathcal{A}_5$ is the branch divisor of P. At a general point $(A,\Theta) \in P(\mathcal{Q})$ the fiber of P has the structure of the set of lines on a one-nodal cubic surface, that is, $P^{-1}(A,\Theta) \cap \mathcal{Q}$ consists of 6 ramification points corresponding to the 6 lines through the node. The remaining 15 points of $P^{-1}(A,\Theta)$ are in correspondence with the 15 lines on the one-nodal cubic surface not passing through the node. Since $\deg(P)=27$ it follows that P has simple ramification and $\mathcal{Q}$ is reduced. Donagi [Don92, p. 93] established that $\mathcal{Q}$ is irreducible by showing that the monodromy acts transitively on a general fiber of $P_{|\mathcal{Q}}$. We sketch a different proof which uses the irreducibility of the moduli space of polarized Nikulin surfaces. We summarize these results as follows:

Proposition 6.2. *Set-theoretically, the branch divisor of the map P is equal to the closure N_0' of $P(\mathcal{Q})$ in $\mathcal{A}_5$. At the level of cycles, $P_*[\mathcal{Q}]=6[N_0']$.*

We turn our attention to the geometry of $\mathcal{Q}$. First we compute the class of its closure in $\widetilde{\mathcal{R}}_6$, then we link it to Prym-Brill-Noether theory:

Theorem 6.3. *The ramification divisor $\mathcal{Q}\subset\mathcal{R}_6$ is irreducible. The class of its closure $\widetilde{\mathcal{Q}}$ in $\widetilde{\mathcal{R}}_6$ equals*

$$[\widetilde{\mathcal{Q}}]=7\lambda-\delta_0^{'}-\frac{3}{2}\delta_0^{\mathrm{ram}}-c_{\delta_0^{''}}\delta_0^{''}\in CH^1(\widetilde{\mathcal{R}}_6),$$

where we have the estimate $c_{\delta_0^{''}}\geq 4$.

Proof. The irreducibility of $\mathcal{Q}$ follows from [FV11, Theorem 0.5], where it is proved that $\mathcal{Q}$ can be realized as the image of a projective bundle over the *irreducible* moduli space $\mathcal{F}_6^{\mathfrak{N}}$ of polarized Nikulin $K3$ surfaces of genus 6. To estimate the class of the closure $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ in $\widetilde{\mathcal{R}}_6$, we set up two tautological vector bundles $\mathcal{N}_1$ and $\mathcal{N}_2$ over $\widetilde{\mathcal{R}}_6$ having fibers

$$\mathcal{N}_1(X,\eta):=H^0(X,\omega_X\otimes\eta)\quad\text{and}\quad\mathcal{N}_2(X,\eta):=H^0(X,\omega_X^{\otimes 2}\otimes\eta^{\otimes 2})$$

over a point $(X,\eta)\in\widetilde{\mathcal{R}}_6$. There is a morphism $\phi:\mathrm{Sym}^2(\mathcal{N}_1)\to\mathcal{N}_2$ between vector bundles of the same rank given by multiplication of Prym-canonical forms, and we denote by $\mathcal{Z}$ the degeneracy locus of ϕ. Using [FL10, Proposition 1.7] we have the following formulas in $CH^1(\widetilde{\mathcal{R}}_6)$

$$c_1(\mathcal{N}_1)=\lambda-\frac{1}{4}\delta_0^{\mathrm{ram}}\quad\text{and}\quad c_1(\mathcal{N}_2)=13\lambda-\delta_0^{'}-\delta_0^{''}-3\delta_0^{\mathrm{ram}},$$

thus $[\mathcal{Z}]=c_1(\mathcal{N}_2)-6c_1(\mathcal{N}_1)=7\lambda-\delta_0^{'}-\delta_0^{''}-\frac{3}{2}\delta_0^{\mathrm{ram}}$. By definition, $\mathcal{Q}=\mathcal{Z}\cap\mathcal{R}_6$. Furthermore, ϕ is non-degenerate at a general point of $\Delta_0^{'}$ and Δ_0^{ram}, hence the difference $\mathcal{Z}-\widetilde{\mathcal{Q}}$ is an effective divisor supported only on $\Delta_0^{''}$.

Assume now that $(X,\eta)\in\Delta_0^{''}$ is a generic point corresponding to a normalization map $\nu:C\to X$, where $[C,x,y]\in\mathcal{M}_{5,2}$ and $x,y\in C$ are distinct points such that $\nu(x)=\nu(y)$. Since $\nu^*(\eta)=\mathcal{O}_C$, we obtain an identification $H^0(X,\omega_X\otimes\eta)=H^0(C,K_C)$ whereas $H^0(X,\omega_X^{\otimes 2}\otimes\eta^{\otimes 2})$ is a codimension one subspace of $H^0(C,K_C^{\otimes 2}(2x+2y))$ described by a residue condition at x and y. It is straightforward to check that the kernel

$$\mathrm{Ker}\,\phi(X,\eta)=\mathrm{Ker}\left\{\mathrm{Sym}^2H^0(C,K_C)\to H^0(C,K_C^{\otimes 2})\right\}$$

has dimension 3. Thus $[\mathcal{Z}]-[\widetilde{\mathcal{Q}}]-3\delta_0''$ is effective supported on Δ_0'', which implies that $c_{\delta_0''}\geq 4$. $\square$

Remark 6.4. We shall prove later that in fact $c_{\delta_0''}=4$.

Even though the locus $\mathcal{Q}$ is defined in terms of syzygies of Prym-canonical curves, its points have a characterization in terms of stable singularities of Prym theta divisors.

Theorem 6.5. *The theta divisor of a Prym variety $P(C,\eta)\in\mathcal{A}_5$ has a stable singularity if and only if P ramifies at the point (C,η), that is,*

$$\mathcal{Q}=\Big\{(C,\eta)\in\mathcal{R}_6:\mathrm{Sing}^{\mathrm{st}}_{(C,\eta)}(\Xi)\neq\emptyset\Big\}.$$

Proof. Let us denote by $\mathcal{W}:=\{(C,\eta)\in\mathcal{R}_6:V_3(C,\eta)\neq\emptyset\}$ the Prym-Brill-Noether locus corresponding to stable singularities of Prym theta divisors. Our aim is to show that $\mathcal{W}=\mathcal{Q}$; we begin by establishing the inclusion $\mathcal{W}\subset\mathcal{Q}$. First note that if $[C]\in\mathcal{M}_6$ is trigonal, for any two-torsion point $\eta\in\mathrm{Pic}^0(C)[2]-\{\mathcal{O}_C\}$ we can write $K_C\otimes\eta=A\otimes A'$, where $A\in W^1_3(C)$ and $A'\in W^1_7(C)$. This implies that $(C,\eta)\in\mathcal{Q}$.

Fix now $(C,\eta)\in\mathcal{R}_6$ and a line bundle $L\in V_3(C,\eta)$. If $h^0(\tilde{C},L)\geq 6$, then $\tilde{C}$ (and hence C as well) must be hyperelliptic, so $(C,\eta)\in\mathcal{Q}$ by the previous remark. We may thus assume that $h^0(\tilde{C},L)=4$ and consider the associated Pfaffian quadric $Q_L\in\mathrm{Sym}^2H^0(C,K_C\otimes\eta)$. If $Q_L\neq 0$, then it contains the Prym-canonical model $\varphi_{K_C\otimes\eta}(C)$, in particular $(C,\eta)\in\mathcal{Q}$. If $Q_L\equiv 0$, then there exists $M\in\mathrm{Pic}(C)$ with $h^0(C,M)\geq 3$ and an effective divisor D on $\tilde{C}$, such that $L=f^*(M)\otimes\mathcal{O}_{\tilde{C}}(B)$. If $\deg(M)\leq 4$ then C is hyperelliptic, hence $(C,\eta)\in\mathcal{Q}$. If $\deg(M)=5$, then $B=0$ and C is a smooth plane quintic such that $h^0(C,M\otimes\eta)=1$. It is known, see [Don92, Section 4.3], that in this case $P(C,\eta)$ is the intermediate Jacobian of a cubic threefold and the differential $(dP)_{(C,\eta)}$ has corank 2, thus once more $(C,\eta)\in\mathcal{Q}$.

