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The global geometry of the moduli space of curves

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1. Introduction

ABSTRACT. We survey the progress made in the last decade in understanding the birational geometry of the moduli space of stable curves. Topics that are being discusses include the cones of ample and effective divisors, Kodaira dimension and minimal models of $\overline{\mathcal{M}}_g$.

For a complex projective variety X, one way of understanding its birational geometry is by describing its cones of *ample* and *effective* divisors

$$\mathrm{Ample}(X) \subset \mathrm{Eff}(X) \subset N^1(X)_{\mathbb{R}}.$$ [1]

The closure in $N^1(X)_{\mathbb{R}}$ of $\mathrm{Ample}(X)$ is the cone $\mathrm{Nef}(X)$ of *numerically effective* divisors, i.e. the set of all classes $e \in N^1(X)_{\mathbb{R}}$ such that $C \cdot e \geq 0$ for all curves $C \subset X$. The interior of the closure $\overline{\mathrm{Eff}(X)}$ is the cone of *big* divisors on X. Loosely speaking, one can think of the nef cone as parametrizing *regular* contractions [2] from X to other projective varieties, whereas the effective cone accounts for *rational* contractions of X. For arbitrary varieties of dimension ≥ 3 there is little connection between $\mathrm{Nef}(X)$ and $\mathrm{Eff}(X)$ (for surfaces there is Zariski decomposition which provides a unique way of writing an effective divisor as a combination of a nef and a "negative" part and this relates the two cones, see e.g. [**L1**]). Most questions in higher dimensional geometry can be phrased in terms of the ample and effective cones. For instance, a smooth projective variety X is of general type precisely when $K_X \in \mathrm{int}(\overline{\mathrm{Eff}(X)})$.

The question of describing the ample and the effective cone of $\overline{\mathcal{M}}_g$ goes back to Mumford (see e.g. [**M1**], [**H2**]). Moduli spaces of curves with their inductive structure given by the boundary stratification are good test cases for many problems coming from higher dimensional birational geometry. The first major result result on the global geometry of $\overline{\mathcal{M}}_g$ was the celebrated theorem of Harris, Mumford and Eisenbud that $\overline{\mathcal{M}}_g$ is of general type for all $g \geq 24$ (cf. [**HM**], [**H1**], [**EH3**]). This result disproved a famous conjecture of Severi's who, based on evidence coming from small genus, predicted that $\mathcal{M}_g$ is unirational for all g. The space $\overline{\mathcal{M}}_g$

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[1]Throughout this paper we use the formalism of $\mathbb{R}$-divisors and we say that a class $e \in N^1(X)_{\mathbb{R}}$ is *effective* (resp. *ample*) if e is represented by a $\mathbb{R}$-divisor D on X which is effective (resp. ample).

[2]Recall that a contraction of X is a morphism $f : X \to Y$ with connected fibres.

being of general type, implies for instance that the general curve of genus $g \geq 24$ does not appear in any non-trivial linear system on any non-ruled surface.

The main aim of this paper is to discuss what is currently known about the ample and the effective cones of $\overline{\mathcal{M}}_{g,n}$. Conjecturally, the ample cone has a very simple description being dual to the cone spanned by the irreducible components of the locus in $\overline{\mathcal{M}}_{g,n}$ that consists of curves with $3g-4+n$ nodes (cf. **[GKM]**). The conjecture has been verified in numerous cases and it predicts that for large g, despite being of general type, $\overline{\mathcal{M}}_g$ behaves from the point of view of Mori theory just like a Fano variety, in the sense that the cone of curves is polyhedral, generated by rational curves. In the case of the effective cone the situation is more complicated. In **[FP]** we showed that the Harris-Morrison Slope Conjecture which singled out the Brill-Noether divisors on $\overline{\mathcal{M}}_g$ as those of minimal slope, is false. In this paper we describe a very general construction of geometric divisors on $\overline{\mathcal{M}}_g$ which provide counterexamples to the Slope Conjecture in infinitely many genera (see Theorem 3.2). Essentially, we construct an effective divisor of exceptionally small slope on $\overline{\mathcal{M}}_g$ for $g = s(2s+si+i+1)$, where $s \geq 2, i \geq 0$. For $s=1$, we recover the formula for the class of the Brill-Noether divisor first computed by Harris and Mumford in **[HM]**. The divisors constructed in **[EH3]**, **[FP]**, **[F2]** and **[Kh]** turn out to be particular cases of this construction.

In spite of all the counterexamples, it still seems reasonable to believe that a "Weak" Slope Conjecture on $\overline{\mathcal{M}}_g$ should hold, that is, there should be a universal lower bound on the slopes of all effective divisors on $\overline{\mathcal{M}}_g$ which is independent of g. This fact would highlight a fundamental difference between $\mathcal{M}_g$ and $\mathcal{A}_g$ and would provide a modern solution to the *Schottky problem* (see Subsection 2.2 for more details). In Section 3 we announce a proof that $\overline{\mathcal{M}}_{22}$ is of general type and we describe the Kodaira type of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves.

2. Divisors on $\overline{\mathcal{M}}_{g,n}$

For non-negative integers g and n such that $2g-2+n>0$ we denote by $\overline{\mathcal{M}}_{g,n}$ the moduli stack of n-pointed stable curves of genus g. The stack $\overline{\mathcal{M}}_{g,n}$ has a stratification given by *topological type*, the codimension k strata being the components of the closure in $\overline{\mathcal{M}}_{g,n}$ of the locus of curves with k nodes. The 1-dimensional strata are also called F*-curves* and it is easy to see that each F-curve is isomorphic to either $\overline{\mathcal{M}}_{0,4}$ or to $\overline{\mathcal{M}}_{1,1}$. It is straightforward to list all F-curves on a given $\overline{\mathcal{M}}_{g,n}$. For instance, F-curves on $\overline{\mathcal{M}}_{0,n}$ are in $1:1$ correspondence with partitions (n_1, n_2, n_3, n_4) of n, the corresponding F-curve being the image of the map $\nu : \overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,n}$ which takes a rational 4-pointed curve $[R, x_1, x_2, x_3, x_4]$ to a rational n-pointed curve obtained by attaching a fixed rational (n_i+1)-pointed curve at the point x_i.

The codimension 1 strata in the topological stratification are the boundary divisors on $\overline{\mathcal{M}}_{g,n}$ which are indexed as follows: For $0 \leq i \leq g$ and $S \subset \{1, \ldots, n\}$, we denote by $\delta_{i:S}$ the class of the closure of the locus of nodal curves $C_1 \cup C_2$, where C_1 is a smooth curve of genus i, C_2 is a smooth curve of genus $g-i$ and such that the marked points sitting on C_1 are precisely those labeled by S. We also have the class δ_0 corresponding to irreducible pointed curves with a single node. Apart from boundary divisor classes, we also introduce the tautological classes

$\{\psi_i = c_1(\mathbb{L}_i)\}_{i=1}^n$ corresponding to the n marked points. Here $\mathbb{L}_i$ is the line bundle over the moduli stack with fibre $\mathbb{L}_i[C, x_1, \ldots, x_n] := T_{x_i}(C)^\vee$ over each point $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$. Finally, we have the Hodge class defined as follows: If $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the universal curve, we set $\lambda := c_1(\mathbb{E}) \in \text{Pic}(\overline{\mathcal{M}}_g)$, where $\mathbb{E} := \pi_*(\omega_\pi)$ is the rank g Hodge bundle on $\overline{\mathcal{M}}_g$. A result of Harer and Arbarello-Cornalba (cf. [**AC**]) says that $\lambda, \psi_1, \ldots, \psi_n$ together with the boundary classes δ_0 and $\delta_{i:S}$ generate $\text{Pic}(\overline{\mathcal{M}}_{g,n})$. This result has been extended to arbitrary characteristic by Moriwaki (cf. [**Mo2**]). When $g \geq 3$ these classes form a basis of $\text{Pic}(\overline{\mathcal{M}}_{g,n})$.

2.1. The ample cone of $\overline{\mathcal{M}}_{g,n}$.

In this section we describe the ample cone of $\overline{\mathcal{M}}_{g,n}$. Historically speaking, the study of ample divisors on $\overline{\mathcal{M}}_g$ began when Cornalba and Harris proved that the $\mathbb{Q}$- class $a\lambda - \delta_0 - \cdots - \delta_{[g/2]}$ is ample on $\overline{\mathcal{M}}_g$ if and only if $a > 11$ (cf. [**CH**]). Later, Faber completely determined $\text{Ample}(\overline{\mathcal{M}}_3)$: a class $D \equiv a\lambda - b_0\delta_0 - b_1\delta_1 \in \text{Pic}(\overline{\mathcal{M}}_3)$ is nef if and only if

$$2b_0 - b_1 \geq 0, \ \ b_1 \geq 0 \text{ and } \ a - 12b_0 + b_1 \geq 0 \ \ \text{(cf. [**Fa**])}.$$

He pointed out that the numbers appearing in the left hand side of these inequalities are intersection numbers of D with certain F-curves in $\overline{\mathcal{M}}_3$ thus raising for the first time the possibility that the F-curves might generate the Mori cone of curves $NE_1(\overline{\mathcal{M}}_{g,n})$. The breakthrough in this problem came when Gibney, Keel and Morrison proved that strikingly, $NE_1(\overline{\mathcal{M}}_{g,n})$ is the sum of the cone generated by F-curves and the cone $NE_1(\overline{\mathcal{M}}_{0,g+n})$. In this way, computing the nef cone of $\overline{\mathcal{M}}_{g,n}$ for any $g > 0$ always boils down to a problem in genus 0!

THEOREM 2.1. *([GKM]) If $j : \overline{\mathcal{M}}_{0,g+n} \to \overline{\mathcal{M}}_{g,n}$ is the "flag map" given by attaching fixed elliptic tails to the first g marked points of every $(g+n)$-pointed stable rational curve, then a divisor D on $\overline{\mathcal{M}}_{g,n}$ is nef if and only if $j^*(D)$ is nef on $\overline{\mathcal{M}}_{0,g+n}$ and $D \cdot C \geq 0$ for every F-curve C in $\overline{\mathcal{M}}_{g,n}$.*

This reduction to genus 0 then makes the following conjecture very plausible:

CONJECTURE 2.2. ([GKM]) The Mori cone $NE_1(\overline{\mathcal{M}}_{g,n})$ is generated by F-curves. A divisor D on $\overline{\mathcal{M}}_{g,n}$ is ample if and only if $D \cdot C > 0$ for every F-curve C in $\overline{\mathcal{M}}_{g,n}$.

The conjecture reflects the expectation that the extremal rays of $\overline{\mathcal{M}}_{g,n}$ should have modular meaning. Since F-curves can be easily listed, this provides an explicit (conjectural) description of the ample cone. For instance, on $\overline{\mathcal{M}}_g$, the conjecture predicts that a divisor $D \equiv a\lambda - b_0\delta_0 - \cdots - b_{[g/2]}\delta_{[g/2]} \in \text{Pic}(\overline{\mathcal{M}}_g)$ is ample if and only if the following inequalities are satisfied:

$$a - 12b_0 + b_1 > 0, \ \ 2b_0 > b_i > 0 \text{ for all } \ i \geq 1,$$

$$b_i + b_j > b_{i+j} \ \text{ for all } i, j \geq 1 \ \text{ with } \ i + j \leq g - 1$$

and

$$b_i + b_j + b_k + b_l > b_{i+j} + b_{i+k} + b_{i+l} \text{ for all } \ i, j, k, l \geq 1 \text{ with } \ i + j + k + l = g.$$

Here we have the usual convention $b_i = b_{g-i}$. Conjecture 2.2 has been checked on $\overline{\mathcal{M}}_g$ for all $g \leq 24$ in [**KMc**], [**FG**] and [**G**]. In fact, Gibney has reduced the conjecture on a given $\overline{\mathcal{M}}_g$ to an entirely combinatorial question which can be checked by computer. Recently, Coskun, Harris and Starr have reduced the calculation of the ample cone of the moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^r, d)$ to Conjecture 2.2 for $\overline{\mathcal{M}}_{0,n+d}$ (cf. **[CHS]**). In **[GKM]** it is also pointed out that Conjecture 2.2 would be implied by an older conjecture of Fulton motivated by the fact that $\overline{\mathcal{M}}_{0,n}$ has many of the geometric features of a toric variety (without being a toric variety, of course):

CONJECTURE 2.3. Any divisor class D on $\overline{\mathcal{M}}_{0,n}$ satisfying $C \cdot D \geq 0$ for all F-curves C in $\overline{\mathcal{M}}_{0,n}$ can be expressed as an effective combination of boundary classes.

