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HIGHER RANK BRILL–NOETHER THEORY ON SECTIONS OF $K3$ SURFACES

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We discuss the role of $K3$ surfaces in the context of Mercat's conjecture in higher rank Brill–Noether theory. Using liftings of Koszul classes, we show that Mercat's conjecture in rank 2 fails for any number of sections and for any gonality stratum along a Noether–Lefschetz divisor inside the locus of curves lying on $K3$ surfaces. Then we show that Mercat's conjecture in rank 3 fails even for curves lying on $K3$ surfaces with Picard number 1. Finally, we provide a detailed proof of Mercat's conjecture in rank 2 for general curves of genus 11, and describe explicitly the action of the Fourier–Mukai involution on the moduli space of curves.

Keywords: Mercat conjecture; Brill–Noether theory; Lazarsfeld–Mukai bundle; Koszul cohomology.

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1. Introduction

The Clifford index $\text{Cliff}(C)$ of an algebraic curve C is the second most important invariant of C after the genus, measuring the complexity of the curve in its moduli space. Its geometric significance is amply illustrated for instance in the statement

$$K_{p,2}(C, K_C) = 0 \Leftrightarrow p < \text{Cliff}(C)$$

of Green's Conjecture [6] on syzygies of canonical curves. It has been a long-standing problem to find an adequate generalization of $\text{Cliff}(C)$ for higher rank vector bundles. A definition in this sense has been proposed by Lange and Newstead [13]: If $E \in \mathcal{U}_C(n, d)$ denotes a semistable vector bundle of rank n and degree d on a curve C of genus g, one defines its Clifford index as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C, E) + 2 \geq 0,$$

and then the *higher Clifford indices* of C are defined as the quantities

$$\mathrm{Cliff}_n(C) := \min\{\gamma(E) : E \in \mathcal{U}_C(n,d), d \le n(g-1), h^0(C,E) \ge 2n\}.^{\mathrm{a}}$$

Note that $\mathrm{Cliff}_1(C) = \mathrm{Cliff}(C)$ is the classical Clifford index of C. By specializing to sums of line bundles, it is easy to check that $\mathrm{Cliff}_n(C) \le \mathrm{Cliff}(C)$ for all $n \ge 1$. Mercat [18] proposed the following interesting conjecture, which we state in the form of [13, Conjecture 9.3], linking the newly-defined invariants $\mathrm{Cliff}_n(C)$ to the classical geometry of C:

$$(M_n) : \mathrm{Cliff}_n(C) = \mathrm{Cliff}(C).$$

Mercat's conjecture (M_2) holds for various classes of curves, in particular general k-gonal curves of genus $g > 4k-4$, or arbitrary smooth plane curves, see [13]. In [5, Theorem 1.7], we have verified (M_2) for a general curve $[C] \in \mathcal{M}_g$ with $g \le 16$. More generally, the statement (M_2) is a consequence of the *Maximal Rank Conjecture* (see [5, Conjecture 2.2]), therefore it is expected to be true for a general curve $[C] \in \mathcal{M}_g$. However, for every genus $g \ge 11$ there exist curves $[C] \in \mathcal{M}_g$ with maximal Clifford index $\mathrm{Cliff}(C) = [\frac{g-1}{2}]$ carrying stable rank 2 vector bundles E with $h^0(C,E) = 4$ and $\gamma(E) < \mathrm{Cliff}(C)$, see [5, Theorems 3.6 and 3.7; 15, Theorem 1.1] for an improvement. For these curves, the inequality $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ holds.

Obvious questions emerging from this discussion are whether such results are specific to (i) rank 2 bundles with 4 sections, or to (ii) curves with maximal Clifford index $[\frac{g-1}{2}]$. First we prove that under general circumstances, curves on $K3$ surfaces carry rank 2 vector bundles E with a prescribed (and exceptionally high) number of sections invalidating Mercat's inequality $\gamma(E) \ge \mathrm{Cliff}(C)$.

Theorem 1.1. *We fix integers $p \ge 1$ and $a \ge 2p+3$. There exists a smooth curve C of genus $2a+1$ and Clifford index $\mathrm{Cliff}(C) = a$, lying on a $K3$ surface $C \subset S \subset \boldsymbol{P}^{2p+2}$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot C \oplus \mathbb{Z}\cdot H$, where $H^2 = 4p+2$, $H\cdot C = \deg(C) = 2a+2p+1$, as well as a stable rank 2 vector bundle $E \in \mathcal{SU}_C(2, \mathcal{O}_C(H))$, such that $h^0(C,E) = p+3$. In particular $\gamma(E) = a - \frac{1}{2} < \mathrm{Cliff}(C)$ and Mercat's conjecture (M_2) fails for C.*

It is well-known cf. [20, 24], that a curve $[C] \in \mathcal{M}_{2a+1}$ lying on a $K3$ surface S possesses a rank 2 vector bundle $F \in \mathcal{SU}_C(2, K_C)$ with $h^0(C,F) = a+2$. In particular, $\gamma(F) = a \ge \mathrm{Cliff}(C)$ (with equality if $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$), hence such bundles satisfy condition (M_2). Let us consider the $K3$ locus in the moduli space of curves

$$\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

[a] The invariant $\mathrm{Cliff}_n(C)$ is denoted in the paper [13] by $\gamma'_n(C)$. Since the appearance of [13], it has become abundantly clear that $\mathrm{Cliff}_n(C)$, defined as above, is the most relevant Clifford type invariant for rank n vector bundles on C. Accordingly, the notation $\mathrm{Cliff}_n(C)$ seems appropriate.

When $g = 11$ or $g \geq 13$, the variety $\mathcal{K}_g$ is irreducible and $\dim(\mathcal{K}_g) = 19 + g$, see [2, Theorem 5]. For integers $r, d \geq 1$ such that $d^2 > 4(r-1)g$ and $2r - 2 \nmid d$, we define the *Noether–Lefschetz* divisor inside the locus of sections of $K3$ surfaces

$$\mathfrak{NL}^r_{g,d} := \left\{ [C] \in \mathcal{K}_g \,\middle|\, \begin{array}{l} C \text{ lies on a } K3 \text{ surface } S, \mathrm{Pic}(S) \supset \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H, \\ H \in \mathrm{Pic}(S) \text{is nef}, H^2 = 2r-2, C \cdot H = d, C^2 = 2g-2 \end{array} \right\}.$$

A consequence of Theorem 1.1 can be formulated as follows.