Therefore $\mathcal{W}\subset\mathcal{Q}$. We claim that $\mathcal{W}$ has at least a divisorial component, which follows by exhibiting a point $(C,\eta)\in\mathcal{R}_6$ and a line bundle $L\in V_3(C,\eta)$ such that μ_L^- is surjective. Assuming this for a moment, we conclude that $\mathcal{W}=\mathcal{Q}$ by invoking the irreducibility of $\mathcal{Q}$.

To finish the proof we use a realization of Prym curves $(C,\eta)\in\mathcal{R}_6$ with $V_3(C,\eta)\neq\emptyset$ resembling [FV11, Section 2]. For a line bundle $L\in V_3(C,\eta)$ with $h^0(\tilde{C},L)=4$, if $\mu_L^+:\mathrm{Sym}^2H^0(\tilde{C},L)\to H^0(C,K_C)$ denotes the i^*-invariant part of the Petri map, one has the following commutative diagram:

$$\begin{tikzcd}
\tilde{C} \arrow[r, "{(L,i^*L)}"] \arrow[dd, "f"'] & \mathbf{P}^3\times\mathbf{P}^3 \arrow[dd, "q"'] \arrow[dr] & \\
 & & \mathbf{P}^{15}=\mathbf{P}\left(H^0(L)^\vee\otimes H^0(L)^\vee\right) \arrow[dl, dashed] \\
C \arrow[r, "\mu_L^+"] & \mathbf{P}^9=\mathbf{P}(\mathrm{Sym}^2H^0(L)^\vee) &
\end{tikzcd}$$

In this diagram $q : \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbf{P}^9$ is the map $a \otimes b \mapsto a \otimes b + b \otimes a$ into the projective space of symmetric tensors. Reversing this construction, if $\iota \in \mathrm{Aut}(\mathbf{P}^3 \times \mathbf{P}^3)$ denotes the involution interchanging the two factors, the complete intersection of $\mathbf{P}^3 \times \mathbf{P}^3$ with 4 *general* ι^*-invariant hyperplanes in $H^0(\mathcal{I}_{\mathbf{P}^3 \times \mathbf{P}^3}(1,1))^+$ and one *general* ι^*-anti-invariant hyperplane in $H^0(\mathcal{I}_{\mathbf{P}^3 \times \mathbf{P}^3}(1,1))^-$ is a smooth curve $\tilde{C} \subset \mathbf{P}^3 \times \mathbf{P}^3$; the automorphism $\iota_{|C}$ induces a double cover $f : \tilde{C} \to C$ such that $\mathrm{Ker}(\mu_L^-)$ has 1-dimensional kernel corresponding to the unique element in $H^0(\mathcal{I}_{\mathbf{P}^3 \times \mathbf{P}^3}(1,1))^-$. For more details on this type of argument, we refer to [FV11]. $\square$

7. The antiramification divisor of the Prym map

In this section we describe geometrically the *antiramification divisor* $\mathcal{U}$ of the map $P : \mathcal{R}_6 \to \mathcal{A}_5$, defined via the equality of divisors

$$P^*(N_0') = 2\mathcal{Q} + \mathcal{U}. \tag{10}$$

For a general curve $[C] \in \mathcal{GP}^1_{6,4}$, if $M \in W^1_4(C)$ denotes the pencil such that $\mu_0(M)$ is not injective, we let $x + y \in C_2$ be the support of the unique section of $K_C \otimes M^{\otimes(-2)}$. We consider the four line bundles

$$L_{u,v} := f^*M \otimes \mathcal{O}_{\tilde{C}}(x_u + y_v) \in \mathrm{Nm}_f^{-1}(K_C),$$

where $1 \le u, v \le 2$ and $f(x_u) = x$, $f(y_v) = y$. Using the parity flipping lemma of [Mum74], exactly two of the quantities $h^0(\tilde{C}, L_{u,v})$ are equal to 2, the other being equal to 3, that is, $\mathrm{Sing}_f^{\mathrm{ex}}(\Xi)$ contains at least two points. Hence $\pi^*(\mathcal{GP}^1_{6,4}) \subset \mathcal{U}$. Using Theorem 6.3, equality (10), and the formula for $[\mathcal{GP}^1_{6,4}] \in CH^1(\mathcal{M}_6)$, we compute

$$[\mathcal{U}] = P^*([N_0']) - 2[\mathcal{Q}] = 108\lambda - 14\lambda = \pi^*([\mathcal{GP}^1_{6,4}]) \in CH^1(\mathcal{R}_6).$$

Since the λ-coefficient of any non-trivial effective divisor class on $\mathcal{R}_6$ must be strictly positive, we obtain the following result:

Proposition 7.1. *We have the following equality of divisors on $\mathcal{R}_6$:*

$$\mathcal{U} = \pi^*(\mathcal{GP}^1_{6,4}).$$

We now determine the pull-back of $\overline{N_0'}$ under the map $P : \widetilde{\mathcal{R}}_6 \dashrightarrow \overline{\mathcal{A}}_5$. As usual, $\widetilde{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ inside $\widetilde{\mathcal{R}}_6$.

Theorem 7.2. *We have the following equality of divisors on $\widetilde{\mathcal{R}}_6$:*

$$P^*(\overline{N_0'}) = 2\widetilde{\mathcal{Q}} + \widetilde{\mathcal{U}} + 20\Delta_0''.$$

Proof. We use the formula $[\overline{N_0'}] = 108\lambda_1 - 14D$, as well as Theorem 6.3 and formula (9) in order to note that the effective class $P^*(\overline{N_0'}) - 2\overline{\mathcal{Q}} - \pi^*(\overline{\mathcal{GP}}^1_{6,4})$ is supported only on the boundary divisor Δ_0''.

We now prove that the multiplicity of Δ_0'' in $P^*(\overline{N_0'})$ equals 20, or equivalently $\mathrm{mult}_{\Delta_0''}(P^*(\overline{N_0})) = 40$, since $P(\Delta_0'') = \mathcal{J}_5 \nsubseteq \theta_{\mathrm{null}}$. Let $\widetilde{\mathcal{A}}_5 := \mathrm{Bl}_{\mathcal{J}_5}(\mathcal{A}_5)$

be the blowup of $\mathcal{A}_5$ along the Jacobian locus and denote by $\mathcal{E} \subset \widetilde{\mathcal{A}_5}$ the exceptional divisor. Then $\mathcal{E}$ is a $\mathbf{P}^2$-bundle over $\mathcal{J}_5$ with the fiber over a point $(\mathrm{Jac}(C), \Theta_C) \in \mathcal{J}_5$ being identified with the space $\mathbf{P}(I_2(K_C)^\vee)$ of pencils of quadrics containing the canonical curve $C \subset \mathbf{P}^4$. One can lift the Prym map to a map $\tilde{P} : \widetilde{\mathcal{R}}_6 \dashrightarrow \widetilde{\mathcal{A}}_5$ by setting for a general point $(C_{xy}, \eta) \in \Delta_0^{''}$

$$\tilde{P}(C_{xy}, \eta) := ((\mathrm{Jac}(C), \Theta_C), q_{xy}) \in \widetilde{\mathcal{A}}_5,$$

where $q_{xy} \in \mathbf{P}(I_2(K_C)^\vee)$ is the pencil of quadrics containing the union $C \cup \langle x, y \rangle \subset \mathbf{P}^4$. Furthermore, $\tilde{P}^*(\mathcal{E}) = \Delta_0^{\mathrm{ram}}$, showing that

$$\mathrm{mult}_{\Delta_0^{''}} P^*(\overline{N_0}) = \mathrm{mult}_{\mathcal{J}_5}(N_0).$$