Fulton's conjecture is true on $\overline{\mathcal{M}}_{0,n}$ for $n \leq 6$ (cf. **[FG]**). Note that it is not true that every effective divisor on $\overline{\mathcal{M}}_{0,n}$ is equivalent to an effective combination of boundary divisors (cf. **[Ve]**): If $\xi : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_3$ denotes the map which identifies three pairs of marked points on a genus 0 curve, then the pull-back of the hyperelliptic locus $\xi^*(\overline{\mathcal{M}}^1_{3,2})$ is an effective divisor on $\overline{\mathcal{M}}_{0,6}$ for which there exists an explicit curve $R \subset \overline{\mathcal{M}}_{0,6}$ not contained in the boundary of $\overline{\mathcal{M}}_{0,6}$ such that $R \cdot \xi^*(\overline{\mathcal{M}}^1_{3,2}) < 0$. Thus $\xi^*(\overline{\mathcal{M}}_{0,6})$ is not an effective combination of boundary divisors.

REMARK 2.4. In low genus one can show that Ample$(\overline{\mathcal{M}}_g)$ is "tiny" inside the much bigger cone Eff$(\overline{\mathcal{M}}_g)$ which underlies the fact that regular contractions of $\overline{\mathcal{M}}_g$ do not capture the rich birational geometry of $\overline{\mathcal{M}}_g$ (For instance, the only divisorial contraction of $\overline{\mathcal{M}}_{g,n}$ with relative Picard number 1 is the blow-down of the elliptic tails, see **[GKM]**). The difference between the two cones can be vividly illustrated on $\overline{\mathcal{M}}_3$: we have seen that Nef$(\overline{\mathcal{M}}_3)$ is generated by the classes $\lambda, 12\lambda - \delta_0$ and $10\lambda - \delta_0 - 2\delta_1$ (cf. **[Fa]**), whereas it is easy to show that Eff$(\overline{\mathcal{M}}_3)$ is much larger, being spanned by δ_0, δ_1 and the class of the hyperelliptic locus $h = 9\lambda - \delta_0 - 3\delta_1$.

Theorem 2.1 has a number of important applications to the study of regular morphisms from $\overline{\mathcal{M}}_{g,n}$ to other projective varieties. For instance it is known that for $g \geq 2$, $\overline{\mathcal{M}}_g$ has no non-trivial fibrations (that is, morphisms with connected fibres to lower dimensional varieties). Any fibration of $\overline{\mathcal{M}}_{g,n}$ must factor through one of the forgetful maps $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,i}$ for some $i < n$ (see **[GKM]**, Corollary 0.10). If $f : \overline{\mathcal{M}}_{g,n} \to X$ is a birational morphism to a projective variety, it is known that the exceptional locus Exc(f) is contained in the boundary of $\overline{\mathcal{M}}_{g,n}$. In particular such a projective variety X is a new compactification of $\mathcal{M}_g$. (The use of such a result is limited however by the fact that there are very few known examples of regular morphism from $\overline{\mathcal{M}}_{g,n}$). Theorem 2.1 can be directly applied to show that many types of divisors D on $\overline{\mathcal{M}}_{g,n}$ which non-negatively meet all F-curves are actually nef. For instance one has the following result (cf. **[GKM]**, Proposition 6.1):

THEOREM 2.5. *If $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ is a divisor on $\overline{\mathcal{M}}_g$ such that $b_i \geq b_0$ for all $1 \leq i \leq [g/2]$ and $C \cdot D \geq 0$ for any F-curve C, then D is nef.*

REMARK 2.6. Since any regular morphism $f : \overline{\mathcal{M}}_{g,n} \to X$ to a projective variety is responsible for a semi-ample line bundle $L := f^*(\mathcal{O}_X(1))$ rather than a nef one, it is a very interesting question to try to characterize semi-ample line bundles on $\overline{\mathcal{M}}_{g,n}$. Surprisingly, this question is easier to handle in positive characteristic due to the following result of Keel (cf. [**K**], Theorem 0.2): If L is a nef line bundle on a projective variety X over a field of positive characteristic, then L is semi-ample if and only if the restriction of L to its *exceptional locus* Exc(L) is semi-ample. Recall that if L is a nef line bundle on X, then

$$\text{Exc}(L) := \bigcup\{Z \subset X : Z \text{ is an irreducible subvariety with } L^{\dim(Z)} \cdot Z = 0\}.$$

An easy application of Keel's Theorem is that the tautological class $\psi \in \text{Pic}(\overline{\mathcal{M}}_{g,1})$ is semi-ample in positive characteristic but fails to be so in characteristic 0 (see [**K**], Corollary 3.1). No example of a nef line bundle on $\overline{\mathcal{M}}_g$ which fails to be semi-ample is known although it is expected there are many such examples.

REMARK 2.7. It makes sense of course to ask what is the nef cone of the moduli space of abelian varieties. Shepherd-Barron computed the nef cone of the first Voronoi compactification $\mathcal{A}_g^I$ of $\mathcal{A}_g$ (cf. [**SB**]). Precisely, $NE_1(\mathcal{A}_g^I)$ is generated by two curve classes C_1 and C_2, where C_1 is any exceptional curve in the contraction of $\mathcal{A}_g^I$ to the Satake compactification of $\mathcal{A}_g$, while $C_2 = \{[X \times E]\}_{[E]\in\mathcal{A}_1}$, where $[X] \in \mathcal{A}_{g-1}$ is a fixed ppav of dimension $g-1$ and E is a moving elliptic curve. Hulek and Sankaran have determined the nef cone of the second Voronoi compactification $\mathcal{A}_4^{II}$ of $\mathcal{A}_4$ (cf. [**HS**]).

Towards the canonical model of $\overline{\mathcal{M}}_{g,n}$

A somewhat related question concerns the canonical model of $\overline{\mathcal{M}}_g$. Since the variety $\overline{\mathcal{M}}_g$ is of general type for large g, a result from [**BCHM**] implies the finite generation of the canonical ring $R(\overline{\mathcal{M}}_g) := \oplus_{n\geq 0} H^0(\overline{\mathcal{M}}_g, nK_{\overline{\mathcal{M}}_g})$ and the existence of a canonical model of the moduli space $\overline{\mathcal{M}}_g^{can} := \text{Proj}\big(R(\overline{\mathcal{M}}_g)\big)$. It is natural to ask for a modular interpretation of the canonical model. Very interesting ongoing work of Hassett and Hyeon provides the first steps towards understanding $\overline{\mathcal{M}}_g^{can}$ (see [**Ha1**], [**HH**], but also [**HL**] where the Minimal Model Program for $\overline{\mathcal{M}}_3$ is completed). Precisely, if

$$\delta := \delta_0 + \cdots + \delta_{[g/2]} \in \text{Pic}(\overline{\mathcal{M}}_g)$$

denotes the total boundary of $\overline{\mathcal{M}}_g$ and $K_{\overline{M}_g} = 13\lambda - 2\delta$ is the canonical class of the moduli stack, for each rational number $0 \leq \alpha \leq 1$ we introduce the *log canonical model*

$$\overline{\mathcal{M}}_g^{can}(\alpha) := \text{Proj}\Big(\oplus_{n\geq 0} H^0\big(\overline{\mathcal{M}}_g, n(K_{\overline{M}_g} + \alpha\delta)\big)\Big).$$

Then $\overline{\mathcal{M}}_g^{can}(\alpha) = \overline{\mathcal{M}}_g$ for $9/11 \leq \alpha \leq 1$ because of the already mentioned result of Cornalba and Harris [**CH**], whereas $\lim_{\alpha\to 0} \overline{\mathcal{M}}_g^{can}(\alpha) = \overline{\mathcal{M}}_g^{can}$. The first interesting question is what happens to $\overline{\mathcal{M}}_g^{can}(\alpha)$ when $\alpha = 9/11$ since in this case there there exists a curve $R \subset \overline{\mathcal{M}}_g$ such that $(K_{\overline{M}_g} + \frac{9}{11}\delta) \cdot R = 0$ (precisely, R corresponds to a pencil of plane cubics with a section which is attached to a fixed pointed curve of genus $g-1$). It turns out that for $7/10 < \alpha \leq 9/11$, the moduli space $\overline{\mathcal{M}}_g^{can}(\alpha)$ exists and it is identified with the space $\overline{\mathcal{M}}_g^{\text{ps}}$ of *pseudo-stable* curves in which cusps are allowed but elliptic tails are ruled out. The morphism $\overline{\mathcal{M}}_g \stackrel{|11\lambda-\delta|}{\longrightarrow} \overline{\mathcal{M}}_g^{\text{ps}}$ is a divisorial contraction of the boundary divisor Δ_1. The next (substantially more involved) step is to understand what happens when $\alpha = 7/10$. It turns out that

$\overline{\mathcal{M}}_g^{can}(7/10)$ exists as the quotient by SL_{3g-3} of the Chow variety of bicanonical curves $C \subset \mathbf{P}^{3g-4}$, whereas the model $\overline{\mathcal{M}}_g^{can}(7/10-\epsilon)$ for $0 \leq \epsilon << 1$ exists and it is obtained from $\overline{\mathcal{M}}_g^{can}(7/10+\epsilon)$ by an explicit flip and it parameterizes curves with nodes, cusps and tacnodes as singularities. As $\alpha \to 0$, one expects of course worse and worse singularities like higher-order tacnodes.

2.2. The effective cone of $\overline{\mathcal{M}}_g$.

Following Harris and Morrison [**HMo**], we define the slope function on the effective cone $s : \mathrm{Eff}(\overline{\mathcal{M}}_g) \to \mathbb{R} \cup \{\infty\}$ by the formula

$$s(D) := \inf \{\frac{a}{b} : a, b > 0 \text{ such that } a\lambda - b\delta - D \equiv \sum_{i=0}^{[g/2]} c_i\delta_i, \text{ where } c_i \geq 0\}.$$

From the definition it follows that $s(D) = \infty$ unless $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ with $a, b_i \geq 0$ for all i. Moreover, it is well-known that $s(D) < \infty$ for any D which is the closure of an effective divisor on $\mathcal{M}_g$. In this case one has that $s(D) = a/\min_{i=0}^{[g/2]} b_i$. We denote by $s(\overline{\mathcal{M}}_g)$ the *slope of the moduli space* $\overline{\mathcal{M}}_g$, defined as

$$s(\overline{\mathcal{M}}_g) := \inf \{s(D) : D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)\}.$$

CONJECTURE 2.8. (Harris, Morrison, [**HMo**]) We have the inequality

$$s(D) \geq 6 + \frac{12}{g+1}$$

for all effective divisors $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$, with equality if $g+1$ is composite and D is a combination of Brill-Noether divisors.