Corollary 1.2. *We fix integers $p \geq 1$ and $a \geq 2p+3$ and set $g := 2a+1$. Then Mercat's conjecture (M_2) fails generically along the Noether–Lefschetz locus $\mathfrak{NL}^{2p+2}_{g,2a+2p+1}$ inside $\mathcal{K}_g$, that is,* $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ *for a general point $[C] \in \mathfrak{NL}^{2p+2}_{g,2a+2p+1}$.*

It is natural to wonder whether it is necessary to pass to a Noether–Lefschetz divisor in $\mathcal{K}_g$, or perhaps, all curves $[C] \in \mathcal{K}_g$ give counterexamples to conjecture (M_2). To see that this is not always the case and all conditions in Theorem 1.1 are necessary, we study in detail the case $g = 11$. Mukai [21] proved that a general curve $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, thus, $\mathcal{M}_{11} = \mathcal{K}_{11}$.

Theorem 1.3. *For a general curve $[C] \in \mathcal{M}_{11}$ one has the equality* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$, *that is, Mercat's conjecture holds generically on $\mathcal{M}_{11}$. Furthermore, the locus*

$$\{[C] \in \mathcal{M}_{11} : \mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)\}$$

can be identified with the Noether–Lefschetz divisor $\mathfrak{NL}^4_{11,13}$ on $\mathcal{M}_{11}$.

In Sec. 5, we describe in detail the divisor $\mathfrak{NL}^4_{11,13}$ and discuss, in connection with Mercat's conjecture, the action of the Fourier–Mukai involution $FM : \mathcal{F}_{11} \to \mathcal{F}_{11}$ on the moduli space of polarized $K3$ surfaces of genus 11. The automorphism FM acts on the set of Noether–Lefschetz divisors and in particular it (i) fixes the 6-gonal locus $\mathcal{M}^1_{11,6}$ and it maps the divisor $\mathfrak{NL}^4_{11,13}$ which corresponds to certain elliptic $K3$ surfaces, to the Noether–Lefschetz divisor corresponding to $K3$ surfaces carrying a rational curve of degree 3.

Next we turn our attention to the conjecture (M_n) for $n \geq 3$. It was observed in [12] that Mukai's description [22] of a general curve of genus 9 in terms of linear sections of a certain rational homogeneous variety, and especially the connection to rank 3 Brill–Noether theory, can be used to construct, on a general curve $[C] \in \mathcal{M}_9$, a stable vector bundle $E \in \mathcal{SU}_C(3, K_C)$ such that $h^0(C, E) = 6$. In particular $\gamma(E) = \frac{10}{3} < \mathrm{Cliff}(C)$, that is, Mercat's conjecture (M_3) fails for a general curve $[C] \in \mathcal{M}_9$. A similar construction is provided in [12] for a general curve of genus 11. In what follows we outline a construction illustrating that the results from [12] are part of a larger picture and curves on $K3$ surfaces carry vector bundles E of rank at least 3 with $\gamma(E) < \mathrm{Cliff}(C)$.

Let S be a $K3$ surface and $C \subset S$ a smooth curve of genus g. We choose a linear series $A \in W^r_d(C)$ of minimal degree such that the Brill–Noether number $\rho(g, r, d)$

is non-negative, that is, $d := r + [\frac{r(g+1)}{r+1}]$. The *Lazarsfeld bundle* M_A on C is defined as the kernel of the evaluation map, that is,

$$0 \to M_A \to H^0(C, A) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}_C} A \to 0.$$

As usual, we set $Q_A := M_A^\vee$, hence $\mathrm{rank}(Q_A) = r$ and $\det(Q_A) = A$. Following a procedure that already appeared in [16, 20, 24], we note that C carries a vector bundle of rank $r+1$ with canonical determinant and unexpectedly many global sections.

Theorem 1.4. *For a curve $C \subset S$ and $A \in W^r_d(C)$ as above there exists a globally generated vector bundle E on C with* $\mathrm{rank}(E) = r+1$ *and* $\det(E) = K_C$, *expressible as an extension*

$$0 \to Q_A \to E \to K_C \otimes A^\vee \to 0,$$

satisfying the condition $h^0(C, E) = h^0(C, A) + h^0(C, K_C \otimes A^\vee) = g - d + 2r + 1.$ *If moreover $r \leq 2$ and* $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, *then the above extension is nontrivial.*

When $r = 1$ the rank 2 bundle E constructed in Theorem 1.4 is well-known and plays an essential role in [24]. In this case $\gamma(E) \geq [\frac{g-1}{2}]$. For $r = 2$ and $g = 9$ (in which case $A \in W^2_8(C)$), or for $g = 11$ (and then $A \in W^2_{10}(C)$), Theorem 1.4 specializes to the construction in [12]. When $\mathrm{rank}(E) = 3$, we observe by direct calculation that $\gamma(E) < [\frac{g-1}{2}]$. In view of providing counterexamples to Mercat's conjecture (M_3), it is thus important to determine whether E is stable.

Theorem 1.5. *Fix $C \subset S$ as above with $g = 7, 9$ or $g \geq 11$ such that* $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, *as well as $A \in W^2_d(C)$, where $d := [\frac{2g+8}{3}]$. Then any globally generated rank 3 vector bundle E on C lying nontrivially in the extension*

$$0 \to Q_A \to E \to K_C \otimes A^\vee \to 0,$$

and with $h^0(C, E) = h^0(C, A) + h^0(C, K_C \otimes A^\vee) = g - d + 5$, is stable.

As a corollary, we note that for sufficiently high genus Mercat's statement (M_3) fails to hold for *any* smooth curve of maximal Clifford index lying on a $K3$ surface.