To estimate the latter multiplicity we consider a general one-parameter family $j : U \to \mathcal{A}_5$ from a disc $U \ni 0$ such that $j(0) = (\mathrm{Jac}(C), \Theta_C)$, with $[C] \in \mathcal{M}_5$ being a general curve. Let $\Theta_U := U \times_{\mathcal{A}_5} \Theta \to U$ be the relative theta divisor over U. The image of the differential $(dj)_0(T_0(U))$ can be viewed as a hyperplane $h \subset \mathbf{P}\left(\mathrm{Sym}^2 H^0(K_C)\right)$. The variety Θ_U has ordinary double points at those points $(0, L) \in \Theta_U$ where $L \in \mathrm{Sing}(\Theta_C) = W_4^1(C)$ is a singularity such that its tangent cone $Q_L \in \mathbf{P}I_2(K_C)$ belongs to h. Since the assignment $W_4^1(L) \ni L \mapsto Q_L \in \mathbf{P}I_2(K_C)$ is an unramified double cover over a smooth plane quintic, we find that Θ_U has 10 nodes. Using the theory of Milnor numbers for theta divisors as explained in [SV85] we obtain

$$\mathrm{mult}_{\mathcal{J}_5}(N_0) = \chi(\theta_{\mathrm{gen}}) - \chi(W_4(C)) + 10,$$

where $\chi(\theta_{\mathrm{gen}}) = 5! = 120$ is the topological Euler characteristic of a general (smooth) theta divisor of genus 5. We finally determine $\chi(W_4(C))$, using the resolution $C_4 \to W_4(C)$. From the Macdonald formula, see [ACGH85], $\chi(C_4) = (-1)^{g-1}\binom{2g-2}{g-1}_{|g=5} = 70$, whereas $\chi(W_4^1(C)) = -20$, because $g(W_4^1(C)) = 11$. Therefore $\chi(C_4^1) = 2\chi(W_4^1(C)) = -40$. We find that $\chi(W_4(C)) = \chi(C_4) - \chi(C_4^1) + \chi(W_4^1(C)) = 90$, thus $\mathrm{mult}_{\mathcal{J}_5}(N_0) = 120 - 90 + 10 = 40$. $\square$

Corollary 7.3. *We have the following formula in $CH^1(\widetilde{\mathcal{R}}_6)$:*

$$[\widetilde{\mathcal{Q}}] = 7\lambda - \delta_0^{'} - 4\delta_0^{''} - \frac{3}{2}\delta_0^{\mathrm{ram}}.$$

We exploit the geometry of the ramification and antiramification divisors of the Prym map and determine the pushforward of divisor classes on $\overline{\mathcal{R}}_6$:

Theorem 7.4. *The pushforwards of tautological divisor classes via the rational Prym map $P : \overline{\mathcal{R}}_6 \dashrightarrow \overline{\mathcal{A}}_5$ are as follows:*

$$P_*(\lambda) = 18 \cdot 27\lambda_1 - 57D, \quad P_*(\delta_0^{\mathrm{ram}}) = 4(17 \cdot 27\lambda_1 - 57D),$$

$$P_*(\delta_0^{'}) = 27D, \quad P_*(\delta_0^{''}) = P_*(\delta_i) = P_*(\delta_{i:g-i}) = 0 \qquad \textit{for } 1 \leq i \leq g-1.$$

We point out that even though P is not a regular map, it can be extended in codimension 1 such that P is the morphism induced at the level of coarse moduli spaces by a proper morphism of stacks, see e.g. [Don92] p.63-64.

Furthermore P^{-1} contracts no divisors, in particular, we can pushforward divisors under P and use the push-pull formula. Perhaps the most novel aspect of Theorem 7.4 is the calculation of the class of the divisor $P_*(\Delta_0^{\mathrm{ram}})$ consisting of Prym varieties corresponding to ramified double covers $\widetilde{C} \to C$ of genus 5 curves with two branch points.

Proof. We write the following formulas in $CH^1(\overline{\mathcal{A}}_5)$:

$$27\lambda_1 = P_*P^*\lambda_1 = P_*(\lambda) - \frac{1}{4}P_*(\delta_0^{\mathrm{ram}}),$$

$$6 \cdot (108\lambda_1 - 14D) = 6[\overline{N_0'}] = P_*([\overline{\mathcal{Q}}]) = 7P_*(\lambda) - \frac{3}{2}P_*(\delta_0^{\mathrm{ram}}) - 27P_*(\delta_0^{'}).$$

From [GSM11] it follows that $P_*(\delta_0^{'}) = 27D$, whereas obviously $P_*(\delta_0^{''}) = 0$, which suffices to solve the system of equations for coefficients. □

8. The slope of $\mathcal{A}_5$

Using the techniques developed in previous chapters, we determine the slope of the perfect cone compactification $\overline{\mathcal{A}}_5$ of $\mathcal{A}_5$ (note also that by the appendix by K. Hulek to [GSM11] this slope is the same for all toroidal compactification). We begin with some preliminaries. Let D be a $\mathbf{Q}$-divisor on a normal $\mathbf{Q}$-factorial variety X. We say that D is rigid if $|mD| = \{mD\}$ for all sufficiently large and divisible integers m. Equivalently, the Kodaira-Iitaka dimension $\kappa(X, D)$ equals zero.

We denote by $\mathbf{B}(D) := \bigcap_m \mathrm{Bs}(|mD|)$ the *stable base locus* of D. We say that D is *movable* if $\operatorname{codim} \mathbf{B}(D) \geq 2$.

Recall that one defines the slope of $\overline{\mathcal{A}}_g$ as $s(\overline{\mathcal{A}}_g) := \inf_{E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)} s(E)$. In a similar fashion one defines the *moving slope* of $\overline{\mathcal{A}}_g$ as the slope of the cone of moving divisors on $\overline{\mathcal{A}}_g$, that is,

$$s'(\overline{\mathcal{A}}_g) := \inf \left\{ s(E) : E \in \mathrm{Eff}(\overline{\mathcal{A}}_g),\ E \text{ is movable} \right\}.$$

Thus $s'(\overline{\mathcal{A}}_g)$ measures the minimal slope of a divisor responsible for a non-trivial map from $\overline{\mathcal{A}}_g$ to a projective variety. It is known that $s(\overline{\mathcal{A}}_4) = 8$ [SM92], and as an immediate consequence of the result about the slope of $\mathcal{M}_4$ we have that $s'(\overline{\mathcal{A}}_4) = s(\overline{\theta}_{\mathrm{null}}) = \frac{17}{2}$. In the next case, that of dimension $g = 5$, the formula $[\overline{N_0'}] = 108\lambda_1 - 14D$ yields the upper bound $s(\overline{\mathcal{A}}_5) \leq \frac{54}{7}$. A lower bound for the slope $s(\overline{\mathcal{A}}_5)$ was recently obtained in [GSM11].

We shall now prove Theorem 0.7 and establish that

$$\kappa(\overline{\mathcal{A}}_5, \overline{N_0'}) = 0,$$

in particular showing that $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. To prove Theorem 0.7 we translate the problem into a question on the linear series $|P^*(\overline{N_0'})|$ on $\overline{\mathcal{R}}_6$. One can show that each of the components of $P^*(\overline{N_0'})$ is an extremal divisor on $\overline{\mathcal{R}}_6$, however their sum could well have positive Kodaira dimension. Of crucial importance is a uniruled parametrization of $\overline{\mathcal{Q}}$ using sextics with a totally tangent conic.

We fix general points $q_1, \ldots, q_4 \in \mathbf{P}^2$, then set $S := \mathrm{Bl}_{\{q_i\}_{i=1}^4}(\mathbf{P}^2) \to \mathbf{P}^2$ and denote by $\{E_{q_i}\}_{i=1}^4$ the corresponding exceptional divisors. We make the identification $\mathbf{P}^{15} := \left|\mathcal{O}_S(6)(-2\sum_{i=1}^4 E_{q_i})\right|$, then consider the space of 4-nodal sextics having a totally tangent conic

$$\mathcal{X} := \left\{(\Gamma, Q) \in \mathbf{P}^{15} \times |\mathcal{O}_S(2)| : \Gamma \cdot Q = 2d, \text{ where } d \in (\Gamma_{\mathrm{reg}})_6\right\}.$$

A parameter count shows that $\mathcal{X}$ is pure of dimension 14. We define the rational map $v : \mathcal{X} \dashrightarrow \mathcal{R}_6$

$$v(\Gamma, Q) := \left(C, \eta := \nu^* \left(\mathcal{O}_\Gamma(1)(-d)\right)\right) \in \mathcal{R}_6,$$

where $\nu : C \to \Gamma$ is the normalization map. The image $v(\mathcal{X})$ is expected to be a divisor on $\mathcal{R}_6$, and we show that this is indeed the case — this construction yields another geometric characterization of points in $\mathcal{Q}$.

Theorem 8.1. *The closure of $v(\mathcal{X})$ inside $\mathcal{R}_6$ is equal to $\mathcal{Q}$, that is, a general Prym curve $(C, \eta) \in \mathcal{Q}$ possesses a totally tangent conic.*

Proof. We carry out a class calculation on $\overline{\mathcal{R}}_6$ and the result will be a consequence of the extremality properties of the class $[\overline{\mathcal{Q}}] \in \mathrm{Eff}(\overline{\mathcal{R}}_6)$. We work on a partial compactification $\mathcal{R}_6'$ of $\mathcal{R}_6$ that is even smaller than $\widetilde{\mathcal{R}}_6$.