Let us recall the definition of the classical Brill-Noether divisors. We fix a genus $g \geq 3$ such that there exist $r, d \geq 1$ with $\rho(g, r, d) := g - (r+1)(g-d+r) = -1$ (in particular, $g+1$ has to be composite). We define the following geometric subvariety of $\mathcal{M}_g$

$$\mathcal{M}_{g,d}^r := \{[C] \in \mathcal{M}_g : C \text{ has a linear series of type } \mathfrak{g}_d^r\}.$$

Since $\rho(g, r, d)$ is the expected dimension of the determinantal variety $W_d^r(C)$ of $\mathfrak{g}_d^r$'s on a fixed curve C of genus g (see [**ACGH**]), one would naively expect that $\mathcal{M}_{g,d}^r$ is a divisor on $\mathcal{M}_g$. In fact we have a stronger result due to Eisenbud and Harris (cf. [**EH2**], [**EH3**]):

THEOREM 2.9. *The locus $\mathcal{M}_{g,d}^r$ is an irreducible divisor on $\mathcal{M}_g$ whenever $\rho(g, r, d) = -1$. Moreover, the class of the compactification in $\overline{\mathcal{M}}_g$ of the Brill-Noether divisor is given by the formula*

$$\overline{\mathcal{M}}_{g,d}^r \equiv c_{g,d,r}\Big((g+3)\lambda - \frac{g+1}{6}\,\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big),$$

where $c_{g,d,r}$ is an explicitly given constant.

Thus $s(\overline{\mathcal{M}}_{g,d}^r) = 6 + 12/(g+1)$ and then the Slope Conjecture singles out the Brill-Noether divisors on $\overline{\mathcal{M}}_g$ as those having minimal slope. Apart from the evidence coming from low genus, the conjecture was mainly based on the large number of calculations of classes of geometric divisors on $\overline{\mathcal{M}}_g$ (see e.g. [**EH3**], [**H1**]).

REMARK 2.10. If $D \equiv a\lambda - b_0\delta_0 - \cdots - b_{[g/2]}\delta_{[g/2]}$ is a nef divisor on $\overline{\mathcal{M}}_g$, then combining the inequalities $a - 12b_0 + b_1 \geq 0$ and $2b_0 - b_1 \geq 0$ obtained by intersecting D with two F-curves, we obtain that $s(D) \geq a/b_0 \geq 10$. On the other hand, the class $10\lambda - \delta_0 - 2\delta_1$ is nef on $\overline{\mathcal{M}}_g$ (cf. **[GKM]**), hence

$$\text{liminf}\{s(D) : D \in \text{Nef}(\overline{\mathcal{M}}_g)\} = 10.$$

This once again illustrates that nef divisors contribute little to the birational geometry of $\overline{\mathcal{M}}_g$.

As explained in the original paper **[HMo]**, the Slope Conjecture is intimately related to the problem of determining the Kodaira dimension of $\mathcal{M}_g$. Recall first the computation of the canonical class of $\overline{\mathcal{M}}_g$:

THEOREM 2.11. *(Harris-Mumford)*

$$K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}.$$

PROOF. If $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the universal curve, then by Kodaira-Spencer theory we have that $\Omega^1_{\overline{\mathcal{M}}_g} = \pi_*(\omega_\pi \otimes \Omega_\pi)$, where Ω_π is the cotangent sheaf while ω_π is the dualizing sheaf. Then apply Grothendieck-Riemann-Roch to the universal curve, to obtain that the canonical class of the *moduli stack* is equal to

$$K_{\overline{M}_g} = 13\lambda - 2(\delta + \ldots + \delta_{[g/2]}).$$

To obtain the formula for the canonical class $K_{\overline{\mathcal{M}}_g}$ of the *coarse moduli space* we use that the natural map from the stack to the coarse moduli space is simply branched along the boundary divisor Δ_1. □

Since the Hodge class λ is big and nef, it follows that $\overline{\mathcal{M}}_g$ is of general type whenever $s(\overline{\mathcal{M}}_g) < s(K_{\overline{\mathcal{M}}_g}) = 13/2$. Since $s(\overline{\mathcal{M}}^r_{g,d}) = 6 + 12/(g+1) < 13/2 \Longleftrightarrow g \geq 24$, we obtain the main result from **[HM]** and **[EH3]**, namely that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$ (Strictly speaking, this argument works only for those g for which $g+1$ is composite. In the remaining cases, when there are no Brill-Noether divisors on $\overline{\mathcal{M}}_g$, one has to use the locus where the classical Petri Theorem fails, see **[EH3]**). The Slope Conjecture would immediately imply the following statement:

CONJECTURE 2.12. The Kodaira dimension of $\overline{\mathcal{M}}_g$ is $-\infty$ for all $g < 23$.

The unirationality of $\overline{\mathcal{M}}_{14}$

Severi proved that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$. The cases $g = 11, 12, 13$ were settled by Sernesi and then Chang and Ran (cf. **[Se]**, **[CR1]**). Moreover, it is known that $\overline{\mathcal{M}}_{15}$ is rationally connected (cf. **[BV]**) and that $\kappa(\overline{\mathcal{M}}_{16}) = -\infty$ (cf. **[CR2]**). Optimal bounds for rationality of $\overline{\mathcal{M}}_{g,n}$ when $g \leq 6$ are provided in **[CF]**. Verra has recently settled the long standing case of $\overline{\mathcal{M}}_{14}$ proving the following theorem (cf. **[Ver]**):

THEOREM 2.13. *The moduli space $\mathcal{M}_{14}$ is unirational.*

Sketch of proof. We denote by $H_{d,g,r}$ the Hilbert scheme of curves $C \subset \mathbf{P}^r$ with $g(C) = g$ and $\deg(C) = d$. The key observation is that if $[C] \in \mathcal{M}_{14}$ is suitably general, then $\dim W^6_{18}(C) = 0$ and if $C \overset{|L|}{\hookrightarrow} \mathbf{P}^6$ is the embedding given by any linear series $L \in W^6_{18}(C)$, then

$$\dim\ \text{Ker}\{\text{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})\} = 5,$$

that is, C lies precisely on 5 quadrics $Q_1, \ldots, Q_5$. Writing that

$$\cap_{i=1}^5 Q_i = C \cup R,$$

one finds that the residual curve R is smooth with $\deg(R) = 14$ and $g(R) = 8$. Significantly, $h^1(\mathcal{O}_R(1)) = 0$ (that is, non-special) and the Hilbert scheme $H_{14,8,6}$ is a lot easier to study than the scheme $H_{18,14,6}$ we started with and which parameterizes curves $C \subset \mathbf{P}^6$ with $h^1(\mathcal{O}_C(1)) = 2$. Using Mukai's result that a generic canonical curve of genus 8 is a linear section of $G(2,6) \subset \mathbf{P}^{14}$, one proves that $H_{14,8,6}$ is unirational. If $\mathcal{G}_5 \to H_{14,8,6}$ denotes the Grassmann bundle consisting of pairs $[R,V]$ with $[R] \in H_{14,8,6}$ and $V \in G\big(5, H^0(I_R(2))\big)$, then $\mathcal{G}_5$ is unirational (because $H_{14,8,6}$ is so), and there exists a dominant rational map $\mathcal{G}_5 - - > H_{18,14,6}$ which sends $[R,V]$ to $[C]$, where $C \cup R$ is the scheme in $\mathbf{P}^6$ defined by V. By standard Brill-Noether theory, the forgetful morphism $H_{18,14,6} - - > \mathcal{M}_{14}$ is dominant, hence the composition $\mathcal{G}_5 - - > \mathcal{M}_{14}$ is dominant as well. This shows that $\mathcal{M}_{14}$ is unirational. □

The Slope Conjecture is also connected to the Schottky problem of describing geometrically $\mathcal{M}_g$ in the Torelli embedding $t : \mathcal{M}_g \to \mathcal{A}_g$ given by

$$[C] \mapsto [\mathrm{Jac}(C), \Theta_C].$$

The map t can be extended to a rational map $t : \overline{\mathcal{M}}_g - - > \mathcal{A}_g^{part}$ well-defined at least in codimension 1, where $\mathcal{A}_g^{part}$ is Mumford's partial compactification of rank 1 degenerations obtained by blowing-up the open subvariety $\mathcal{A}_g \cup \mathcal{A}_{g-1}$ inside the Satake compactification of $\mathcal{A}_g$ (cf. **[M2]**). One has that $\mathrm{Pic}(\mathcal{A}_g^{part}) \otimes \mathbb{Q} = \mathbb{Q} \cdot \lambda \oplus \mathbb{Q} \cdot \delta$, where $\lambda := c_1(\mathbb{E})$ is the Hodge class corresponding to modular forms of weight one and $\delta = [\mathcal{A}_g^{part} - \mathcal{A}_g]$ is the class of the irreducible boundary divisor. Note that $t^*(\lambda) = \lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ while $t^*(\delta) = \delta_0$. The quasi-projective variety $\mathcal{A}_g^{part}$ is as good as any projective model of $\mathcal{A}_g$ when it comes to codimension 1 problems like determining the Kodaira dimension of of $\mathcal{A}_g^{part}$ or describing $\mathrm{Eff}(\mathcal{A}_g^{part})$. In particular, one can define the *slope* of $\mathcal{A}_g$ as being

$$s(\mathcal{A}_g) = s(\mathcal{A}_g^{part}) := \inf\{s(D) = \frac{a}{b} : D \equiv a\lambda - b\,\delta \in \mathrm{Eff}(\mathcal{A}_g^{part})\}.$$

THEOREM 2.14. *(Tai, [T]) We have that* $lim_{g \to \infty} s(\mathcal{A}_g) = 0$.

If we combine Tai's estimate with the Slope Conjecture, it follows that any Siegel modular form of slope less than $6 + 12/(g+1)$ would automatically vanish on $\mathcal{M}_g$ thus providing a Schottky relation. Note that any weaker estimate of the form $s(\overline{\mathcal{M}}_g) \geq \epsilon$ for g large, where $\epsilon > 0$ is a constant independent on g, would suffice to obtain the same conclusion. It is then very tempting to ask whether the modular forms of slope $\geq \epsilon$ cut out precisely $\mathcal{M}_g$. A positive answer to this question would represent a completely novel solution to the Schottky problem.

Unfortunately, the Slope Conjecture (at least in its original form), turns out to be false. The first counterexample was constructed in **[FP]** and we start by giving a geometric reinterpretation to the Slope Conjecture which will turn out to be crucial in constructing counterexamples:

PROPOSITION 2.15. *Let D be an effective divisor on $\overline{\mathcal{M}}_g$. If $s(D) < 6 + 12/(g+1)$, then D contains the closure of the locus $\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C$ sits on a $K3$ surface$\}$.*

PROOF. Clearly, we may assume that D is the closure of an effective divisor on $\mathcal{M}_g$. We consider a Lefschetz pencil of curves of genus g lying on a general $K3$ surface of degree $2g-2$ in $\mathbf{P}^g$. This gives rise to a curve B in the moduli space $\overline{\mathcal{M}}_g$. Since $\mathcal{K}_g$ is the image of a $\mathbf{P}^g$-bundle over the irreducible moduli space of polarized $K3$ surfaces of degree $2g-2$, the pencils B fill up the entire locus $\mathcal{K}_g$. We have that $\lambda \cdot B = g+1$, $\delta_0 \cdot B = 6g+18$ and $\delta_i \cdot B = 0$ for $i \geq 1$. The first two intersection numbers are computed using the classical formula for the number of singular fibres in a pencil (see e.g. **[GH]**, p. 508) while the last assertion is obvious since a Lefschetz pencil contains no reducible curves. We can write that $\delta \cdot B/\lambda \cdot B = 6 + 12/(g+1) > s(D)$, which implies that $D \cdot B < 0$ hence $B \subset D$. By varying B and S we obtain that $\overline{\mathcal{K}}_g \subset D$. □

REMARK 2.16. Proposition 2.15 shows that the Slope Conjecture would be implied by the curve $B \subset \overline{\mathcal{M}}_g$ being nef. It is also proved in **[FP]** that if

$$D \equiv a\lambda - b_0\delta_0 - \cdots - b_{[g/2]}\delta_{[g/2]} \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$$

is a divisor class such that $a/b_0 \leq 71/10$, then $b_i \geq b_0$ for all $1 \leq i \leq 11$. At least for $g \leq 23$, the statement of the Slope Conjecture is thus equivalent to B being a nef curve.