Corollary 1.6. *We fix an integer $g = 9$ or $g \geq 11$ and a curve $[C] \in \mathcal{K}_g$. Then the inequality* $\mathrm{Cliff}_3(C) < [\frac{g-1}{2}]$ *holds. In particular, Mercat's conjecture (M_3) fails generically along $\mathcal{K}_g$.*

We close the Introduction by thanking Lange and Newstead for making a number of very pertinent comments on the first version of this paper.

2. Higher Rank Vector Bundles with Canonical Determinant

In this section we treat Mercat's conjecture (M_3) and prove Theorems 1.4 and 1.5. We begin with a curve C of genus g lying on a smooth $K3$ surface S such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, and fix a linear series $A \in W^2_d(C)$ of minimal degree $d := [\frac{2g+8}{3}]$.

Under such assumptions both A and $K_C \otimes A^\vee$ are base-point-free. From the onset, we point out that the existence of vector bundles of higher rank on C having exceptional Brill–Noether behavior has been repeatedly used in [16, 20, 24]. Our aim is to study these bundles from the point of view of Mercat's conjecture and discuss their stability.

We define the *Lazarsfeld–Mukai* sheaf $\mathcal{F}_A$ via the following exact sequence on S:

$$0 \to \mathcal{F}_A \to H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}_S} A \to 0.$$

Since A is base-point-free, $\mathcal{F}_A$ is locally free. We consider the vector bundle $\mathcal{E}_A := \mathcal{F}_A^\vee$ on S, which by dualizing, sits in an exact sequence

$$0 \to H^0(C,A)^\vee \otimes \mathcal{O}_S \to \mathcal{E}_A \to K_C \otimes A^\vee \to 0. \tag{2.1}$$

Since $K_C \otimes A^\vee$ is assumed to be base-point-free, the bundle $\mathcal{E}_A$ is globally generated. It is well-known (and follows from the sequence (2.1), that $c_1(\mathcal{E}_A) = \mathcal{O}_S(C)$ and $c_2(\mathcal{E}_A) = d$.

Proof of Theorem 1.4. We write down the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^0(C,A)\otimes\mathcal{O}_S(-C) & \xrightarrow{=} & H^0(C,A)\otimes\mathcal{O}_S(-C) & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{F}_A & \to & H^0(C,A)\otimes\mathcal{O}_S & \to A \to 0 \\
 & & \downarrow & & \downarrow & \downarrow = \\
0 \to & & M_A & \to & H^0(C,A)\otimes\mathcal{O}_C & \to A \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

from which, if we set $F_A := \mathcal{F}_A \otimes \mathcal{O}_C$ and $E_A := \mathcal{E}_A \otimes \mathcal{O}_C$, we obtain the exact sequence

$$0 \to M_A \otimes K_C^\vee \to H^0(C,A) \otimes K_C^\vee \to F_A \to M_A \to 0$$

(use that $\mathrm{Tor}^1_{\mathcal{O}_S}(M_A, \mathcal{O}_C) = M_A \otimes K_C^\vee$). Taking duals, we find the exact sequence

$$0 \to Q_A \to E_A \to K_C \otimes A^\vee \to 0. \tag{2.2}$$

Since S is regular, from (2.1) we obtain that $h^0(S,\mathcal{E}_A) = h^0(C,A) + h^0(C, K_C \otimes A^\vee)$ while $H^0(S, \mathcal{E}_A \otimes \mathcal{O}_S(-C)) = 0$, that is,

$$h^0(S,\mathcal{E}_A) \le h^0(C,E_A) \le h^0(C,A) + h^0(C, K_C \otimes A^\vee).$$

Thus the sequence (2.2) is exact on global sections.

We are left with proving that the extension (2.2) is nontrivial. We set $r = 2$ and then $\mathrm{rank}(\mathcal{E}_A) = 3$ and place ourselves in the situation when $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$ (the case $r = 1$ works similarly). By contradiction we assume that $E_A = Q_A \oplus (K_C \otimes A^\vee)$

and denote by $s : E_A \to Q_A$ a retract and by $\tilde{s} : \mathcal{E}_A \to Q_A$ the induced map. We set $\mathcal{M} := \text{Ker}\{\mathcal{E}_A \stackrel{\tilde{s}}{\to} Q_A\}$, hence $\mathcal{M}$ can be regarded as an elementary transformation of the Lazarsfeld–Mukai bundle $\mathcal{E}_A$ along C. By direct calculation we find that

$$c_1(\mathcal{M}) = \mathcal{O}_S(-C) \quad \text{and} \quad c_2(\mathcal{M}) = 2d - 2g + 2,$$

hence the discriminant of $\mathcal{M}$ equals $\Delta(\mathcal{M}) := 6c_2(\mathcal{M}) - 2c_1^2(\mathcal{M}) = 4(3d - 4g + 4) < 0$. Thus the sheaf $\mathcal{M}$ is $\mathcal{O}_S(C)$-unstable. Applying [10, Theorems 7.3.3 and 7.3.4], there exists a subsheaf $\mathcal{M}' \subset \mathcal{M}$ such that if $\xi_{\mathcal{M},\mathcal{M}'} := \frac{c_1(\mathcal{M}')}{\text{rank}(\mathcal{M}')} - \frac{c_1(\mathcal{M})}{\text{rank}(\mathcal{M})} \in \text{Pic}(S)_{\mathbb{R}}$, then

$$\text{(i)} \quad \xi_{\mathcal{M},\mathcal{M}'} \cdot C > 0 \quad \text{and} \quad \text{(ii)} \quad \xi^2_{\mathcal{M},\mathcal{M}'} \geq -\frac{\Delta(\mathcal{M})}{18}.$$

Since $\text{Pic}(S) = \mathbb{Z} \cdot C$, we may write $c_1(\mathcal{M}') = \mathcal{O}_S(aC)$ and also set $r' := \text{rank}(\mathcal{M}')$. The Lazarsfeld–Mukai bundle $\mathcal{E}_A$ is $\mathcal{O}_S(C)$-stable, in particular $\mu_C(\mathcal{M}') \leq \mu_C(\mathcal{E}_A)$, which yields $a \leq 0$. Then from (i) we write that $0 \leq \frac{a}{r'} + \frac{1}{3} \leq \frac{1}{3}$, whereas from (ii) one finds

$$\frac{1}{9} \geq \frac{4(g-1) - 3d}{9(g-1)} \Leftrightarrow d \geq g - 1,$$

which is a contradiction. It follows that the extension (2.2) is nontrivial. □

It is natural to ask when is the above constructed bundle E_A stable. We give an affirmative answer under certain generality assumptions, when $r < 3$.