Let $\mathcal{R}_6' := \mathcal{R}_6^0 \cup \pi^{-1}(\Delta_0^*)$ be the open subvariety of $\overline{\mathcal{R}}_6$, where $\mathcal{R}_6^0$ consists of smooth Prym curves (C, η) for which $\dim W_6^2(C) = 0$ and $h^0(C, L \otimes \eta) = 1$ for every $L \in W_6^2(C)$, whereas $\Delta_0^* \subset \Delta_0$ is the locus of curves $[C_{xy}]$, where $[C] \in \mathcal{M}_5 - \mathcal{M}_{5,3}^1$ and $x, y \in C$. Observe that $\mathrm{codim}(\widetilde{\mathcal{R}}_6 - \mathcal{R}_6', \widetilde{\mathcal{R}}_6) = 2$, in particular we can identify $CH^1(\mathcal{R}_6')$ and $CH^1(\widetilde{\mathcal{R}}_6)$. Over the Deligne-Mumford stack $\mathbf{R}_6'$ of Prym curves coarsely represented by the scheme $\mathcal{R}_6'$ (observe that $\mathbf{R}_6'$ is an open substack of $\mathbf{R}_6$), we consider the finite cover

$$\sigma : \mathfrak{G}_6^2 \to \mathbf{R}_6',$$

where $\mathfrak{G}_6^2$ is the Deligne-Mumford stack that classifies triples (C, η, L), with $(C, \eta) \in \mathcal{R}_6'$ and $L \in W_6^2(C)$. Note that a curve $[C_{xy}] \in \Delta_0'$ carries no non-locally free sheaves $F \in \overline{\mathrm{Pic}^6}(C_{xy})$ with $h^0(C_{xy}, F) \geq 5$, for F would correspond to a $\mathfrak{g}_5^2$ on the normalization C of C_{xy}, a contradiction. The universal curve $p : \mathcal{C} \to \mathfrak{G}_6^2$ is equipped both with a universal Prym bundle $\mathcal{P} \in \mathrm{Pic}(\mathcal{C})$ and a universal Poincaré line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ such that $\mathcal{L}_{|p^{-1}(C,\eta,L)} = L$, for any $(C, \eta, L) \in \mathfrak{G}_6^2$. We form the codimension 1 tautological classes

$$\mathfrak{a} := p_*\left(c_1(\mathcal{L})^2\right), \quad \mathfrak{b} := p_*\left(c_1(\mathcal{L}) \cdot c_1(\omega_p)\right) \in CH^1(\mathfrak{G}_6^2), \tag{11}$$

and the sheaves $\mathcal{V}_i := p_*(\mathcal{L}^{\otimes i})$, where $i = 1, 2$. Both $\mathcal{V}_1$ and $\mathcal{V}_2$ are locally free. The dependence of $\mathfrak{a}$ and $\mathfrak{b}$ on the choice of $\mathcal{L}$ is discussed in [FL10].

Using the isomorphism $CH^1(\mathbf{R}'_6) = CH^1(\mathcal{R}'_6)$, one can write the following formulas in $CH^1(\mathcal{R}'_6)$, see [FL10, page 776]:
(12)
$$\sigma_*(\mathfrak{a}) = -48\lambda+7\pi^*(\delta_0),\ \sigma_*(\mathfrak{b}) = 36\lambda-3\pi^*(\delta_0),\ \sigma_*(c_1(\mathcal{V})) = -22\lambda+3\pi^*(\delta_0).$$

We also introduce the sheaf $\mathcal{E} := p_*(\mathcal{P}\otimes\mathcal{L})$. Since $R^1p_*(\mathcal{P}\otimes\mathcal{L}) = 0$ (this is the point where we use $H^1(C, L\otimes\eta) = 0$ for each $(C, \eta, L) \in \mathcal{R}'_6$), applying Grauert's theorem we obtain that $\mathcal{E}$ is locally free and via Grothendieck-Riemann-Roch we compute its Chern classes. Taking into account that $p_*(c_1(\mathcal{P})^2) = \delta_0^{\mathrm{ram}}/2$ and $p_*(c_1(\mathcal{L})\cdot c_1(\mathcal{P})) = 0$, see [FL10, Proposition 1.6], one computes

$$c_1(\mathcal{E}) = \lambda - \frac{\delta_0^{\mathrm{ram}}}{4} + \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} \in CH^1(\mathfrak{G}^2_6). \tag{13}$$

Similarly, by GRR we find that $c_1(\mathcal{V}_2) = \lambda - \mathfrak{b} + 2\mathfrak{a}$.

After this preparation we return to the problem of describing the closure $\widetilde{v(\mathcal{X})}$ of $v(\mathcal{X})$ in $\mathcal{R}'_6$. For a point $(C, \eta, L) \in \mathfrak{G}^2_6$, the two-torsion point η is induced by a conic totally tangent to the image of $\nu : C \stackrel{|L|}{\to} \Gamma \subset \mathbf{P}^2$, if and only if the map given by multiplication followed by projection

$$\chi(C, \eta, L) : H^0(C, L\otimes\eta)\otimes H^0(C, L\otimes\eta) \to H^0(C, L^{\otimes 2})/\mathrm{Sym}^2 H^0(C, L)$$

is not an isomorphism. Working over the stack we obtain a morphism of vector bundles over $\mathfrak{G}^2_6$

$$\chi : \mathcal{E}^{\otimes 2} \to \mathcal{V}_2/\mathrm{Sym}^2(\mathcal{V}_1),$$

such that the class of $\widetilde{v(\mathcal{X})}$ is (up to multiplicity) equal to

$$\sigma_* c_1\Big(\frac{\mathcal{V}_2}{\mathrm{Sym}^2(\mathcal{V}_1)} - \mathcal{E}^{\otimes 2}\Big) = 35\lambda - 5(\delta_0^{'} + \delta_0^{''}) + \frac{5}{2}\delta_0^{\mathrm{ram}} = 5[\mathcal{Z}],$$

where we have used both (12) and (13). We recall that the cycle $\mathcal{Z}$ was defined in the proof of Theorem 6.3 as a subvariety of the larger space $\widetilde{\mathcal{R}}_6$ with the property that $\mathcal{Z}\cap\mathcal{R}'_6 = \mathcal{Q}\cap\mathcal{R}'_6$. Thus the class $[\widetilde{v(\mathcal{X})}] \in CH^1(\mathcal{R}'_6)$ is proportional (up to the divisor class $\delta_0^{''}$) to the class $[\widetilde{\mathcal{Q}}]$. It is proved in [FV11, Proposition 3.6] that if D is an effective divisor on $\mathcal{R}_6$ such that $[\overline{D}] = \alpha[\overline{\mathcal{Q}}] + \beta\ \delta_0^{''}$, then one has the set-theoretic equality $D = \mathcal{Q}$. Thus we conclude that the closure of $v(\mathcal{X})$ in $\mathcal{R}_6$ is precisely $\mathcal{Q}$. $\square$

Theorem 8.2. *Through a general point of the ramification divisor $\overline{\mathcal{Q}}$ there passes a rational curve $R \subset \overline{\mathcal{R}}_6$ with the following numerical features:*

$$R\cdot\lambda = 6,\ R\cdot\delta_0^{'} = 35,\ R\cdot\delta_0^{''} = 0,\ R\cdot\delta_0^{\mathrm{ram}} = 6,\ R\cdot\delta_i = R\cdot\delta_{i:5-i} = 0,$$

for $i = 1, \ldots, 4$. In particular $R\cdot\overline{\mathcal{Q}} < 0$ and $R\cdot\overline{\mathcal{U}} = 0$.