REMARK 2.17. The pencils B fill up $\overline{\mathcal{M}}_g$ for $g \leq 11, g \neq 10$ (cf. **[Mu]**), hence Proposition 2.15 gives a short proof of the Slope Conjecture on $\overline{\mathcal{M}}_g$ for these values. For those g such that $\mathcal{K}_g \subsetneq \mathcal{M}_g$, Proposition 2.15 suggests how to search for counterexamples to the Slope Conjecture: one has to come up with divisorial geometric properties which are a relaxation of the condition that a curve lie on a $K3$ surface. The first case where one can test the Slope Conjecture is on $\overline{\mathcal{M}}_{10}$, where contrary to the naive dimension count, $\mathcal{K}_{10}$ is a divisor: The moduli space of polarized $K3$ surfaces of genus g depends on 19 parameters, hence the expected dimension of $\mathcal{K}_g$ is $\min(19+g, 3g-3)$ which would suggest that any $[C] \in \mathcal{M}_{10}$ lies on a $K3$ surface. However, Mukai has proved that $K3$ surfaces of genus 10 appear as codimension 3 linear sections of a certain rational homogeneous variety $X_5 \subset \mathbf{P}^{13}$ corresponding to the Lie group G_2 (cf. **[Mu]**). Therefore, if $[C] \in \mathcal{M}_{10}$ lies on a $K3$ surface, then C lies on ∞^3 $K3$ surfaces and $\mathcal{K}_{10}$ is a divisor on $\overline{\mathcal{M}}_{10}$. The Slope Conjecture holds on $\overline{\mathcal{M}}_{10}$ if and only if it holds for $\overline{\mathcal{K}}_{10}$.

THEOREM 2.18. *([FP]) The divisor $\overline{\mathcal{K}}_{10}$ provides a counterexample to the Slope Conjecture. Its class is given by the formula $\overline{\mathcal{K}}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5$, hence $s(\overline{\mathcal{K}}_{10}) = 7$.*

The proof of this theorem does not use the original definition of $\overline{\mathcal{K}}_{10}$. Instead, we show that $\overline{\mathcal{K}}_{10}$ has a number of other interpretations, in particular, we can geometrically characterize the points from $\overline{\mathcal{K}}_{10}$ in ways that make no reference to $K3$ surfaces and use these descriptions to compute the class of $\overline{\mathcal{K}}_{10}$.

THEOREM 2.19. *([FP]) The divisor $\mathcal{K}_{10}$ has two other incarnations as a geometric subvariety of $\mathcal{M}_{10}$:*

(1) *The locus of curves $[C] \in \mathcal{M}_{10}$ carrying a semistable rank two vector bundle E with $\wedge^2(E) = K_C$ and $h^0(C, E) \geq 7$.*
(2) *The locus of curves $[C] \in \mathcal{M}_{10}$ for which there exists $L \in W^4_{12}(C)$ such that the multiplication map $\mathrm{Sym}^2 H^0(L) \to H^0(L^{\otimes 2})$ is not an isomorphism.*

Proof of Theorem 2.18. We use the second description from Theorem 2.19 and studying degenerations of multiplication maps, we can explicitly describe the pull-backs $j_i^*(\overline{\mathcal{K}}_{10})$ where $j_i^* : \overline{\mathcal{M}}_{i,1} \to \overline{\mathcal{M}}_{10}$ is the map obtained by attaching a fixed tail of genus $10-i$ at the marked point of each genus i curve. It turns out that these pull-backs are sums of "pointed" Brill-Noether divisors on $\overline{\mathcal{M}}_{i,1}$. Since these classes have been computed in **[EH2]**, we get enough relations in the coefficients of $[\overline{\mathcal{K}}_{10}]$ that enable us to fully determine its class. □

REMARK 2.20. Note that if $B \subset \overline{\mathcal{M}}_{10}$ is the pencil appearing in Proposition 2.15, then $\overline{\mathcal{K}}_{10} \cdot B = -1$. The Slope Conjecture fails on $\overline{\mathcal{M}}_{10}$ precisely because of the failure of B to be a nef curve.

REMARK 2.21. We note that a general curve $[C] \in \mathcal{M}_{10}$ possesses finitely many (precisely 42) linear series $\mathfrak{g}^4_{12} = K_C(-\mathfrak{g}^1_6)$, and these $\mathfrak{g}^1_6$'s are the pencils of minimal degree on C. If C lies on a $K3$ surface S, the exceptional rank 2 vector bundle E which appears in Theorem 2.19 is a Lazarsfeld-Mukai bundle obtained as a restriction to C of a rank 2 bundle on S which is the elementary transformation along C given by the evaluation map $H^0(\mathfrak{g}^1_6) \otimes \mathcal{O}_S \to \mathfrak{g}^1_6$. These bundles have played an important role in Voisin's recent proof of Green's Conjecture on syzygies of canonical curves (cf. **[V1]**, **[V2]**).

The counterexample constructed in Theorem 2.18 now raises at least three questions:

- Is the divisor $\overline{\mathcal{K}}_{10}$ an isolated counterexample? (After all, the condition that a curve lie on a $K3$ surface is divisorial only for $g = 10$, and even on $\overline{\mathcal{M}}_{10}$ this condition gives rise to a divisor almost by accident, due to the somewhat miraculous existence of Mukai's rational 5-fold $X_5 \subset \mathbf{P}^{13}$).
- If the answer to the first question is no and the Slope Conjecture fails systematically, are there integers $g \leq 23$ and divisors $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ such that $s(D) < 13/2$, so that $\overline{\mathcal{M}}_g$ of general type, thus contradicting Conjecture 2.12?
- In light of the application to the Schottky problem, is there still a lower bound on $s(\overline{\mathcal{M}}_g)$? Note that we know that $s(\overline{\mathcal{M}}_g) \geq \mathrm{O}(1/g)$ for large g (cf. **[HMo]**).

In the remaining part of this paper we will provide adequate answers to the first two of these questions.

3. Constructing divisors of small slope using syzygies

We describe a general recipe of constructing effective divisors on $\overline{\mathcal{M}}_g$ having very small slope. In particular, we obtain an infinite string of counterexamples to the Slope Conjecture. Everything in this section is contained in **[F2]** and **[F3]** and we outline the main ideas and steps in these calculations. The key idea is to reinterpret the second description of the divisor $\overline{\mathcal{K}}_{10}$ (see Theorem 2.19) as a failure of $[C] \in \mathcal{M}_{10}$ to satisfy the Green-Lazarsfeld property (N_0) in the embedding given by one of the finitely many linear series $\mathfrak{g}^4_{12}$ on C. We will be looking at loci in $\mathcal{M}_g$ consisting of curves that have exceptional syzygy properties with respect to certain $\mathfrak{g}^r_d$'s.

Suppose that $C \overset{|L|}{\hookrightarrow} \mathbf{P}^r$ is a curve of genus g embedded by a line bundle $L \in \mathrm{Pic}^d(C)$. We denote by $I_{C/\mathbf{P}^r}$ the ideal of C in $\mathbf{P}^r$ and consider its minimal resolution of free graded $S = \mathbb{C}[x_0, \ldots, x_r]$-modules

$$0 \to F_{r+1} \to \cdots \to F_2 \to F_1 \to I_{C/\mathbf{P}^r} \to 0.$$

Following Green and Lazarsfeld we say that the pair (C, L) satisfy the property (N_i) for some integer $i \geq 1$, if $F_j = \oplus S(-j-1)$ for all $j \leq i$ (or equivalently in terms of graded Betti numbers, $b_{i,l}(C) = 0$ for all $l \geq 2$). Using the computation of $b_{j,l}(C)$ in terms of Koszul cohomology, there is a well-known cohomological interpretation of property (N_i): If M_L is the vector bundle on C defined by the exact sequence

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_C \to L \to 0,$$

then (C, L) satisfies property (N_i) if and only if for all $j \geq 1$, the natural map

$$u_{i,j} : \wedge^{i+1} H^0(L) \otimes H^0(L^{\otimes j}) \to H^0(\wedge^i M_L \otimes L^{\otimes (j+1)})$$

is surjective (cf. e.g. **[L2]**).

Our intention is to define a determinantal syzygy type condition on a generically finite cover of $\overline{\mathcal{M}}_g$ parametrizing pairs consisting of a curve and a $\mathfrak{g}^r_d$. We fix integers $i \geq 0$ and $s \geq 1$ and set

$$r := 2s + si + i, \ g := rs + s \text{ and } d := rs + r.$$

We denote by $\mathfrak{G}^r_d$ the stack parametrizing pairs $[C, L]$ with $[C] \in \mathcal{M}_g$ and $L \in W^r_d(C)$ and denote by $\sigma : \mathfrak{G}^r_d \to \mathcal{M}_g$ the natural projection. Since $\rho(g, r, d) = 0$, by general Brill-Noether theory, the general curve of genus g has finitely many $\mathfrak{g}^r_d$'s and there exists a unique irreducible component of $\mathfrak{G}^r_d$ which maps onto $\mathcal{M}_g$.

We define a substack of $\mathfrak{G}^r_d$ consisting of those pairs (C, L) which fail to satisfy property (N_i). In **[F3]** we introduced two vector bundles $\mathcal{A}$ and $\mathcal{B}$ over $\mathfrak{G}^r_d$ such that for a curve $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^r$ corresponding to a point $(C, L) \in \mathfrak{G}^r_d$, we have that

$$\mathcal{A}(C, L) = H^0(\mathbf{P}^r, \wedge^i M_{\mathbf{P}^r}(2)) \text{ and } \mathcal{B}(C, L) = H^0(C, \wedge^i M_L \otimes L^2).$$

There is a natural vector bundle morphism $\phi : \mathcal{A} \to \mathcal{B}$ given by restriction. From Grauert's Theorem we see that both $\mathcal{A}$ and $\mathcal{B}$ are vector bundles over $\mathfrak{G}^r_d$ and from Bott's Theorem we compute their ranks

$$\text{rank}(\mathcal{A}) = (i+1)\binom{r+2}{i+2} \text{ and } \text{rank}(\mathcal{B}) = \binom{r}{i}\Big(-\frac{id}{r} + 2d + 1 - g\Big)$$

(use that M_L is a stable bundle, hence $H^1(\wedge^i M_L \otimes L^{\otimes 2}) = 0$, while rank($\mathcal{B}$) can be computed from Riemann-Roch). It is easy to check that for our numerical choices we have that rank($\mathcal{A}$) = rank($\mathcal{B}$).

THEOREM 3.1. *The cycle*

$$\mathcal{U}_{g,i} := \{(C, L) \in \mathfrak{G}^r_d : (C, L) \textit{ fails property } (N_i)\},$$

is the degeneracy locus of vector bundle map $\phi : \mathcal{A} \to \mathcal{B}$ over $\mathfrak{G}^r_d$.