We fix a $K3$ surface S such that $\text{Pic}(S) = \mathbb{Z} \cdot C$ and as before, set $d := [\frac{2g+8}{3}]$. Under these assumptions, it follows from [16] that C satisfies the Brill–Noether theorem. We prove the stability of every globally generated non-split bundle E sitting in an extension of the form (2.2) and having a maximal number of sections.

Proof of Theorem 1.5. We first discuss the possibility of a destabilizing sequence

$$0 \to F \to E \to B \to 0,$$

where F is a vector bundle of rank 2 and $\deg(F) \geq \frac{4}{3}(g-1)$. Since E is globally generated, it follows that B is globally generated as well, hence $h^0(C, B) \geq 2$, in particular $\deg(B) \geq (g+2)/2$ and hence $\deg(F) \leq \frac{3}{2}g - 3$. Since $\deg(B) \leq \frac{2}{3}(g-1)$ and C is Brill–Noether general, it follows that $h^0(C, B) = 2$, therefore $h^0(C, F) \geq g - d + 3$. There are two cases to distinguish, depending on whether F possesses a subpencil or not.

Assume first that F has no subpencils. We apply [23, Lemma 3.9] to find that $h^0(C, \det(F)) \geq 2h^0(C, F) - 3 \geq 2g - 2d + 3$. Writing down the inequality

$$\rho(g, 2g - 2d + 2, \deg(F)) \geq 0$$

and using that $\deg(F) < \frac{3}{2}g - 3$, we obtain a contradiction. If on the other hand, F has a subpencil, then as pointed out in [5, Lemma 3.2], $\gamma(F) \geq \text{Cliff}(C)$, but again this is a contradiction. This shows that E cannot have a rank 2 destabilizing subsheaf.

We are left with the possibility of a destabilizing short exact sequence

$$0 \to B \to E \to F \to 0,$$

where B is a line bundle with $\deg(B) \geq \frac{2}{3}(g-1)$ and F is a rank 2 bundle. The bundle Q_A is well-known to be stable and based on slope considerations, B cannot be a subbundle of Q_A, that is, necessarily $H^0(C, K_C \otimes A^\vee \otimes B^\vee) \neq 0$. Since the bundle E is not decomposable, it follows that $\deg(B) \leq \deg(K_C \otimes A^\vee) - 1 = 2g - 3 - d$. Furthermore $h^1(C, B) \geq 3$.

If F is not stable, we reason along the lines of [12, Proposition 3.5] and pull-back a destabilizing line subbundle of F to obtain a rank 2 subbundle $F' \subset E$ such that

$$\deg(F') \geq \deg(B) + \frac{1}{2}(\deg(E) - \deg(B)) \geq \frac{4}{3}(g-1),$$

which is the case we have already ruled out. So we may assume that F is stable. We write $h^0(C, B) = a + 1$, hence $h^0(C, F) \geq g - d - a + 4$. Assume first that F admits no subpencils. Then from [23, Lemma 3.9] we find the following estimate for the number of sections of the line bundle $\det(F) = K_C \otimes B^\vee$,

$$h^0(C, K_C \otimes B^\vee) \geq 2h^0(C, F) - 3 \geq 2g - 2d - 2a + 5,$$

which, after applying Riemann–Roch to B, leads to the inequality

$$3a \geq g - 2d + 5 + \deg(B).$$

Combining this estimate with the Brill–Noether inequality $\rho(g, a, \deg(B)) \geq 0$ and substituting the actual value of d, we find that $3a + 3 \geq g$. On the other hand $a \leq h^0(C, K_C \otimes A^\vee) - 2 = g - d < \frac{g-3}{3}$, and this is a contradiction.

Finally, if F admits a subpencil, then $\gamma(F) \geq \mathrm{Cliff}(C)$. Combining this with the classical Clifford inequality for B, we find that $\gamma(E) \geq \mathrm{Cliff}(C)$, which again is a contradiction. We conclude that the rank 3 bundle E must be stable. □

3. Rank 2 Bundles and Koszul Classes

The aim of this section is to prove Theorem 1.1. We shall construct rank 2 vector bundles on curves using a connection between vector bundles on curves and Koszul cohomology of line bundles, cf. [1, 25]. Let us recall that for a smooth projective variety X, a sheaf $\mathcal{F}$ and a globally generated line bundle L on X, the Koszul cohomology group $K_{p,q}(X; \mathcal{F}, L)$ is defined as the cohomology of the complex:

$$\bigwedge^{p+1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q-1}) \stackrel{d_{p+1,q-1}}{\to} \bigwedge^{p} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q}) \stackrel{d_{p,q}}{\to} \bigwedge^{p-1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q+1}).$$

Most of the time $\mathcal{F} = \mathcal{O}_X$, and then one writes $K_{p,q}(X; \mathcal{O}_X, L) := K_{p,q}(X, L)$.

A Koszul class $[\zeta] \in K_{p,1}(X, L)$ is said to have rank $\leq n$, if there exists a subspace $W \subset H^0(X, L)$ with $\dim(W) = n$ and a representative $\zeta \in \wedge^p W \otimes H^0(X, L)$. The smallest number n with this property is the rank of the syzygy $[\zeta]$.