Assuming for the moment Theorem 8.2, we explain how it implies Theorem 0.7. Assume that $E \in \mathrm{Eff}(\overline{\mathcal{A}}_5)$ with $s(E) \le s(\overline{N'_0})$. First note that one can assume that $\overline{N'_0} \not\subseteq \mathrm{supp}(E)$, for else, we can replace E by an effective divisor of the form $E' := E - \alpha\overline{N_0'}$ with $\alpha > 0$ and still $s(E') \leq s(\overline{N_0'})$. After rescaling by a positive factor, we can write $E \equiv \overline{N_0'} - \epsilon\lambda_1 \in \mathrm{Eff}(\overline{\mathcal{A}}_5)$, where $\epsilon \geq 0$. Clearly we have $P^*(E) \in \mathrm{Eff}(\overline{\mathcal{R}}_6)$; observe that since $\overline{N_0'}$ is not a component of E, the ramification divisor $\overline{\mathcal{Q}}$ cannot be a component of $P^*(E)$ either. Thus $R \cdot P^*(E) \geq 0$, that is,

$$0 \leq R \cdot P^*(E) = R \cdot (2\overline{\mathcal{Q}} + \overline{\mathcal{U}} + 20\delta_0^{''}) - \epsilon R \cdot \left(\lambda - \frac{\delta_0^{\mathrm{ram}}}{4}\right) = -4 - \frac{9\epsilon}{2},$$

which is a contradiction. Thus $s(E) = s(\overline{N_0'})$ and E must be *equal* to a multiple of $\overline{N_0'}$.

Proof of Theorem 8.2. We retain the notation from Theorem 8.1 and fix a general element $(C, \eta) \in \mathcal{Q}$ corresponding to a sextic curve $\Gamma \subset \mathbf{P}^2$ having nodes at $q_1, \ldots, q_4$. From Theorem 8.1 we may assume that there exists a conic $Q \subset \mathbf{P}^2$ such that $Q \cdot \Gamma = 2(p_1 + \cdots + p_6)$, where $p_1, \ldots, p_6 \in \Gamma_{\mathrm{reg}}$. Since the points $q_1, \ldots, q_4 \in \mathbf{P}^2$ are distinct and no three are collinear, it follows that $[C] \notin \overline{\mathcal{GP}}^1_{6,4}$, and this holds even when C has nodal singularities.

To construct the pencil $R \subset \overline{\mathcal{R}}_6$, we reverse this construction and start with a conic $Q \subset \mathbf{P}^2$ and six general points $p_1, \ldots, p_6 \in Q$ on it. On the blowup S' of $\mathbf{P}^2$ at the 10 points $q_1, \ldots, q_4, p_1, \ldots, p_6$, we denote by $\{E_{p_i}\}_{i=1}^6$ and by $\{E_{q_j}\}_{j=1}^4$ respectively, the exceptional divisors. For $1 \leq i \leq 6$, let $l_i \in E_{p_i}$ be the point corresponding to the tangent line $T_{p_i}(Q)$. If $\tilde{S}$ is the blowup of S' at $l_1, \ldots, l_6$, by slight abuse of notation we denote by E_{p_i}, E_{l_i} and E_{q_j} the exceptional divisors on $\tilde{S}$ (respectively the proper transforms of exceptional divisors on S'). Then $\dim \left|\mathcal{O}_{\tilde{S}}(6)\left(-2\sum_{j=1}^4 E_{q_j} - \sum_{i=1}^6 (E_{p_i} + E_{l_i})\right)\right| = 3$, and we choose a general pencil in this linear system. This pencil induces a curve $R \subset \overline{\mathcal{R}}_6$. Note that the pencil contains one distinguished element, consisting of the union of Q and two conics Q_1 and Q_2 passing through $q_1, \ldots, q_4$. Considering the pushforward $\pi_*(R) \subset \overline{\mathcal{M}}_6$, after a routine calculation we find:

$$R \cdot \lambda = 6,\ R \cdot (\delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}) = \pi_*(R) \cdot \delta_0 = 47,\ \pi_*(R) \cdot \delta_i = 0 \ \text{ for } i = 1, 2, 3.$$

In particular, as expected $R \cdot \overline{\mathcal{U}} = 0$. The points in $R \cap \Delta_0^{\mathrm{ram}}$ correspond to the case when the underlying level 2 structure is not locally free, which happens precisely when one of the points p_i becomes singular. For each $1 \leq i \leq 6$, there is one such curve in R, hence $R \cdot \delta_0^{\mathrm{ram}} = 6$, thus $R \cdot \delta_0^{'} = 35$ and therefore using Corollary 7.3 we find:

$$R \cdot \overline{\mathcal{Q}} = 7R \cdot \lambda - R \cdot \delta_0^{'} - \frac{3}{2} R \cdot \delta_0^{\mathrm{ram}} = 42 - 35 - 9 = -2.$$

$\square$

Since the divisor $\overline{N_0'}$ is rigid, we obtain that $s'(\overline{\mathcal{A}}_5) > s(\overline{N_0'})$. Concerning the value of the moving slope $s'(\overline{\mathcal{A}}_5)$, we make the following prediction:

Conjecture 8.3. $s'(\overline{\mathcal{A}}_5) = \frac{70}{9}$.

The inequality $s'(\overline{\mathcal{A}}_5) \geq \frac{70}{9}$ follows from the first part of the proof of Theorem 8.2. Let E be as before, in particular $\overline{N_0'} \not\subseteq \mathrm{supp}(E)$. After rescaling by a positive factor, we can write $[E] = [\overline{N_0'}] + \epsilon\lambda_1 \in \mathrm{Eff}(\overline{\mathcal{A}}_5)$, where $\epsilon \geq 0$. Since $R \cdot P^*(E) \geq 0$, we find that

$$0 \leq R \cdot P^*(E) = R \cdot (2\overline{\mathcal{Q}} + \overline{\mathcal{U}} + 20\delta_0^{''}) + \epsilon R \cdot \left(\lambda - \frac{\delta_0^{\mathrm{ram}}}{4}\right) = -4 + \frac{9\epsilon}{2}.$$

Hence $\epsilon \geq \frac{8}{9}$ and this gives $s'(\overline{\mathcal{A}}_5) \geq \frac{70}{9}$. Thus Conjecture 8.3 boils down to constructing a movable effective divisor on $\overline{\mathcal{A}}_5$ having slope $\frac{70}{9}$.

Remark 8.4. Theorem 0.7 implies that the divisor $\overline{N_0'}$ can be contracted via a birational map having $\overline{\mathcal{A}}_5$ as its source. Especially from the point of view of the Minimal Model Program for $\overline{\mathcal{A}}_5$, it would be interesting to find a new compactification of the moduli space of ppav $\mathcal{A}_5^*$, and a birational map $f : \overline{\mathcal{A}}_5 \dashrightarrow \mathcal{A}_5^*$ such that f contracts $\overline{N_0'}$.

9. The Prym realization of the components of H

For each irreducible component of $H = N_0^{'4}$ in $\mathcal{A}_5$, we describe an explicit codimension 2 subvariety of $\mathcal{R}_6$ which dominates it via the Prym map. As a consequence, we prove that H consists of two irreducible components, both unirational and of dimension 13. We define two subvarieties of $\mathcal{R}_6$ corresponding to Prym curves (C, η) such that $\varphi_{K_C \otimes \eta}$ lies on a quadric of rank at most 4, cutting a (Petri special) pencil on C. Depending on the degree of this pencil, we denote these loci by $\mathcal{Q}_4$ and $\mathcal{Q}_5$ respectively.

Definition 9.1. We denote by $\mathcal{Q}_5$ the closure in $\mathcal{R}_6$ of the locus of curves $(C, \eta) \in \mathcal{R}_6$ such that C carries two vanishing theta characteristics $\theta_1, \theta_2 \in W_5^1(C)$ with $\eta = \theta_1 \otimes \theta_2^\vee$.

Equivalently, $K_C \otimes \eta = \theta_1 \otimes \theta_2$, which implies that the Prym-canonical model of C lies on a quadric $Q \subset \mathbf{P}^4$ of rank 4, whose rulings induce θ_1 and θ_2 respectively.

Definition 9.2. We denote by $\mathcal{Q}_4$ the closure in $\mathcal{R}_6$ of the locus of curves $(C, \eta) \in \mathcal{R}_6$ such that $\eta \in W_4^1(C) - W_4(C)$ and $K_C \otimes \eta$ is very ample.

Equivalently, $K_C \otimes \eta = A \otimes A'$, where $A \in W_4^1(C)$ and $A' \in W_6^1(C)$, and then the image $\varphi_{K_C \otimes \eta}(C)$ lies on a quadric $Q \subset \mathbf{P}^4$ of rank at most 4, whose rulings cut out A and A' respectively.