Thus $\mathcal{Z}_{g,i} := \sigma(\mathcal{U}_{g,i})$ is a virtual divisor on $\mathcal{M}_g$ when $g = s(2s + si + i + 1)$. In **[F3]** we show that we can extend the determinantal structure of $\mathcal{Z}_{g,i}$ over $\overline{\mathcal{M}}_g$ in such a way that whenever $s \geq 2$, the resulting virtual slope violates the Harris-Morrison Conjecture. One has the following statement:

THEOREM 3.2. *If $\sigma : \widetilde{\mathfrak{G}}^r_d \to \overline{\mathcal{M}}_g$ is the compactification of $\mathfrak{G}^r_d$ given by limit linear series, then there exists a natural extension of the vector bundle map $\phi : \mathcal{A} \to \mathcal{B}$ over $\widetilde{\mathfrak{G}}^r_d$ such that $\overline{\mathcal{Z}}_{g,i}$ is the image of the degeneracy locus of ϕ. The class of the pushforward to $\overline{\mathcal{M}}_g$ of the virtual degeneracy locus of ϕ is given by*

$$\sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) \equiv a\lambda - b_0\, \delta_0 - b_1\, \delta_1 - \cdots - b_{[g/2]}\, \delta_{[g/2]},$$

where $a, b_0, \ldots, b_{[g/2]}$ *are explicitly given coefficients such that* $b_1 = 12b_0 - a$, $b_i \geq b_0$ *for* $1 \leq i \leq [g/2]$ *and*

$$s\big(\sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}))\big) = \frac{a}{b_0} = 6\frac{f(s,i)}{(i+2)\ sg(s,i)},\ \textit{with}$$

$$f(s,i) = (i^4+24i^2+8i^3+32i+16)s^7+(i^4+4i^3-16i-16)s^6-(i^4+7i^3+13i^2-12)s^5-(i^4+2i^3+i^2+14i+24)s^4+(2i^3+2i^2-6i-4)s^3+(i^3+17i^2+50i+41)s^2+(7i^2+18i+9)s+2i+2$$

and

$$g(s,i) = (i^3+6i^2+12i+8)s^6+(i^3+2i^2-4i-8)s^5-(i^3+7i^2+11i+2)s^4- \\ -(i^3-5i)s^3+(4i^2+5i+1)s^2+(i^2+7i+11)s+4i+2.$$

Furthermore, we have that $6 < \frac{a}{b_0} < 6 + \frac{12}{g+1}$ *whenever* $s \geq 2$. *If the morphism* ϕ *is generically non-degenerate, then* $\overline{\mathcal{Z}}_{g,i}$ *is a divisor on* $\overline{\mathcal{M}}_g$ *which gives a counterexample to the Slope Conjecture for* $g = s(2s + si + i + 1)$.

REMARK 3.3. Theorem 3.2 generalizes all known examples of effective divisors on $\overline{\mathcal{M}}_g$ violating the Slope Conjecture. For $s = 2$ and $g = 6i + 10$ (that is, in the case $h^1(L) = 2$ when $\mathfrak{G}^r_d$ is isomorphic to a Hurwitz scheme parametrizing covers of $\mathbf{P}^1$), we recover our result from **[F2]**. We have that

$$s(\overline{\mathcal{Z}}_{6i+10,i}) = \frac{3(4i+7)(6i^2+19i+12)}{(12i^2+31i+18)(i+2)}.$$

For $i = 0$ we recover the main result from **[Kh]** originally proved using a completely different method:

COROLLARY 3.4. *(Khosla) For* $g = s(2s+1), r = 2s, d = 2s(s+1)$ *the slope of the virtual class of the locus of those* $[C] \in \overline{\mathcal{M}}_g$ *for which there exists* $L \in W^r_d(C)$ *such that the embedded curve* $C \subset \mathbf{P}^r$ *sits on a quadric hypersurface, is*

$$s(\overline{\mathcal{Z}}_{s(2s+1),0}) = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}.$$

REMARK 3.5. In the case $s = 1, g = 2i + 3$ when $\mathfrak{g}^r_d = \mathfrak{g}^{g-1}_{2g-2}$ is the canonical system, our formula reads

$$s(\overline{\mathcal{Z}}_{2i+3,i}) = \frac{6(i+3)}{i+2} = 6 + \frac{12}{g+1}.$$

Remembering that $\mathcal{Z}_{2i+3,i}$ is the locus of curves $[C] \in \mathcal{M}_{2i+3}$ for which K_C fails property (N_i), from Green's Conjecture for generic curves (cf. **[V1]**, **[V2]**) we obtain the set-theoretic identification identification between $\mathcal{Z}_{2i+3,i}$ and the locus $\mathcal{M}^1_{2i+3,i+2}$ of $(i+2)$-gonal curves. Thus $\mathcal{Z}_{2i+3,i}$ is a Brill-Noether divisor! Theorem 3.2 provides a new way of calculating the class of the compactification of the Brill-Noether divisor which was first computed by Harris and Mumford (cf. **[HM]**).

Theorem 3.2 is proved by extending the determinantal structure of $\mathcal{Z}_{g,i}$ over the boundary divisors in $\overline{\mathcal{M}}_g$. We can carry this out outside a locus of codimension ≥ 2 in $\overline{\mathcal{M}}_g$. We denote by $\widetilde{\mathcal{M}}_g := \mathcal{M}^0_g \cup \big(\cup_{j=0}^{[g/2]} \Delta^0_j\big)$ the locally closed subset of $\overline{\mathcal{M}}_g$ defined as the union of the locus $\mathcal{M}^0_g$ of smooth curves carrying no linear systems $\mathfrak{g}^r_{d-1}$ or $\mathfrak{g}^{r+1}_d$ to which we add the open subsets $\Delta^0_j \subset \Delta_j$ for $1 \leq j \leq [g/2]$

consisting of 1-nodal genus g curves $C \cup_y D$, with $[C] \in \mathcal{M}_{g-j}$ and $[D, y] \in \mathcal{M}_{j,1}$ being Brill-Noether general curves, and the locus $\Delta_0^0 \subset \Delta_0$ containing 1-nodal irreducible genus g curves $C' = C/q \sim y$, where $[C, q] \in \mathcal{M}_{g-1}$ is a Brill-Noether general pointed curve and $y \in C$, together with their degenerations consisting of unions of a smooth genus $g-1$ curve and a nodal rational curve. One can then extend the finite covering $\sigma : \mathfrak{G}^r_d \to \mathcal{M}_g^0$ to a proper, generically finite map

$$\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathcal{M}}_g$$

by letting $\widetilde{\mathfrak{G}}^r_d$ be the stack of limit $\mathfrak{g}^r_d$'s on the treelike curves from $\widetilde{\mathcal{M}}_g$ (see **[EH1]**, Theorem 3.4 for the construction of the space of limit linear series).

One method of computing $[\overline{\mathcal{Z}}_{g,i}]$ is to intersect the locus $\overline{\mathcal{Z}}_{g,i}$ with standard test curves in the boundary of $\overline{\mathcal{M}}_g$ which are defined as follows: we fix a Brill-Noether general curve C of genus $g-1$, a general point $q \in C$ and a general elliptic curve E. We define two 1-parameter families

(1)
$$C^0 := \{C/y \sim q : y \in C\} \subset \Delta_0 \subset \overline{\mathcal{M}}_g \text{ and } C^1 := \{C \cup_y E : y \in C\} \subset \Delta_1 \subset \overline{\mathcal{M}}_g.$$

It is well-known that these families intersect the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ as follows:

$$C^0 \cdot \lambda = 0,\ C^0 \cdot \delta_0 = -(2g-2),\ C^0 \cdot \delta_1 = 1 \text{ and } C^0 \cdot \delta_a = 0 \text{ for } a \geq 2, \text{ and}$$

$$C^1 \cdot \lambda = 0,\ C^1 \cdot \delta_0 = 0,\ C^1 \cdot \delta_1 = -(2g-4),\ C^1 \cdot \delta_a = 0 \text{ for } a \geq 2.$$

Before we proceed we review the notation used in the theory of limit linear series (see **[EH1]** as a general reference). If X is a treelike curve and l is a limit $\mathfrak{g}^r_d$ on X, for a component Y of X we denote by $l_Y = (L_Y, V_Y \subset H^0(L_Y))$ the Y-aspect of l. For a point $y \in Y$ we denote by by $\{a_s^{l_Y}(C)\}_{s=0\ldots r}$ the *vanishing sequence* of l at y and by $\rho(l_Y, y) := \rho(g, r, d) - \sum_{i=0}^r (a_i^{l_Y}(y) - i)$ the adjusted Brill-Noether number with respect to y. We have the following description of the curves $\sigma^*(C^0)$ and $\sigma^*(C^1)$:

PROPOSITION 3.6. *(1) Let $C_y^1 = C \cup_y E$ be an element of Δ_1^0. If (l_C, l_E) is a limit $\mathfrak{g}^r_d$ on C_y^1, then $V_C = H^0(L_C)$ and $L_C \in W^r_d(C)$ has a cusp at y. If $y \in C$ is a general point, then $l_E = \big(\mathcal{O}_E(dy), (d-r-1)y + |(r+1)y|\big)$, that is, l_E is uniquely determined. If $y \in C$ is one of the finitely many points for which there exists $L_C \in W^r_d(C)$ such that $\rho(L_C, y) = -1$, then $l_E(-(d-r-2)y)$ is a $\mathfrak{g}^r_{r+2}$ with vanishing sequence at y being $\geq (0, 2, 3, \ldots, r, r+2)$. Moreover, at the level of 1-cycles we have the identification $\sigma^*(C^1) \equiv X + \nu\, T$, where*

$$X := \{(y, L) \in C \times W^r_d(C) : h^0(C, L(-2y)) \geq r\}$$

and T is the curve consisting of $\mathfrak{g}^r_{r+2}$'s on E with vanishing $\geq (0, 2, \ldots, r, r+2)$ at the fixed point $y \in E$ while ν is a positive integer.
(2) Let $C_y^0 = C/y \sim q$ be an element of Δ_0^0. Then limit linear series of type $\mathfrak{g}^r_d$ on C_y^0 are in 1:1 correspondence with complete linear series L on C of type $\mathfrak{g}^r_d$ satisfying the condition $h^0(C, L \otimes \mathcal{O}_C(-y-q)) = h^0(C, L) - 1$. Thus there is an isomorphism between the cycle $\sigma^*(C^0)$ of $\mathfrak{g}^r_d$'s on all curves C_y^0 with $y \in C$, and the smooth curve

$$Y := \{(y, L) \in C \times W^r_d(C) : h^0(C, L(-y-q)) \geq r\}.$$

Throughout the papers **[F2]** and **[F3]** we use a number of facts about intersection theory on Jacobians which we now quickly review. Let C be a Brill-Noether general curve of genus $g-1$ (recall that $g = rs + s$ and $d = rs + s$, where

$r=2s+si+i$). Then $\dim W^r_d(C)=r$ and every $L\in W^r_d(C)$ corresponds to a complete and base point free linear series. We denote by $\mathcal{L}$ a Poincaré bundle on $C\times \mathrm{Pic}^d(C)$ and by $\pi_1: C\times \mathrm{Pic}^d(C)\to C$ and $\pi_2: C\times \mathrm{Pic}^d(C)\to \mathrm{Pic}^d(C)$ the projections. We define the cohomology class $\eta=\pi_1^*([\text{point}])\in H^2(C\times \mathrm{Pic}^d(C))$, and if $\delta_1,\ldots,\delta_{2g}\in H^1(C,\mathbb{Z})\cong H^1(\mathrm{Pic}^d(C),\mathbb{Z})$ is a symplectic basis, then we set

$$\gamma:=-\sum_{\alpha=1}^{g}\Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha})-\pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big).$$