Next we discuss a connection due to Voisin [25] and expanded in [1], between rank 2 vector bundles on curves and syzygies. Let E be a rank 2 bundle on a smooth curve C with $h^0(C,E) \geq p+3 \geq 4$ and set $L := \det(E)$. Let

$$\lambda : \wedge^2 H^0(C,E) \to H^0(C,L)$$

be the determinant map, and we assume that there exists linearly independent sections $e_1 \in H^0(C,E)$ and $e_2, \ldots, e_{p+3} \in H^0(C,E)$, such that the map

$$\lambda(e_1 \wedge -) \, : \, \langle e_2, \ldots, e_{p+3} \rangle \to H^0(C,L)$$

in injective onto its image. Such an assumption is automatically satisfied for instance if E admits no subpencils. We introduce the subspace

$$W := \langle s_2 := \lambda(e_1 \wedge e_2), \ldots, s_{p+3} := \lambda(e_1 \wedge e_{p+3}) \rangle \subset H^0(C,L).$$

By assumption, $\dim(W) = p+2$. Following [1, 25], we define the tensor

$$\zeta(E) := \sum_{i<j} (-1)^{i+j} \; s_2 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge \hat{s}_j \wedge \cdots \wedge s_{p+3} \otimes \lambda(e_i \wedge e_j) \in \wedge^p W \otimes H^0(C,L).$$

One checks that $d_{p,1}(\zeta(E)) = 0$, hence $[\zeta(E)] \in K_{p,1}(C,L)$ is a nontrivial Koszul class of rank at most $p+2$. Conversely, starting with a nontrivial class $[\zeta] \in K_{p,1}(C,L)$ represented by an element ζ of $\wedge^p W \otimes H^0(C,L)$ where $\dim(W) = p+2$, Aprodu and Nagel [1, Theorem 3.4] constructed a rank 2 vector bundle E on C with $\det(E) = L$, $h^0(C,E) \geq p+3$ and such that $[\zeta(E)] = [\zeta]$. This correspondence sets up a dictionary between the Brill–Noether loci in $\{E \in \mathcal{SU}_C(2,L) : h^0(C,E) \geq p+3\}$ and Koszul classes of rank at most $p+2$ in $K_{p,1}(C,L)$.

Let us now fix integers $p \geq 1$ and $a \geq 2p+3$. Using the surjectivity of the period mapping, see e.g. [11, Theorem 1.1], one can construct a smooth $K3$ surface $S \subset \mathbf{P}^{2p+2}$ of degree $4p+2$ containing a smooth curve $C \subset S$ of degree $d := 2a+2p+1$ and genus $g := 2a+1$. The surface S can be chosen with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, where $H^2 = 4p+2$, $H \cdot C = d$ and $C^2 = 4a$. The smooth curve $H \subset C$ is the hyperplane section of S and has genus $g(H) = 2p+2$. The following observation is trivial.

Lemma 3.1. *Keeping the notation above, we have that* $H^0(S, \mathcal{O}_S(H-C)) = 0$.

Proof. It is enough to notice that H is nef and $(H-C) \cdot H = 2p - 2a + 1 < 0$. □

We consider the decomposable rank 2 bundle $K_H = A \oplus (K_H \otimes A^\vee)$ on H, where $A \in W^1_{p+2}(H)$. Via the Green–Lazarsfeld non-vanishing theorem [7] (or equivalently, applying [1]), one obtains a nonzero Koszul class of rank $p+1$

$$\beta := [\zeta(A \oplus (K_H \otimes A^\vee))] \in K_{p,1}(H, K_H).$$

Since S is a regular surface, there exist an exact sequence

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(H)) \to H^0(H, K_H) \to 0,$$

which induces an isomorphism [6, Theorem (3.b.7)]

$$\mathrm{res}_H : K_{p,1}(S, \mathcal{O}_S(H)) \cong K_{p,1}(H, K_H).$$

By construction, the nontrivial class $\alpha := \mathrm{res}_H^{-1}(\beta) \in K_{p,1}(S, \mathcal{O}_S(H))$ has rank at most $\mathrm{rank}(\beta) + 1 = p + 2$. Using [6, Theorem (3.b.1)], we write the following exact sequence in Koszul cohomology:

$$\cdots \to K_{p,1}(S; -C, H) \to K_{p,1}(S, H) \to K_{p,1}(C, H \otimes \mathcal{O}_C) \to K_{p-1,2}(S; -C, H) \to \cdots.$$

Since $H^0(S, \mathcal{O}_S(H - C)) = 0$, it follows that $K_{p,1}(S; -C, H) = 0$, in particular the nonzero class $\alpha \in K_{p,1}(S, H)$ can be viewed as a Koszul class of rank at most $p+2$ inside the group $K_{p,1}(C, \mathcal{O}_C(H))$. This class corresponds to a *stable* rank 2 bundle on C.

Proposition 3.2. *Let $C \subset S \subset \boldsymbol{P}^{2p+2}$ as above and $L := \mathcal{O}_C(1) \in \mathrm{Pic}^{2a+2p+1}(C)$. Then there exists a stable vector bundle $E \in \mathcal{SU}_C(2, L)$ with $h^0(C, E) = p + 3$.*

Proof. From [1] we know that there exists a rank 2 vector bundle E on C with $\det(E) = L$ such that $[\zeta(E)] = \alpha \in K_{p,1}(C, L)$, in particular $h^0(C, E) \geq p+3$. Geometrically, E is the restriction to C of the Lazarsfeld–Mukai bundle $\mathcal{E}_A$ on S corresponding to a pencil $A \in W^1_{p+2}(H)$. In particular, E is globally generated, being the restriction of a globally generated bundle on S. We also know that $\mathrm{Cliff}(C) = a$ (to be proved in Proposition 3.3). Since $\gamma(E) \leq a - \frac{1}{2} < \mathrm{Cliff}(C)$, it follows that E admits no subpencils (If $B \subset E$ is a subpencil, then $h^0(C, L \otimes B^\vee) \geq 2$ because E is globally generated. It is easily verified that both B and $L \otimes B^\vee$ contribute to $\mathrm{Cliff}(C)$, which brings about a contradiction). Assume now that

$$0 \to B \to E \to L \otimes B^\vee \to 0$$

is a destabilizing sequence, where $B \in \mathrm{Pic}(C)$ has degree at least $a+p+1$. As already pointed out, $h^0(C, B) \leq 1$, hence $h^0(C, L \otimes B^\vee) \geq p+2$. If $h^1(C, L \otimes B^\vee) \leq 1$, then $p + 2 \leq h^0(C, L \otimes B^\vee) \leq 1 + \deg(L \otimes B^\vee) - 2a$, which leads to a contradiction. If on the other hand $h^1(C, L \otimes B^\vee) \geq 2$, then $\mathrm{Cliff}(L \otimes B^\vee) \leq a - p - 2 < a$, which is impossible. Thus E is a stable vector bundle. □