Remark 9.3. Along the same lines, one can consider the locus $\mathcal{Q}_3$ of curves $(C, \eta) \in \mathcal{R}_6$ such that $K_C \otimes \eta = A \otimes A'$, where $A \in W_3^1(C)$ and $A' \in W_7^1(C)$. Observe that $\mathcal{Q}_3 = \pi^{-1}(\mathcal{M}_{6,3}^1)$, where $\mathcal{M}_{6,3}^1$ is the trigonal locus inside $\mathcal{M}_6$. In particular, $\mathrm{codim}(\mathcal{Q}_3, \mathcal{R}_6) = 2$. However from the trigonal construction [Don92, Section 2.4], it follows that $P(\mathcal{Q}_3) = \mathcal{J}_5$, that is, P blows-down $\mathcal{Q}_3$ and thus $\mathcal{Q}_3$ plays no further role in describing the components of H in $\mathcal{A}_5$.

First we show that $\mathcal{Q}_4$ lies both in the ramification and the antiramification divisor of the Prym map:

Proposition 9.4. *$\mathcal{Q}_4 \subseteq \mathcal{Q} \cap \mathcal{U}$.*

Proof. We choose a point $(C, \eta) \in \mathcal{Q}_4$ general in a component of $\mathcal{Q}_4$ and write $\eta = M \otimes \mathcal{O}_C(-D)$, where $M \in W^1_4(C)$ and $D \in C_4$ is an effective divisor. Then we compute

$$h^0(C, K_C \otimes \eta(-D)) = h^0(K_C \otimes M^\vee) = 3,$$

that is, $\ell := \langle D \rangle$ is a 4-secant line to the Prym canonical model $\varphi_{K_C \otimes \eta}(C)$. Moreover ℓ is contained in the rank 4 quadric Q whose rulings cut out on C the pencils M and $K_C \otimes \eta \otimes M^\vee$ respectively. The line ℓ is not contained in a plane of Q belonging to the ruling Λ that cuts out on C the pencil M, for else it would follow that $\eta = 0$. Then ℓ is unisecant to the planes in Λ and if $d_M \in |M|$ is a general element, then $\langle D + d_M \rangle$ is a hyperplane in $\mathbf{P}^4$. Thus

$$K_C \otimes \eta = \mathcal{O}_C(d_M + D + x + y),$$

where $x, y \in C$, that is, $H^0(C, K_C \otimes M^{\otimes(-2)}) \neq 0$, and $[C] \in \mathcal{GP}^1_{6,4}$. $\square$

Proposition 9.5. *The locus $\mathcal{Q}_4$ is unirational and of dimension 13.*

Proof. Since $\mathcal{Q}_4 \subset \mathcal{Q} \cap \mathcal{U}$, we use the fact that *every* curve $[C] \in \mathcal{GP}^1_{6,4}$ is a quadratic section of a nodal quintic del Pezzo surface. In the course of proving Theorem 8.2 we observed that a general Prym curve $(C, \eta) \in \mathcal{U}$ is characterized by the existence of a totally tangent conic. We show that a similar description carries over to the case of 1-nodal del Pezzo surfaces.

We fix collinear points $q_1, q_2, q_3 \in \mathbf{P}^2$, a general point $q_4 \in \mathbf{P}^2$, then denote by $\ell := \langle q_1, q_2, q_3 \rangle \subset \mathbf{P}^2$, by $S' := \mathrm{Bl}_{\{q_i\}_{i=1}^4}(\mathbf{P}^2) \to \mathbf{P}^2$ the surface whose image by the linear system $\left|\mathcal{O}_{S'}(2)\big(-\sum_{i=1}^4 E_{q_i}\big)\right|$ is a 1-nodal del Pezzo quintic. Set $\mathbf{P}^3_{S'} := \left|\mathcal{O}_{S'}(3)\big(-\sum_{i=1}^3 E_{q_i} - 2E_{q_4}\big)\right|$. Note that $\mathrm{Aut}(S') = \mathbf{C}^*$. We consider the 10-dimensional rational variety

$$\mathcal{V} := \{(Q, p_1, \ldots, p_5) : Q \in |\mathcal{O}_{\mathbf{P}^2}(2)|,\ p_1, \ldots, p_5 \in Q\}$$

and the rational map $p : \mathcal{V} \dashrightarrow \mathbf{P}^2$ given by $p((Q, p_1, \ldots, p_5)) := p_6$, where p_6 is the residual point of intersection of Q with the unique cubic $E \in |\mathcal{O}_{\mathbf{P}^2}(3)|$ passing through $q_1, \ldots, q_4, p_1, \ldots, p_5$. We consider the linear system

$$\mathbf{P}_{(Q,p_1,\ldots,p_5)} := \Big\{\Gamma \in \Big|\mathcal{O}_{S'}(6)\big(-2\sum_{i=1}^4 E_{q_i}\big)\Big| : \Gamma \cdot Q = 2(p_1 + \cdots + p_6)\Big\}.$$

Claim: For a general $(Q, p_1, \ldots, p_5) \in \mathcal{V}$, we have dim $\mathbf{P}_{(Q,p_1,\ldots,p_5)} = 4$, that is, the points $q_1, \ldots, q_4, p_1, \ldots, p_6$ fail to impose one independent condition on 4-nodal sextic curves.

Since $\ell + Q + \mathbf{P}^3_{S'} \subset \mathbf{P}_{(Q,p_1,\ldots,p_5)}$, to conclude that $\dim \mathbf{P}_{(Q,p_1,\ldots,p_5)} \geq 4$, it suffices to find *one* curve $\Gamma \in \mathbf{P}_{(Q,p_1,\ldots,p_5)}$ that does not have ℓ as a component. We choose $(Q, p_1, \ldots, p_5) \in \mathcal{V}$ general enough that the corresponding cubic E is smooth. Then $2E \in \mathbf{P}_{(Q,p_1,\ldots,p_5)}$ and obviously $\ell \nsubseteq 2E$. To finish the proof of the claim, we exhibit a point $(Q, p_1, \ldots, p_5) \in \mathcal{V}$ such that $\dim(\mathbf{P}_{(Q,p_1,\ldots,p_5)}) = 4$. We specialize to the case $p_1 \in \ell$ and let Q_2 be the conic determined by $p_2, \ldots, p_5$ and q_4. The cubic E must equal $\ell + Q_2$ and $p_6 \in \ell \cap Q$, so that $E \cdot Q = p_1 + \cdots + p_6$. Then $\mathbf{P}_{(Q,p_1,\ldots,p_5)} = \ell + \mathbf{P}'$, where

$$\mathbf{P}' := \Big\{ Y \in \Big| \mathcal{O}_{S'}(5)\big(-\sum_{i=1}^{3} E_{q_i} - 2E_{q_4}\big) \Big| : Y \cdot Q_2 = p_1 + p_6 + 2(p_2 + p_3 + p_4 + p_5) \Big\}.$$

Because $p_2, \ldots, p_5 \in \mathbf{P}^2$ are general, $\dim(\mathbf{P}') = 20 - 3 - 3 - 2 - 8 = 4$, which completes the proof of the claim.

We now consider the $\mathbf{P}^4$-bundle $\mathcal{P} := \big\{(Q, p_1, \ldots, p_5, \Gamma) : \Gamma \in \mathbf{P}_{(Q,p_1,\ldots,p_5)}\big\}$, together with the map $u : \mathcal{P} \dashrightarrow \mathcal{R}_6$, given by

$$u((Q, p_1, \ldots, p_5, \Gamma)) := \big(C, \eta := \mathcal{O}_C(1)(-p_1 - \cdots - p_6)\big),$$

where $C \subset S'$ is the normalization of Γ. Then $M := \mathcal{O}_C(2)(-\sum_{i=1}^4 E_{q_i}) \in W^1_4(C)$ is Petri special and $|M \otimes \eta| \cong |\mathcal{O}_{S'}(3)(-\sum_{i=1}^4 E_{q_i} - \sum_{j=1}^6 p_j)| \neq \emptyset$, hence $u(\mathcal{P}) \subset \mathcal{Q}_4$. Therefore there is an induced map $\bar{u} : \mathcal{P} /\!/ \operatorname{Aut}(S') \dashrightarrow \mathcal{Q}_4$ between 13-dimensional varieties. Since every curve $(C, \eta) \in \mathcal{Q}_4$ has a totally tangent conic and can be embedded in S', it follows that any $M \in W^1_4(C)$ with $h^0(C, M \otimes \eta) \geq 1$ appears in the way described above, which finishes the proof. $\square$

Another distinguished codimension 2 cycle in $\mathcal{R}_6$ is the locus

$$\mathcal{Q}'_4 := \{(C, \eta) \in \mathcal{R}_6 : \eta \in W_2(C) - W_2(C)\}$$

of Prym curves (C, η) for which $\varphi_{K_C \otimes \eta}$ fails to be very ample. Writing $\eta = \mathcal{O}_C(a + b - p - q)$, with $a, b, p, q \in C$, then $M := \mathcal{O}_C(2a + 2b) \in W^1_4(C)$ and the 2-nodal image curve $\varphi_{K_C \otimes \eta}(C)$ lies on a pencil of quadrics in $\mathbf{P}^4$, thus also on a singular quadric of type $(4, 6)$. We show however, that this quadric is *not* the projectivized tangent cone of a quadratic singularity $L \in \operatorname{Sing}^{\mathrm{st}}_{(C,\eta)}(\Xi)$, hence points in $\mathcal{Q}'_4$ do not constitute a component of $P^{-1}(H)$.