We have the formula (cf. [**ACGH**], p. 335) $c_1(\mathcal{L})=d\eta+\gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{L})$. We also record that $\gamma^3=\gamma\eta=0$, $\eta^2=0$ and $\gamma^2=-2\eta\pi_2^*(\theta)$. On $W^r_d(C)$ we have the tautological rank $r+1$ vector bundle $\mathcal{E}:=(\pi_2)_*(\mathcal{L}_{|C\times W^r_d(C)})$. The Chern numbers of $\mathcal{E}$ can be computed using the Harris-Tu formula (cf. [**HT**]): if we write $\sum_{i=0}^r c_i(\mathcal{E}^\vee)=(1+x_1)\cdots(1+x_{r+1})$, then for every class $\zeta\in H^*(\mathrm{Pic}^d(C),\mathbb{Z})$ one has the formula [3]

$$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\,\zeta=\det\Big(\frac{\theta^{g-1+r-d+i_j-j+l}}{(g-1+r-d+i_j-j+l)!}\Big)_{1\le j,l\le r+1}\zeta.$$

If we use the expression of the Vandermonde determinant, we get the formula

$$\det\Big(\frac{1}{(a_j+l-1)!}\Big)_{1\le j,l\le r+1}=\frac{\Pi_{j>l}\,(a_l-a_j)}{\Pi_{j=1}^{r+1}\,(a_j+r)!}.$$

By repeatedly applying this we get all intersection numbers on $W^r_d(C)$ which we need:

LEMMA 3.7. *If $c_i:=c_i(\mathcal{E}^\vee)$ we have the following identities in $H^*(W^r_d(C),\mathbb{Z})$:*

(1) $c_{r-1}\theta=\frac{r(s+1)}{2}c_r$.
(2) $c_{r-2}\theta^2=\frac{r(r-1)(s+1)(s+2)}{6}c_r$.
(3) $c_{r-2}c_1\theta=\frac{r(s+1)}{2}\big(1+\frac{(r-2)(r+2)(s+2)}{3(s+r+1)}\big)c_r$.
(4) $c_{r-1}c_1=(1+\frac{(r-1)(r+2)(s+1)}{2(s+r+1)})c_r$.
(5) $c_r=\frac{1!\,2!\cdots(r-1)!\,(r+1)!}{(s-1)!\,(s+1)!\,(s+2)!\cdots(s+r)!}\theta^{g-1}$.

For each integers $0\le a\le r$ and $b\ge 2$ we shall define vector bundles $\mathcal{G}_{a,b}$ and $\mathcal{H}_{a,b}$ over $\widetilde{\mathfrak{G}}^r_d$ with fibres

$$\mathcal{G}_{a,b}(C,L)=H^0(C,\wedge^a M_L\otimes L^{\otimes b})\ \text{ and }\ \mathcal{H}_{a,b}(C,L)=H^0(\mathbf{P}^r,\wedge^a M_{\mathbf{P}^r}(b))$$

for each $(C,L)\in\mathfrak{G}^r_d$ giving a map $C\xrightarrow{|L|}\mathbf{P}^r$. Clearly $\mathcal{G}_{i,2|\mathfrak{G}^r_d}=\mathcal{B}$ and $\mathcal{H}_{i,2|\mathfrak{G}^r_d}=\mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are the vector bundles introduced in Proposition 3.1. The question is how to extend this description over the divisors Δ_j^0. For simplicity we only explain how to do this over $\sigma^{-1}(\mathcal{M}_g^0\cup\Delta_0^0\cup\Delta_1^0)$ which will be enough to compute the slope of $\overline{\mathcal{Z}}_{g,i}$. The case of the divisors $\sigma^{-1}(\Delta_j^0)$ where $2\le j\le[g/2]$ is technically more involved and it is dealt with in [**F3**]. We start by extending $\mathcal{G}_{0,b}$ (see [**F3**], Proposition 2.8):

PROPOSITION 3.8. *For each $b\ge 2$ there exists a vector bundle $\mathcal{G}_{0,b}$ over $\widetilde{\mathfrak{G}}^r_d$ of rank $bd+1-g$ whose fibres admit the following description:*

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[3] There is a confusing sign error in the formula (1.4) in [**HT**]: the formula is correct as it is appears in [**HT**], if the x_j's denote the Chern roots of the *dual* of the kernel bundle.

- *For* $(C, L) \in \mathfrak{G}^r_d$, *we have that* $\mathcal{G}_{0,b}\big(C, L\big) = H^0(C, L^{\otimes b})$.
- *For* $t = (C \cup_y E, L) \in \sigma^{-1}(\Delta_1^0)$, *where* $L \in W^r_d(C)$ *has a cusp at* $y \in C$, *we have that*
$$\mathcal{G}_{0,b}(t) = H^0(C, L^{\otimes b}(-2y)) + \mathbb{C} \cdot u^b \subset H^0\big(C, L^{\otimes b}\big),$$
where $u \in H^0(C, L)$ *is any section such that* $\mathrm{ord}_y(u) = 0$.
- *For* $t = (C/y \sim q, L) \in \sigma^{-1}(\Delta_0^0)$, *where* $q, y \in C$ *and* $L \in W^r_d(C)$ *is such that* $h^0(C, L(-y-q)) = h^0(L) - 1$, *we have that*
$$\mathcal{G}_{0,b}(t) = H^0(C, L^{\otimes b}(-y-q)) \oplus \mathbb{C} \cdot u^b \subset H^0(C, L^{\otimes b}),$$
where $u \in H^0(C, L)$ *is a section such that* $\mathrm{ord}_y(u) = \mathrm{ord}_q(u) = 0$.

Having defined the vector bundles $\mathcal{G}_{0,b}$ we now define inductively all vector bundles $\mathcal{G}_{a,b}$ by the exact sequence

$$0 \longrightarrow \mathcal{G}_{a,b} \longrightarrow \wedge^a \mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \stackrel{d_{a,b}}{\longrightarrow} \mathcal{G}_{a-1,b+1} \longrightarrow 0. \tag{2}$$

To define $\mathcal{H}_{a,b}$ is even easier. We set $\mathcal{H}_{0,b} := \mathrm{Sym}^b \mathcal{G}_{0,1}$ for all $b \geq 1$ and we define $\mathcal{H}_{a,b}$ inductively via the exact sequence

$$0 \longrightarrow \mathcal{H}_{a,b} \longrightarrow \wedge^a \mathcal{H}_{0,1} \otimes \mathrm{Sym}^b \mathcal{H}_{0,1} \longrightarrow \mathcal{H}_{a-1,b+1} \longrightarrow 0. \tag{3}$$

The surjectivity of the right map in (3) is obvious, whereas to prove that $d_{a,b}$ is surjective, one employs the arguments from Proposition 3.10 in **[F2]**. There is a natural vector bundle morphism $\phi_{a,b} : \mathcal{H}_{a,b} \to \mathcal{G}_{a,b}$. Moreover $\mathrm{rank}(\mathcal{H}_{i,2}) = \mathrm{rank}(\mathcal{G}_{i,2})$ and the degeneracy locus of $\phi_{i,2}$ is the codimension one compactification of $\mathcal{Z}_{g,i}$.

We now compute the class of the curves X and Y defined in Proposition 3.6 (see **[F3]** Proposition 2.11 for details):

PROPOSITION 3.9. *Let* C *be a Brill-Noether general curve of genus* $g-1$ *and* $q \in C$ *a general point. We denote by* $\pi_2 : C \times W^r_d(C) \to W^r_d(C)$ *the projection and set* $c_i := (\pi_2)^*\big(c_i(\mathcal{E}^\vee)\big)$.
(1) The class of the curve $X = \{(y, L) \in C \times W^r_d(C) : h^0(C, L(-2y)) \geq r\}$ *is given by*
$$[X] = c_r + c_{r-1}(2\gamma + (2d + 2g - 4)\eta) - 6c_{r-2}\, \eta\theta.$$
(2) The class of the curve $Y = \{(y, L) \in C \times W^r_d(C) : h^0(C, L(-y-q)) \geq r\}$ *is given by*
$$[Y] = c_r + c_{r-1}(\gamma + (d-1)\eta) - 2c_{r-2}\, \eta\theta.$$

Sketch of proof. Both X and Y are expressed as degeneracy loci over $C \times W^r_d(C)$ and we compute their classes using the Thom-Porteous formula. For $(y, L) \in C \times W^r_d(C)$ the natural map $H^0(C, L_{|2y})^\vee \to H^0(C, L)^\vee$ globalizes to a vector bundle map $\zeta : J_1(\mathcal{L})^\vee \to (\pi_2)^*(\mathcal{E}^\vee)$. Then $X = Z_1(\zeta)$ and we apply Thom-Porteous. □

We mention the following intersection theoretic result (cf. **[F3]**, Lemma 2.12):

LEMMA 3.10. *For each* $j \geq 2$ *we have the following formulas:*
(1) $c_1(\mathcal{G}_{0,j\ |X}) = -j^2\theta - (2g-4)\eta - j(d\eta + \gamma)$.
(2) $c_1(\mathcal{G}_{0,j\ |Y}) = -j^2\theta + \eta$.

Proof of Theorem 3.2. Since $\mathrm{codim}(\overline{\mathcal{M}}_g - \widetilde{\mathcal{M}}_g, \overline{\mathcal{M}}_g) \geq 2$, it makes no difference whether we compute the class $\sigma_*(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})$ on $\widetilde{\mathcal{M}}_g$ or on $\overline{\mathcal{M}}_g$ and we can write

$$\sigma_*(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = A\lambda - B_0\, \delta_0 - B_1\, \delta_1 - \cdots - B_{[g/2]}\, \delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g),$$

where $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ are the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$. First we note that one has the relation $A - 12B_0 + B_1 = 0$. This can be seen by picking a general curve $[C, q] \in \mathcal{M}_{g-1,1}$ and at the fixed point q attaching to C a Lefschetz pencil of plane cubics. If we denote by $R \subset \overline{\mathcal{M}}_g$ the resulting curve, then $R \cdot \lambda = 1,\ R \cdot \delta_0 = 12,\ R \cdot \delta_1 = -1$ and $R \cdot \delta_j = 0$ for $j \geq 2$. The relation $A - 12B_0 + B_1 = 0$ follows once we show that $\sigma^*(R) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = 0$. To achieve this we check that $\mathcal{G}_{0,b|\sigma^*(R)}$ is trivial and then use (2) and (3). We take $[C \cup_q E]$ to be an arbitrary curve from R, where E is an elliptic curve. Using that limit $\mathfrak{g}^r_d$ on $C \cup_q E$ are in $1:1$ correspondence with linear series $L \in W^r_d(C)$ having a cusp at q (this being a statement that holds independent of E) and that $\mathcal{G}_{0,b|\sigma^*(\Delta_1^0)}$ consists on each fibre of sections of the genus $g-1$ aspect of the limit $\mathfrak{g}^r_d$, the claim now follows.