We are left with showing that the curve $C \subset S$ constructed above has maximal Clifford index a. Note that the corresponding statement when $p = 1$ has been proved in [5, Theorem 3.6].

Proposition 3.3. *We fix integers $p \geq 1$, $a \geq 2p+3$ and a $K3$ surface S with Picard lattice $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$ where $C^2 = 4a, H^2 = 4p + 2$ and $C \cdot H = 2a + 2p + 1$. Then $\mathrm{Cliff}(C) = a$.*

Proof. First note that C has Clifford dimension 1, for curves $C \subset S$ of higher Clifford dimension have even genus. Observe also that $h^0(C, \mathcal{O}_C(1)) = 2p+3$ and $h^1(C, \mathcal{O}_C(1)) = 2$, hence $\mathcal{O}_C(1)$ contributes to the Clifford index of C and

$$\mathrm{Cliff}(C) \leq \mathrm{Cliff}(C, \mathcal{O}(1)) = C \cdot H - 2(2p+2) = 2a - 2p - 3 \ (\geq a).$$

Assume by contradiction that $\mathrm{Cliff}(C) < a$. According to [8], there exists an effective divisor $D \equiv mH + nC$ on S satisfying the conditions

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \quad h^0(S, \mathcal{O}_S(C-D)) \geq 2, \quad C \cdot D \leq g-1, \tag{3.1}$$

and with $\mathrm{Cliff}(\mathcal{O}_C(D)) = \mathrm{Cliff}(C)$. By [17, Lemma 2.2], the dimension $h^0(C', \mathcal{O}_{C'}(D))$ stays constant for all smooth curves $C' \in |C|$ and its value equals $h^0(S, D)$. We conclude that $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(D)) = D \cdot C - 2\dim|D|$. We summarize the numerical consequences of the inequalities (3.1):

(i) $md + 2n(g-1) \leq g-1$,
(ii) $(2p+1)m^2 + mnd + n^2(g-1) \geq 0$,
(iii) $(4p+2)m + dn > 2$.

We claim that for any divisor $D \subset S$ verifying (i)–(iii), the following inequality holds:

$$\mathrm{Cliff}(\mathcal{O}_C(D)) = D \cdot C - D^2 - 2 \geq H \cdot C - H^2 - 2 = 2a - 2p - 3 \geq a.$$

This will contradict the assumption $\mathrm{Cliff}(C) < a$. The proof proceeds along the lines of Theorem 3 in [4], with the difference that we must also consider curves with $D^2 = 0$, that is, elliptic pencils which we now characterize. By direct calculation, we note that there are no (-2)-curves in S. Equality holds in (ii) when $m = -n$ or $m = -un$ with $u := 2a/(2p+1)$.

First, we describe the effective divisors $D \subset S$ with self-intersection $D^2 = 0$. Consider the case $m = -un$. If $2p+1$ does not divide a, then $D \equiv 2aH - (2p+1)C$ and $D \cdot C = 2a(2a-2p-1) > g-1$, that is, D does not verify condition (i). If $a = k(2p+1)$, for $k \geq 2$, then $D \equiv 2kH - C$. Notice that $D \cdot C = a(4k-4) + 2k(2p+1) > 2a$ for $k \geq 2$, that is, D does not satisfies (i).

In the case $m = -n$, the effective divisor $D \equiv C - H$, satisfies (i)–(iii) and

$$\mathrm{Cliff}(\mathcal{O}_C(C-H)) = 2a - 2p - 3 \geq a.$$

Case $n < 0$. From (ii) we have either $m < -n$ or $m > -un$. In the first case, by using inequality (iii), we obtain $2 < -(4p+2)n + dn = n(2a-2p-1)$, which is a contradiction since $n < 0$ and $2a > 2p+1$. Suppose $m > -un > 0$. Inequality (i) implies that

$$(-n)\frac{2ad}{2p+1} < -(g-1)(2n-1) = -2a(2n-1),$$

then $(-n)(d - (4p+2)) < 2p+1$ and since $d > 4p+2$, this yields $2a + 2p + 1 = d < 6p + 3$ which contradicts the hypothesis $a \geq 2p+3$.

Case $n > 0$. Again, by condition (ii), we have either that $m < -un$ or $m > -n$. In the first case, using (iii) we write that

$$0 < (4p+2)m + dn < n\left(d - (4p+2)\frac{2a}{2p+1}\right),$$

but one can easily check that $d(2p+1) < 2a(4p+2)$, which yields a contradiction. Suppose now $-n < m < 0$. By (i) we have $2a(2n-1) \leq -md < nd$, so $n < \frac{2a}{4a-d} = \frac{2a}{2a-2p-1} < 2$, since $a \geq 2p+1$. This implies $n = 1$, therefore for $n > 0$ there are no divisors $D \subset S$ with $D^2 > 0$ satisfying the inequalities (i)–(iii).