Proposition 9.6. *We have $\mathcal{Q}'_4 \nsubseteq \mathcal{U}$. In particular, all singularities of the Prym theta divisor corresponding to a general point of Q'_4 are ordinary double points, that is, $P(\mathcal{Q}'_4) \nsubseteq H$.*

Proof. Note that $\mathcal{Q}'_4$ is not contained in $\mathcal{U}$, then use Proposition 5.4. $\square$

Proposition 9.7. *The locus $\mathcal{Q}_4$ dominates via the Prym map the locus H_1, that is, $\overline{P(\mathcal{Q}_4)} \supset H_1$.*

Proof. We start with a point $x_0 = (\tau_0, z_0) \in \mathcal{S}'$, corresponding to a singular point $z_0 \in \Theta_{\tau_0}$ such that $\operatorname{rk} H(x_0) \leq 4$ and x_0 is a general point of a component of $H - \theta_{\mathrm{null}}$. In particular $(A_{\tau_0}, \Theta_{\tau_0})$ can be chosen outside any subvariety of $\mathcal{A}_5$ having codimension at least 3. Since each component of $\mathcal{S}'$ maps generically finite onto N'_0, we find a deformation $\{x_t = (\tau_t, z_t)\}_{t \in T} \subset \mathcal{S}'$, parameterized by an integral curve $T \ni 0$, such that for all $t \in T - \{0\}$, the corresponding theta divisor Θ_t has only a pair of singular points, that is, $\mathrm{Sing}(\Theta_t) = \{\pm z_t\}$. Since $P(\mathcal{Q})$ is dense in N_0', after possibly shrinking T, we can find a family of triples $\{(C_t, \eta_t, L_t)\}_{t\in T}$, such that $(C_t, \eta_t) \in \mathcal{Q}$ for all $t \in T$, while for $t \neq 0$ the line bundle $L_t \in V_3(C_t, \eta_t)$ corresponds to the singularity $z_t \in \mathrm{Sing}(\Xi_t)$. If we set $(C, L, \eta) := (C_0, L_0, \eta_0)$, by semicontinuity we obtain that $h^0(C, L) \geq 4$. Since $\mathrm{rk}\, H(L) = \mathrm{rk}\, Q_L \leq 4$, it follows from Proposition 5.4 that $L \in \mathrm{Sing}^{\mathrm{st}}_{(C,\eta)}(\Xi) \cap \mathrm{Sing}^{\mathrm{ex}}_{(C,\eta)}(\Xi)$, which implies that the Prym-canonical line bundle can be expressed as a sum of two pencils. Since L is not a theta characteristic and $P(C, \eta) \notin \mathcal{J}_5$, we obtain that the Prym-canonical bundle can be expressed as $K_C \otimes \eta = A \otimes A'$, where $A \in W^1_4(C)$. From Proposition 9.6 it follows that $K_C \otimes \eta$ can be assumed to be very ample, that is, $(C, \eta) \in \mathcal{Q}_4$. $\square$

Corollary 9.8. $P(\mathcal{Q}_4)$ *is a unirational component of* H*, different from* θ^4_{null}.

9.1. **A parametrization of θ^4_{null}.** Our aim is to find an explicit unirational parametrization of θ^4_{null}.

Proposition 9.9. $\overline{P(\mathcal{Q}_5)} = \theta^4_{\mathrm{null}}$*, where the the closure is taken inside* $\mathcal{A}_5$.

Proof. This proof resembles that of Proposition 9.7. If $\phi : \mathcal{X}_5 \to \mathcal{A}_5$ denotes the universal abelian variety, recall that we have showed that $\phi_*(\mathcal{S}_{\mathrm{null}} \cap \mathcal{S}') = \theta^4_{\mathrm{null}}$. Thus a point $(\tau, z) \in \mathcal{S}_{\mathrm{null}} \cap \mathcal{S}'$ corresponding to a general point (A_τ, Θ_τ) of a component of θ^4_{null} is a Prym variety $P(C, \eta)$, where $(C, \eta) \in \mathcal{Q} \cap \mathcal{U}$ is a Prym curve such that $z \in \mathrm{Sing}(\Theta_\tau)$ corresponds to a singularity $L \in \mathrm{Sing}^{\mathrm{st}}_{(C,\eta)}(\Xi) \cap \mathrm{Sing}^{\mathrm{ex}}_{(C,\eta)}(\Xi)$. Then $L = f^*(\theta_1)$, where $\theta_1 \in \mathrm{Pic}^5(C)$ is a vanishing theta-null. Since $h^0(\tilde{C}, L) = h^0(C, \theta_1) + h^0(C, \theta_1 \otimes \eta) \geq 4$, we find that $\theta_2 := \theta_1 \otimes \eta$ is another theta characteristic, that is, $(C, \eta) \in \mathcal{Q}_5$. Therefore $\theta^4_{\mathrm{null}} \subseteq \overline{P(\mathcal{Q}_5)}$. The reverse inclusion being obvious, we finish the proof. $\square$

We can now complete the proof of Theorem 0.5. We consider the smooth quadric $Q := \mathbf{P}^1 \times \mathbf{P}^1$ and the linear systems of rational curves

$$\mathbf{P}^7_1 := \left|\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(3,1)\right| \quad \text{and } \mathbf{P}^7_2 := \left|\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,3)\right|.$$

Over $\mathbf{P}^7_1 \times \mathbf{P}^7_2$ we define the $\mathbf{P}^5$-bundle

$$\mathcal{U} := \Big\{(R_1, R_2, \Gamma) : R_i \in \mathbf{P}^7_i \text{ for } i = 1, 2, \quad \Gamma \in |\mathcal{I}^2_{R_1\cdot R_2/Q}(5,5)|\Big\}.$$

There is an induced rational map $\psi : \mathcal{U} \dashrightarrow \mathcal{Q}_5$ given by

$$\psi(R_1, R_2, \Gamma) := (C, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(-1)) \in \mathcal{R}_6,$$

where $\nu : C \to \Gamma$ is the normalization map and $p_1, p_2 : C \to \mathbf{P}^1$ are the composition of ν with the two projections.

A general pair $(R_1, R_2) \in \mathbf{P}^7_1 \times \mathbf{P}^7_2$ corresponds to smooth rational curves such that the intersection cycle $R_1 \cdot R_2 = o_1 + \cdots + o_{10}$ consists of distinct points. For any curve $\Gamma \in |\mathcal{I}^2_{R_1\cdot R_2}(5,5)|$ we have $R_1 \cdot \Gamma = R_2 \cdot \Gamma = 2(o_1 +$

$\cdots + o_{10})$. Since $\nu^* : |\mathcal{I}_{R_1 \cdot R_2}(3,3)| \to |K_C|$ is an isomorphism, it follows that both $p_1^*\mathcal{O}(1)$ and $p_2^*\mathcal{O}(1)$ are vanishing theta-nulls, hence $\psi(\mathcal{U}) \subset \mathcal{Q}_5$.

Theorem 9.10. *The rational map $\psi : \mathcal{U} \dashrightarrow \mathcal{Q}_5$ is generically finite and dominant. In particular $\mathcal{Q}_5$ (and thus $\theta^4_{\mathrm{null}} = \overline{P(\mathcal{Q}_5)}$) is unirational.*

Proof. We start with a point $(C, \theta_1, \theta_2) \in \mathcal{Q}_5$ moving in a 13-dimensional family. In particular, the image Γ of the induced map $\varphi_{(\theta_1,\theta_2)} : C \to \mathbf{P}^1 \times \mathbf{P}^1$ is nodal and we set $\mathrm{Sing}(\Gamma) = \{o_1, \ldots, o_{10}\}$.