Next we determine A, B_0 and B_1 explicitly. We fix a general pointed curve $(C, q) \in \mathcal{M}_{g-1,1}$ and construct the test curves $C^1 \subset \Delta_1$ and $C^0 \subset \Delta_0$. Using the notation from Proposition 3.6, we get that $\sigma^*(C^0) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = c_1(\mathcal{G}_{i,2|Y}) - c_1(\mathcal{H}_{i,2|Y})$ and $\sigma^*(C^1) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = c_1(\mathcal{G}_{i,2|X}) - c_1(\mathcal{H}_{i,2|X})$ (the other component T of $\sigma^*(C^1)$ does not appear because $\mathcal{G}_{0,b|T}$ is trivial for all $b \geq 1$). On the other hand

$$C^0 \cdot \sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) = (2g-2)B_0 - B_1 \text{ and } C^1 \cdot \sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) = (2g-4)B_1,$$

while we already know that $A - 12B_0 + B_1 = 0$. Next we use the relations

$$c_1(\mathcal{G}_{i,2}) = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l}\mathcal{G}_{0,1} \otimes \mathcal{G}_{0,l+2}) = \sum_{l=0}^{i} (-1)^l \binom{r+1}{i-l} c_1(\mathcal{G}_{0,l+2}) +$$

$$+ \sum_{l=0}^{i} (-1)^l \big((l+2)(rs+r) + 1 - rs - s\big) \binom{r}{i-l-1} c_1(\mathcal{G}_{0,1}), \quad \text{and}$$

$$c_1(\mathcal{H}_{i,2}) = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l}\mathcal{G}_{0,1} \otimes \mathrm{Sym}^{l+2}\mathcal{G}_{0,1}) =$$

$$= \sum_{l=0}^{i} (-1)^l \left(\binom{r}{i-l-1} \binom{r+l+2}{l+2} + \binom{r+1}{i-l} \binom{r+l+2}{r+1} \right) c_1(\mathcal{G}_{0,1}),$$

which when restricted to X and Y, enable us (also using Lemma 3.10) to obtain explicit expressions for $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})_{|X}$ and $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})_{|Y}$ in terms of the classes η, θ, γ and $c_1 = \pi_2^*(c_1(\mathcal{E}^\vee))$. Intersecting these classes with $[X]$ and $[Y]$ and using Lemma 3.7, we finally get a linear system of 3 equations in A, B_0 and B_1 which leads to the stated formulas for the first three coefficients. □

Theorem 2.18 produces only virtual divisors on $\overline{\mathcal{M}}_g$ of slope $< 6 + 12/(g+1)$. To get actual divisors one has to show that the vector bundle map $\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$ is generically non-degenerate. This has been carried out for $s = 2, i = 0$ in **[FP]** (relying on earlier work by Mukai), as well in the cases $s = 2, i = 1$ and $s = 2, i = 2$ in **[F2]**, using the program Macaulay. D. Khosla has checked the transversality of ϕ when $s = 3, i = 0$, that is on $\mathcal{M}_{21}$ (cf. **[Kh]**). We generalize this last result as well as **[FP]** by proving that for $i = 0$ and arbitrary s, the map $\phi : \mathcal{H}_{0,2} \to \mathcal{G}_{0,2}$ is always generically non-degenerate. The following result also establishes a proof of the Maximal Rank Conjecture in the case $\rho(g, r, d) = 0$:

THEOREM 3.11. *For an integer $s \geq 2$ we set $r := 2s, d := 2s(s+1)$ and $g := s(2s+1)$. Then the vector bundle map $\phi : \mathcal{H}_{0,2} \to \mathcal{G}_{0,2}$ is generically non-degenerate. In particular*

$$\mathcal{Z}_{g,0} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \text{ such that } C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^r \text{ is not projectively normal}\}$$

is a divisor on $\overline{\mathcal{M}}_g$ of slope

$$s(\overline{\mathcal{Z}}_{g,0}) = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}$$

violating the Slope Conjecture.

Sketch of proof. From Brill-Noether theory it follows that there exists a unique component of $\widetilde{\mathfrak{G}}^r_d$ which maps onto $\widetilde{\mathcal{M}}_g$, therefore it is then enough to produce a Brill-Noether-Petri general smooth curve $C \subset \mathbf{P}^{2s}$ having degree $2s(s+1)$ and genus $s(2s+1)$ such that C does not sit on any quadrics, that is $H^0(\mathcal{I}_{C/\mathbf{P}^{2s}}(2)) = H^1(\mathcal{I}_{C/\mathbf{P}^{2s}}(2)) = 0$. We carry this out inductively: for each $0 \leq a \leq s$, we construct a smooth non-degenerate curve $C_a \subset \mathbf{P}^{s+a}$ with $\deg(C_a) = \binom{s+a+1}{2} + a$ and $g(C_a) = \binom{s+a+1}{2} + a - s$, such that C_a satisfies the Petri Theorem (in particular $H^1(C_a, N_{C_a/\mathbf{P}^{s+a}}) = 0$), and such that the multiplication map

$$\mu_2 : \mathrm{Sym}^2 H^0(C_a, \mathcal{O}_{C_a}(1)) \to H^0(C_a, \mathcal{O}_{C_a}(2))$$

is surjective.

To construct $C_0 \subset \mathbf{P}^s$ we consider the *White surface* $S = \mathrm{Bl}_{\{p_1,\ldots,p_\delta\}}(\mathbf{P}^2) \subset \mathbf{P}^s$ obtained by blowing-up $\mathbf{P}^2$ at general points $p_1, \ldots, p_\delta \in \mathbf{P}^2$ where $\delta = \binom{s+1}{2}$, and embedding it via the linear system $|sh - \sum_{i=1}^{\delta} E_{p_i}|$. Here h is the class of a line on $\mathbf{P}^2$. It is known that $S \subset \mathbf{P}^s$ is a projectively Cohen-Macaulay surface and its ideal is generated by the (3×3)-minors of a certain $(3 \times s)$-matrix of linear forms. The Betti diagram of $S \subset \mathbf{P}^s$ is the same as that of the ideal of (3×3)-minors of a $(3 \times s)$-matrix of indeterminates. In particular, we have that $H^i(\mathcal{I}_{S/\mathbf{P}^s}(2)) = 0$ for $i = 0, 1$. On S we consider a generic smooth curve $C \equiv (s+1)h - \sum_{i=1}^{\delta} E_{p_i}$. We find that the embedded curve $C \subset S \subset \mathbf{P}^s$ has $\deg(C) = \binom{s+1}{2}$ and $g(C) = \binom{s}{2}$. Even though $[C] \in \mathcal{M}_{g(C)}$ itself is not a Petri general curve, the map $H_{d(C),g(C),s} \to \mathcal{M}_{\binom{s}{2}}$ from the Hilbert scheme of curves $C' \subset \mathbf{P}^s$, is smooth and dominant around the point $[C \hookrightarrow \mathbf{P}^s]$. Therefore a generic deformation $[C_0 \hookrightarrow \mathbf{P}^s]$ of $[C \hookrightarrow \mathbf{P}^s]$ will be Petri general and still satisfy the condition $H^1(\mathcal{I}_{C_0/\mathbf{P}^s}(2)) = 0$.

Assume now that we have constructed a Petri general curve $C_a \subset \mathbf{P}^{s+a}$ with all the desired properties. We pick general points $p_1, \ldots, p_{s+a+2} \in C_a$ with the property that if $\Delta := p_1 + \cdots + p_{s+a+2} \in \mathrm{Sym}^{s+a+2} C_a$, then the variety

$$T := \{(M, V) \in W^{s+a+1}_{d(C_a)+s+a+2}(C_a) : \dim\big(V \cap H^0(C_a, M \otimes \mathcal{O}_{C_a}(-\Delta))\big) \geq s+a+1\}$$

of linear series having an $(s+a+2)$-fold point along Δ, has the expected dimension $\rho(g(C_a), s+a+1, d(C_a)+s+a+2) - (s+a+1)^2$. We identify the projective space $\mathbf{P}^{s+a}$ containing C_a with a hyperplane $H \subset \mathbf{P}^{s+a+1}$ and choose a linearly normal elliptic curve $E \subset \mathbf{P}^{s+a+1}$ such that $E \cap H = \{p_1, \ldots, p_{s+a+2}\}$. We set $X := C_a \cup_{\{p_1,\ldots,p_{s+a+2}\}} E \hookrightarrow \mathbf{P}^{s+a+1}$ and then $\deg(X) = p_a(X) + s$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_E(-p_1 - \cdots - p_{s+a+2}) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_a} \longrightarrow 0,$$

we can write that $h^0(\mathcal{O}_X(1)) \le h^0(\mathcal{O}_{C_a}(1)) + h^0(\mathcal{O}_E) = s+a+2$, hence $h^0(\mathcal{O}_X(1)) = s+a+2$ and $h^1(\mathcal{O}_X(1)) = a+1$. One can also write the exact sequence

$$0 \longrightarrow \mathcal{I}_{E/\mathbf{P}^{s+a+1}}(1) \longrightarrow \mathcal{I}_{X/\mathbf{P}^{s+a+1}}(2) \longrightarrow \mathcal{I}_{C_a/H}(2) \longrightarrow 0,$$

from which we obtain that $H^1(\mathcal{I}_{X/\mathbf{P}^{s+a+1}}(2)) = 0$, hence by a dimension count also $H^0(\mathcal{I}_{X/\mathbf{P}^{s+a+1}}(2)) = 0$, that is, X and every deformation of X inside $\mathbf{P}^{s+a+1}$ will lie on no quadrics. In **[F3]** Theorem 1.5 it is proved that $X \hookrightarrow \mathbf{P}^{s+a+1}$ can be deformed to an embedding of a smooth curve C_{a+1} in $\mathbf{P}^{s+a+1}$ such that $H^1(N_{C_{a+1}/\mathbf{P}^{s+a+1}}) = 0$. This enables us to continue the induction and finish the proof. □

4. The Kodaira dimension of $\overline{\mathcal{M}}_g$ and other problems

Since one is able to produce systematically effective divisors on $\overline{\mathcal{M}}_g$ having slope smaller than that of the Brill-Noether divisors, it is natural to ask whether one could diprove Conjecture 2.12, that is, construct effective divisors $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ for $g \le 23$ such that $s(D) < s(K_{\overline{\mathcal{M}}_g}) = 13/2$, which would imply that $\overline{\mathcal{M}}_g$ is of general type. We almost succeeded in this with Theorem 3.2 in the case $s = 2, i = 2, g = 22$: The slope of the (actual) divisor $\overline{\mathcal{Z}}_{22,2} \subset \overline{\mathcal{M}}_{22}$ turns out to be $1665/256 = 6.5039...$, which barely fails to make $\overline{\mathcal{M}}_{22}$ of general type. However, a different syzygy type condition, this time pushed-forward from a variety which maps onto $\overline{\mathcal{M}}_{22}$ with fibres of dimesnion one, produces an effective divisor of slope even smaller than $s(\overline{\mathcal{Z}}_{22,2})$. We have the following result **[F4]**:

THEOREM 4.1. *The moduli space $\overline{\mathcal{M}}_{22}$ is of general type. Precisely, the locus*

$$D_{22} := \{[C] \in \mathcal{M}_{22} : \exists L \in W^6_{25}(C) \text{ such that } C \overset{|L|}{\hookrightarrow} \mathbf{P}^6 \text{ lies on a quadric}\}$$

is a divisor on $\mathcal{M}_{22}$ and the class of its closure in $\overline{\mathcal{M}}_{22}$ equals

$$\overline{D} \equiv c\Big(\frac{17121}{2636}\lambda - \delta_0 - \frac{14511}{2636}\delta_1 - b_2\,\delta_2 - \cdots - b_{11}\,\delta_{11}\Big),$$

where $c > 0$ and $b_i > 1$ for $2 \le i \le 11$. Therefore $s(\overline{D}) = 17121/2636 = 6.49506... < 13/2$.