Case $n = 0$. From (i), one writes $m \leq \frac{g-1}{d} = \frac{2a}{2a+2p+1} < 1$, but this yields to a contradiction since by (iii) it follows that $m > 0$. The proof is thus finished. ☐

4. Curves with Prescribed Gonality and Small Rank 2 Clifford Index

The equality $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ is known to be valid for *arbitrary* k-gonal curves $[C] \in \mathcal{M}^1_{g,k}$ of genus $g > (k-1)(2k-4)$. It is thus of some interest to study Mercat's question for arbitrary curves in a given gonality stratum in $\mathcal{M}_g$ and decide how sharp is this quadratic bound. We shall construct curves C of unbounded genus and relatively small gonality, carrying a stable rank 2 vector bundle E with $h^0(C, E) = 4$ such that $\gamma(E) < \mathrm{Cliff}(C)$. In order to be able to determine the gonality of C, we realize it as a section of a $K3$ surface S in $\mathbf{P}^4$ which is special in the sense of Noether–Lefschetz theory. The pencil computing the gonality is the restriction of an elliptic pencil on the surface. The constraint of having a Picard lattice of rank 2 containing, apart from the hyperplane class, both an elliptic pencil and a curve C of prescribed genus, implies that the discriminant of $\mathrm{Pic}(S)$ must be a perfect square. This imposes severe restrictions on the genera for which such a construction could work.

Theorem 4.1. *We fix integers $a \geq 3$ and $b = 4, 5, 6$. There exists a smooth curve $C \subset \mathbf{P}^4$ with*

$$\deg(C) = 6a + b, \quad g(C) = 3a^2 + ab + 1 \quad \textit{and gonality } \mathrm{gon}(C) = ab,$$

such that C lies on a $(2, 3)$ complete intersection $K3$ surface. In particular $K_{1,1}(C, \mathcal{O}_C(1)) \neq 0$ and conjecture (M_2) fails for C.

Before presenting the proof, we discuss the connection between Theorem 4.1 and conjecture (M_2). For $C \subset S \subset \mathbf{P}^4$ as above, we construct a vector bundle E with $\det(E) = \mathcal{O}_C(1)$ and $h^0(C, E) = 4$, lying in an exact sequence

$$0 \to E \to W \otimes \mathcal{O}_C(1) \to \mathcal{O}_C(2) \to 0,$$

where $W \in G(3, H^0(C, \mathcal{O}_C(1))$ has the property that the quadric $Q \in \mathrm{Sym}^2 H^0(C, \mathcal{O}_C(1))$ induced by S is representable by a tensor in $W \otimes H^0(C, L)$. This construction is a particular procedure of associating vector bundles to nontrivial syzygies, cf. [1]. The proof that E is stable is standard and proceeds along the lines of e.g. [9, Theorem 3.2]. Next we compute the Clifford invariant:

$$\gamma(E) = 3a + \frac{b}{2} < ab - 2 = \text{Cliff}(C),$$

since $b \geq 4$, so not only $\text{Cliff}_2(C) < \text{Cliff}(C)$, but the difference $\text{Cliff}(C) - \text{Cliff}_2(C)$ becomes arbitrarily positive.

Proof. By means of [11, Theorem 6.1], there exist a smooth complete intersection surface $S \subset \mathbf{P}^4$ of type $(2,3)$ such that $\text{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, where $H^2 = 6$, $H \cdot C = d = 6a + b$ and $C^2 = 2(g-1)$ (Note that such a surface exists when $d^2 > 12g$, which is satisfied when $b \geq 4$). The divisor $E := C - aH$ verifies $E^2 = 0$, $E \cdot H = b$ and $E \cdot C = ab$. In particular E is effective. The class E is primitive, hence it follows that $h^0(S, E) = h^0(C, \mathcal{O}_C(E)) = 2$, where the last equality follows by noting that $H^1(S, \mathcal{O}_S(E - C)) = 0$ by Kodaira vanishing. Furthermore, $h^1(C, \mathcal{O}_C(E)) \geq 3a^2 + 2$, that is, $\mathcal{O}_C(E)$ contributes to $\text{Cliff}(C)$ and then we write that

$$\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, \mathcal{O}_C(E)) + 2 = ab.$$

We shall show that $\mathcal{O}_C(E)$ computes the Clifford index of C.

First, we classify the primitive effective divisors $F \equiv mH + nC \subset S$ having self-intersection zero. By solving the equation $(mH + nC)^2 = 0$, where $m, n \in \mathbb{Z}$, we find the following primitive solutions: $E_1 \equiv (3a + b)H - 3C$ for $b \neq 6$ (respectively $E_2 \equiv (a+2)H - C$ for $b = 6$), and $E_3 = E \equiv C - aH$. A simple computation shows that $E_i \cdot C > ab$ for $i = 1, 2$.

Since $\text{Cliff}(C) \leq ab - 2 < [\frac{g-1}{2}]$, the Clifford index of C is computed by a bundle defined on S. Following [8], there exists an effective divisor $D \equiv mH + nC$ on S, satisfying the following numerical conditions:

$$\begin{gathered} h^0(S, D) = h^0(C, \mathcal{O}_C(D)) \geq 2, \quad h^0(S, C - D) \geq 2, \\ D^2 \geq 0 \quad \text{and} \quad D \cdot C \leq g - 1, \end{gathered} \tag{4.1}$$

and such that

$$f(D) := \text{Cliff}(\mathcal{O}_C(D)) + 2 = D \cdot C - D^2 = \text{Cliff}(C) + 2.$$

Furthermore, D can be chosen such that $h^1(S, D) = 0$, cf. [17]. To bound $f(D)$ and show that $f(D) \geq ab$, we distinguish two cases depending on whether $D^2 > 0$ or $D^2 = 0$.

By a complete classification of curves with self-intersection zero, we have already seen that for any elliptic pencil $|D|$ satisfying (4.1), one has $f(D) \geq ab = f(E)$. We are left with the case $D^2 > 0$ and rewrite the inequalities (4.1):

(i) $(6a + b)m + (2n - 1)(3a^2 + ab) \leq 0$,
(ii) $(m + an)(3an + 3m + bn) > 0$,
(iii) $6m + (6a + b)n > 2$,

where (ii) comes from the assumption $D^2 > 0$ and (iii) from the fact that $D \cdot H > 2$. Furthermore,

$$f(m,n) := D \cdot C - D^2 = -6m^2 + m(d - 2nd) + (n - n^2)(2g - 2). \tag{4.2}$$

We prove that for any divisor D satisfying (i)–(iii), the inequality $f(m,n) \geq ab$ holds, from which we conclude that $\mathrm{Cliff}(C) = ab - 2$.