We choose divisors $D, D' \in |\theta_1|$, corresponding to lines $\ell, \ell' \in |\mathcal{O}_Q(1,0)|$ such that $\nu^*(\Gamma \cdot \ell) = D$ and $\nu^*(\Gamma \cdot \ell') = D'$ respectively. Then $D + D' \in |K_C|$, and since the linear system $|\mathcal{I}_{o_1+\cdots+o_{10}}(3,3)|$ cuts out the canonical system on C, it follows that there exists a cubic curve $E \in |\mathcal{O}_Q(3,3)|$ such that

$$E \cdot \Gamma = D + D' + 2\sum_{i=1}^{10} o_i.$$

By Bézout's Theorem, both ℓ and ℓ' must be components of E, that is, we can write $E = \ell + \ell' + R_1$, where $R_1 \in |\mathcal{O}_Q(1,3)|$ is such that $R_1 \cdot \Gamma = 2\sum_{i=1}^{10} o_i$. Switching the roles of θ_1 and θ_2, there exists $R_2 \in |\mathcal{O}_Q(3,1)|$ such that $R_2 \cdot \Gamma = 2\sum_{i=1}^{10} o_i$. It follows that $(R_1, R_2, \Gamma) \in \psi^{-1}\left((C, \theta_1 \otimes \theta_2^\vee)\right)$. The variety $\mathcal{U}$ being a $\mathbf{P}^5$-bundle over $\mathbf{P}_1^7 \times \mathbf{P}_2^7$ is unirational, hence $\mathcal{Q}_5$ is unirational as well, thus finishing the proof. $\square$

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References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.

[ADC84] E. Arbarello and C. De Concini. On a set of equations characterizing Riemann matrices. *Ann. of Math.*, 120(1):119–140, 1984.

[AM67] A. Andreotti and A. L. Mayer. On period relations for abelian integrals on algebraic curves. *Ann. Scuola Norm. Sup. Pisa (3)*, 21:189–238, 1967.

[Bea77] A. Beauville. Prym varieties and the Schottky problem. *Invent. Math.*, 41(2):149–196, 1977.

[CvdG00] C. Ciliberto and G. van der Geer. The moduli space of abelian varieties and the singularities of the theta divisor. *Surv. Differential Geom.*, VII, pages 61–81, Int. Press, Somerville, MA, 2000.

[CvdG08] C. Ciliberto and G. van der Geer. Andreotti-Mayer loci and the Schottky problem. *Doc. Math.*, 13:453–504, 2008.

[Deb92] O. Debarre. Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes. *Ann. Sci. École Norm. Sup. (4)*, 25(6):687–707, 1992.

[Don92] R. Donagi. The fibers of the Prym map. In *Curves, Jacobians, and abelian varieties (Amherst, MA, 1990)*, volume 136 of *Contemp. Math.*, pages 55–125. Amer. Math. Soc., Providence, RI, 1992.

[DS81] R. Donagi and R. Smith. The structure of the Prym map. *Acta Math.*, 146(1-2):25–102, 1981.

[EH87] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, 90:359–387, 1987.

[EL97] L. Ein and R. Lazarsfeld. Singularities of theta divisors and the birational geometry of irregular varieties. *J. Amer. Math. Soc.*, 10(1):243–258, 1997.

[Far05] G. Farkas. Gaussian maps, Gieseker-Petri loci and large theta-characteristics. *J. Reine Angew. Math.*, 581:151–173, 2005.

[FL10] G. Farkas and K. Ludwig. The Kodaira dimension of the moduli space of Prym varieties. *J. European Math. Soc. (JEMS)*, 12(3):755–795, 2010.

[FV11] G. Farkas and A. Verra. Moduli of theta-characteristics via Nikulin surfaces. *Math. Annalen*, 354:465-496, 2012.

[HF06] H. Farkas. Vanishing thetanulls and Jacobians. In *The geometry of Riemann surfaces and abelian varieties*, volume 397 of *Contemp. Math.*, pages 37–53. Amer. Math. Soc., Providence, RI, 2006.

[Fre83] E. Freitag. *Siegelsche Modulfunktionen*, volume 254 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1983.

[vdG99] G. van der Geer. *Cycles on the moduli space of abelian varieties*, pages 65–89. Aspects Math., E33. Vieweg, Braunschweig, 1999.

[GSM07] S. Grushevsky and R. Salvati Manni. Singularities of the theta divisor at points of order two. *Int. Math. Res. Not. IMRN*, (15):Art. ID rnm045, 15, 2007.

[GSM08] S. Grushevsky and R. Salvati Manni. Jacobians with a vanishing theta-null in genus 4. *Israel J. Math.*, 164:303–315, 2008.

[GSM11] S. Grushevsky and R. Salvati Manni. The Prym map on divisors, and the slope of $\mathcal{A}_5$ (with an appendix by K. Hulek). 2011. preprint arXiv:1107.3094.

[Igu72] J.-I. Igusa. *Theta functions*, volume 194 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1972.

[Igu81a] J.-I. Igusa. On the irreducibility of Schottky's divisor. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):531–545 (1982), 1981.

[Igu81b] J.-I. Igusa. Schottky's invariant and quadratic forms. In *E. B. Christoffel (Aachen/Monschau, 1979)*, pages 352–362. Birkhäuser, Basel, 1981.

[dJ10] R. de Jong. Theta functions on the theta divisor. *Rocky Mountain J. Math.*, 40(1):155–176, 2010.

[KSM02] J. Kramer and R. Salvati Manni. An integral characterizing the Andreotti-Mayer locus. *Abh. Math. Sem. Univ. Hamburg*, 72:47–57, 2002.

[Mum74] D. Mumford. Prym varieties. I. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 325–350. Academic Press, New York, 1974.

[Mum83] D. Mumford. On the Kodaira dimension of the Siegel modular variety. In *Algebraic geometry—open problems (Ravello, 1982)*, volume 997 of *Lecture Notes in Math.*, pages 348–375, Berlin, 1983. Springer.

[MV11] M. Matone and R. Volpato. Vector-valued modular forms from the Mumford form, Schottky-Igusa form, product of thetanullwerte and the amazing Klein formula. 2011. preprint arXiv:1102.0006.

[SM92] R. Salvati Manni. Modular forms of the fourth degree. Remark on the paper: "Slopes of effective divisors on the moduli space of stable curves" [Invent. Math. **99** (1990), 321–355] by J. Harris and I. Morrison. Volume 1515 of *Lecture Notes in Math.*, pages 106–111. Springer, Berlin, 1992.

[SB06] N. Shepherd-Barron, Perfect forms and the moduli space of abelian varieties. *Invent. Math.*, 163:25–45, 2006.

[SV85] R. Smith and R. Varley. Components of the locus of singular theta divisors of genus 5. In *Algebraic geometry, Sitges (Barcelona), 1983*, volume 1124 of *Lecture Notes in Math.*, pages 338–416. Springer, Berlin, 1985.

[SV04] R. Smith and R. Varley. A necessary and sufficient condition for Riemann's singularity theorem to hold on a Prym theta divisor. *Compos. Math.*, 140(2):447–458, 2004.

[SV12a] R. Smith and R. Varley. Deformations of isolated even double points of corank one. *Proc. Amer. Math. Soc.* to appear.

[SV12b] R. Smith and R. Varley. A splitting criterion for an isolated singularity at $x = 0$ in a family of even hypersurfaces. *Manuscripta Math.* 137:233–245, 2012.

[Vor1908] G. Voronoi. Nouvelles Applications Des Paramètres continus à la théorie des formes quadratiques. Premier mémoire. Sur quelques propriétés des formes quadratiques positives parfaites. *J. Reine Angew. Math.* 133:79–178, 1909.

[Wel85] G. Welters. A theorem of Gieseker-Petri type for Prym varieties. *Ann. Sci. École Norm. Sup. (4)*, 18(4):671–683, 1985.

[Yos99] K.-I. Yoshikawa. Discriminant of theta divisors and Quillen metrics. *J. Differential Geom.*, 52(1):73–115, 1999.

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