We certainly expect a similar result for $\overline{\mathcal{M}}_{23}$. We have calculated the class of the virtual locus D_{23} consisting of curves $[C] \in \mathcal{M}_{23}$ such that there exists $L \in W^6_{26}(C)$ with the multiplication map $\mu_L : \mathrm{Sym}^2 H^0(L) \to H^0(L^{\otimes 2})$ not being injective. By dimension count we expect this locus to be a divisor on $\overline{\mathcal{M}}_{23}$ and assuming so, we have computed its slope $s(\overline{D}_{23}) = 470749/72725 = 6.47300... < 13/2$. For $g = 23$ this is only a virtual result at the moment, since we cannot rule out the possibility that D_{23} equals the entire moduli space $\mathcal{M}_{23}$. The difficulty lies in the fact that D_{23} as a determinantal variety is expected to be of codimension 3 inside the variety $\mathfrak{G}^6_{26}$ which maps onto $\mathcal{M}_{23}$ with fibres of dimension 2.

The Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$

The problem of describing the Kodaira type of $\overline{\mathcal{M}}_{g,n}$ for $n \ge 1$, has been initiated by Logan in **[Log]**. Using Theorem 2.11 together with the formula $K_{\overline{\mathcal{M}}_{g,n}} = \pi_n^*(K_{\overline{\mathcal{M}}_{g,n-1}}) + \omega_{\pi_n}$, where $\pi_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ is the projection map forgetting the n-th marked point, we find that

$$K_{\overline{\mathcal{M}}_{g,n}} \equiv 13\lambda - 2\delta_0 + \sum_{i=1}^{n} \psi_i - 2 \sum_{i \geq 0, S} \delta_{i:S} - \sum_{S} \delta_{1:S}.$$

To prove that $\overline{\mathcal{M}}_{g,n}$ is of general type one needs an ample supply of explicit effective divisor classes on $\overline{\mathcal{M}}_{g,n}$ such that $K_{\overline{\mathcal{M}}_{g,n}}$ can be expressed as a linear combination with positive coefficients of such an effective divisor, boundary classes and an ample class on $\overline{\mathcal{M}}_{g,n}$ of the type $\sum_{i=1}^{n} a_i\psi_i + b\lambda - \delta_0 - \sum_{i\geq 0,S} \delta_{i:S}$ where $b > 11$ and $a_i > 0$ for $1 \leq i \leq n$. Logan has computed the class of the following effective divisors on $\overline{\mathcal{M}}_{g,n}$ (cf. **[Log]**, Theorems 5.3-5.7): We fix nonnegative integers $a_1, \ldots, a_n$ such that $a_1 + \cdots + a_n = g$ and we define $D_{g:a_1,\ldots,a_n}$ to be the locus of curves $[C, x_1, \ldots, x_n] \in \mathcal{M}_{g,n}$ such that $h^0(C, \mathcal{O}_C(a_1x_1 + \cdots + a_nx_n)) \geq 2$. Then $D_{g:a_1,\ldots,a_n}$ is a divisor on $\overline{\mathcal{M}}_{g,n}$ and one has the following formula in Pic($\overline{\mathcal{M}}_{g,n}$):

$$(4) \quad \overline{D}_{g:a_1,\ldots,a_n} \equiv -\lambda + \sum_{i+1}^{n} \binom{a_i+1}{2}\psi_i - 0 \cdot \delta_0 - \sum_{i<j} \binom{a_i+a_j+1}{2}\delta_{0:\{i,j\}} - \cdots .$$

Note that for $n = 1$ when necessarily $a_1 = g$, we obtain in this way the class of the divisor of Weierstrass points on $\overline{\mathcal{M}}_{g,1}$: $\overline{D}_{g:g} \equiv -\lambda + \binom{g+1}{2} - \sum_{i=1}^{g} \binom{g-i+1}{2}\delta_{i:1}$.

In **[F3]** we introduced a new class of divisors generalizing the loci of higher Weierstrass points in a different way: Fix $g, r \geq 1$ and $0 \leq i \leq g$. We set $n := (2r+1)(g-1) - 2i$ and define the locus

$$\mathfrak{Mrc}^r_{g,i} := \{[C, x_1, \ldots, x_n] \in \mathcal{M}_{g,n} : h^1\big(C, \wedge^i M_{K_C} \otimes K_C^{\otimes(r+1)} \otimes \mathcal{O}_C(-x_1 - \cdots - x_n)\big) \geq 1\}.$$

If we denote by $\Gamma := x_1 + \cdots + x_n \in C_n$, by Serre duality, the condition appearing in the definition of $\mathfrak{Mrc}^r_{g,i}$ is equivalent to

$$h^0\big(C, \wedge^i M^{\vee}_{K_C} \otimes \mathcal{O}_C(\Gamma) \otimes K_C^{\otimes(-r)}\big) \geq 1 \Longleftrightarrow \mathcal{O}_C(\Gamma) \otimes K_C^{\otimes(-r)} \in \Theta_{\wedge^i M^{\vee}_{K_C}},$$

where we recall that for a stable vector bundle E on C having slope $\nu(E) = \nu \in \mathbb{Z}$, its *theta divisor* is the determinantal locus

$$\Theta_E := \{\eta \in \mathrm{Pic}^{g-\mu-1}(C) : h^0(C, E \otimes \eta) \geq 1\}.$$

The main result from **[FMP]** gives an identification $\Theta_{\wedge^i M^{\vee}_{K_C}} = C_{g-i-1} - C_i$, where the right hand side is one of the difference varieties associated to C. Thus one has an alternative description of points in $\mathfrak{Mrc}^r_{g,i}$: a point $(C, x_1, \ldots, x_n) \in \mathfrak{Mrc}^r_{g,i}$ if and only if there exists $D \in C_i$ such that $h^0\big(C, \mathcal{O}_C(\Gamma + D) \otimes K_C^{\otimes(-r)}\big) \geq 1$. For $i = 0$, the divisor $\mathfrak{Mrc}^r_{g,0}$ consists of points $[C, x_1, \ldots, x_{(2r+1)(g-1)}]$ such that $\sum_{i=1}^{(2r+1)(g-1)} x_j \in |K_C^{\otimes r}|$.

THEOREM 4.2. *When $n = (2r+1)(g-1) - 2i$, the locus $\mathfrak{Mrc}^r_{g,i}$ is a divisor on $\mathcal{M}_{g,n}$ and the class of its compactification in $\overline{\mathcal{M}}_{g,n}$ is given by the following formula:*

$$\overline{\mathfrak{Mrc}}^r_{g,i} \equiv \frac{1}{g-1}\binom{g-1}{i}\Big(a\lambda + c\sum_{j=1}^{n} \psi_j - b_0\delta_0 - \sum_{j,s\geq 0,} b_{j:s} \sum_{|S|=s} \delta_{j:S}\Big),$$

where

$$c = rg + g - i - r - 1,\ b_0 = -\frac{1}{g-2}\Big(\binom{r+1}{2}(g-1)(g-2) + i(i+1+2r-rg-g)\Big),$$

$$a = -\frac{1}{g-2}\Big((g-1)(g-2)(6r^2+6r+1) + i(24r+10i+10-10g-12rg)\Big),$$

$$b_{0:s} = \binom{s+1}{2}(g-1) + s(rg-r) - si, \textit{ and } b_{j:s} \geq b_{0:s} \textit{ for } j \geq 1.$$

Using (4) and Theorem 4.2 one obtains the following table for which $\overline{\mathcal{M}}_{g,n}$'s are known to be of general type. In each case the strategy is to show that $K_{\overline{\mathcal{M}}_{g,n}}$ lies in the cone spanned by $\overline{D}_{g:a_1,\ldots,a_n}$, (pullbacks of) $\overline{\mathfrak{Mrc}}^r_{g,i}$, boundary divisors and ample classes:

THEOREM 4.3. *For integers $g = 4, \ldots, 21$, the moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type for all $n \geq f(g)$ where $f(g)$ is described in the following table.*

g	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$f(g)$	16	15	16	15	14	13	11	12	13	11	10	10	9	9	9	7	6	4

We end this paper with a number of questions related to the global geometry of $\overline{\mathcal{M}}_g$.

4.1. The hyperbolic nature of $\overline{\mathcal{M}}_g$. Since $\overline{\mathcal{M}}_g$ is of general type for g large, through a general point there can pass no rational or elliptic curve (On the other hand there are plenty of rational curves in $\overline{\mathcal{M}}_g$ for every g, take any pencil on a surface).

QUESTION 4.4. For large g, find a suitable lower bound for the invariant

$$\gamma_g := \inf\{g(\Gamma) : \Gamma \subset \overline{\mathcal{M}}_g \text{ is a curve passing through a general point } [C] \in \overline{\mathcal{M}}_g\}.$$

Is it true that for large g we have that $\gamma_g \geq C\ \log(g)$, where C is a constant independent of g? At present we do not even seem to be able to rule out the (truly preposterous) possibility that $\gamma_g = 2$. Since Γ will correspond to a fibration $f : S \to \Gamma$ from a surface with fibres being curves of genus g, using the well-known formulas

$$\Gamma \cdot \delta = c_2(S) + 4(g-1)(1-g(\Gamma)) \text{ and } \Gamma \cdot \lambda = \chi(\mathcal{O}_S) + (g-1)(1-g(\Gamma)),$$

the question can easily be turned into a problem about the irregularity of surfaces.

QUESTION 4.5. For large g, compute the invariant

$$n_g := \max\{\dim(Z) : Z \subset \overline{\mathcal{M}}_g, Z \cap \mathcal{M}_g \neq \emptyset,\ Z \text{ is not of general type}\}.$$

For the moduli space $\mathcal{A}_g$, Weissauer proved that for $g \geq 13$ every subvariety of $\mathcal{A}_g$ of codimension $\leq g - 13$ is of general type (cf. **[W]**). It seems reasonable to expect something along the same lines for $\overline{\mathcal{M}}_g$.

QUESTION 4.6. A famous theorem of Royden implies that $\mathcal{M}_{g,n}$ admits no non-trivial automorphisms or unramified correspondences for $2g-2+n \geq 3$ (see e.g. **[M]** Theorem 6.1 and the references cited therein). Recall that a non-trivial unramified correspondence is a pair of distinct finite étale morphisms $\alpha : X \to \mathcal{M}_{g,n}, \beta : X \to \mathcal{M}_{g,n}$. Precisely, Royden proves that the group of holomorphic automorphisms of the Teichmüller space $\mathcal{T}_{g,n}$ is isomorphic to the mapping class group. Using **[GKM]** Corollary 0.12 it follows that this result can be extended to $\overline{\mathcal{M}}_g$ when $g \geq 1$. Any automorphism $f : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ maps the boundary to itself, hence f induces an automorphism of $\mathcal{M}_g$ and then $f = 1_{\overline{\mathcal{M}}_{g,n}}$. One can ask the following questions:

(1) Is there an algebraic proof of Royden's Theorem (in arbitrary characteristic) using only intersection theory on $\overline{\mathcal{M}}_{g,n}$?
(2) Are there any non-trivial *ramified* correspondences of $\mathcal{M}_{g,n}$?
(3) Are there any non-trivial *birational* automorphisms of $\mathcal{M}_g$ for $g \geq 3$? It is known that there exists an integer g_0 such that $\mathcal{A}_g$ admits no birational automorphisms for any $g \geq g_0$ (cf. **[Fr]**).
(4) Is it true that for $n \geq 5$ we have that $\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}) = S_n$? Note that it follows from **[KMc]** Theorem 1.3, that $\mathrm{Aut}\big(\overline{\mathcal{M}}_{0,n}/S_n\big) = \{\mathrm{Id}\}$.

QUESTION 4.7. We fix an ample line bundle $L \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ (say $L = \kappa_1$). Can one attach a modular meaning to the *Seshadri constant* $\epsilon_{\overline{\mathcal{M}}_g}(L, [C])$, where $[C] \in \mathcal{M}_g$ is a general curve?

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