Case $n < 0$. From (iii) we find that $m > 0$. Then $m < -an$ or $3m > -(3a+b)n$. When $m < -an$, from (iii) we have that $2 < 6m + dn < -6an + dn = nb < 0$, which is a contradiction. Suppose $(3a+b)n + 3m > 0$. For a fixed n the function $f(m,n)$ reaches its maximum at $m_0 := \frac{d(1-2n)}{12}$. So when $3m_0 + (3a+b)n \leq 0$, we have $f(m,n) \geq f(\frac{(1-2n)(g-1)}{d}, n)$, since by condition (i), $m \leq \frac{(1-2n)(g-1)}{d}$. A simple computation gives that whenever $n < 0$, one has the inequality:

$$\begin{aligned} f\left(\frac{(1-2n)(g-1)}{d}, n\right) &= (2n^2 - 2n)(g-1)\frac{b^2}{d^2} + (g-1)\left(1 - \frac{6(g-1)}{d^2}\right) \\ &\geq 4(g-1)\frac{b^2}{d^2} + \frac{g-1}{d^2}(18a^2 + b^2 + 6ab) \geq \frac{3a^2 + ab}{2} \geq ab. \end{aligned}$$

Assume now that $3m_0 + (3a+b)n > 0$. Since $m \in (-\frac{(3a+b)n}{3}, \frac{(1-2n)(g-1)}{d}]$, we have

$$f(m,n) \geq \min\left\{ f\left(-\frac{(3a+b)n}{3}, n\right), f\left(\frac{(1-2n)(g-1)}{d}, n\right)\right\}.$$

A direct computation yields

$$f\left(-\frac{(3a+b)n}{3}, n\right) = -n\left(ab + \frac{b^2}{3}\right) \geq ab + \frac{b^2}{3} \geq ab.$$

Case $n > 0$. If $m \geq 0$ we get a contradiction to (i). Suppose $m < 0$, then we have either $3m + (3a+b)n < 0$, or else $m > -an$. The first case contradicts (iii), so it does not appear. Suppose $m > -an$. Reasoning as before, observe that $m_0 < (1 - 2n)(g-1)/d$, where m_0 is the maximum of $f(m,n)$ for a fixed n, and m takes values in the interval $\left(-an, \frac{(1-2n)(g-1)}{d}\right]$. If $-an \geq m_0$, then $f(m,n) \geq f\left(\frac{(1-2n)(g-1)}{d}, n\right)$. Since we are assuming $-an < \frac{(1-2n)(g-1)}{d}$, we have that $n < \frac{3a}{b} + 1$. We use this bound to directly show, like in the previous case, that $f\left(\frac{(1-2n)(g-1)}{d}, n\right) \geq ab$. When $-an < m_0$ we have that

$$f(m,n) \geq \min\left\{ f(-an, n), f\left(\frac{(1-2n)(g-1)}{d}, n\right)\right\}.$$

In this case it is enough to note that $f(-an, n) = nab \geq ab$.

Case $n = 0$. From inequalities (i) and (iii) with $n = 0$, we have $1 \leq m \leq \frac{g-1}{d}$. Note that $f(m,0) = -6m^2 + md$ reaches its maximum at $\frac{d}{12}$. So, since $\frac{g-1}{d} \leq \frac{d}{12}$, we conclude that $f(m,0) \geq f(1,0) = 6a + b - 6$. Finally, we observe that $6a + b - 6 \geq ab$ if and only if $b \leq 6$. This finishes the proof. □

5. The Fourier–Mukai Involution on $\mathcal{F}_{11}$

The aim of this section is to provide a detailed proof of Mercat's conjecture (M_2) in one nontrivial case, that of genus 11, and discuss the connection to Mukai's work [19, 21]. We denote as usual by $\mathcal{F}_g$ the moduli space parametrizing pairs $[S, \ell]$, where S is a smooth $K3$ surface and $\ell \in \mathrm{Pic}(S)$ is a primitive nef line bundle with $\ell^2 = 2g-2$. Furthermore, we introduce the parameter space

$$\mathcal{P}_g := \{[S, C] : S \text{ is a smooth } K3 \text{ surface}, C \subset S \text{ is a smooth curve}, \\ [S, \mathcal{O}_S(C)] \in \mathcal{F}_g\}$$

and denote by $\pi : \mathcal{P}_g \to \mathcal{F}_g$ the projection map $[S, C] \mapsto [S, \mathcal{O}_S(C)]$. If S is a $K3$ surface, following [19], we set $\widetilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ and

$$\widetilde{NS}(S) := H^0(S, \mathbb{Z}) \oplus NS(S) \oplus H^4(S, \mathbb{Z}).$$

We recall the definition of the *Mukai pairing* on $\widetilde{H}(S, \mathbb{Z})$:

$$(\alpha_0, \alpha_2, \alpha_4) \cdot (\beta_0, \beta_2, \beta_4) := \alpha_2 \cup \beta_2 - \alpha_4 \cup \beta_0 - \alpha_0 \cup \beta_4 \in H^4(S, \mathbb{Z}) = \mathbb{Z}.$$

Let now $r, s \geq 1$ be relatively prime integers such that $g = 1 + rs$. For a polarized $K3$ surface $[S, \ell] \in \mathcal{F}_g$ one defines the *Fourier–Mukai dual* $\hat{S} := M_S(r, \ell, s)$, where

$$M_S(r, \ell, s) = \{E : E \text{ is an } \ell-\text{stable sheaf on } S,\ \mathrm{rk}(E) \\ = r,\ c_1(E) = \ell, \chi(S, E) = r + s\}.$$

Setting $v := (r, \ell, s) \in \widetilde{H}(S, \mathbb{Z})$, there is a Hodge isometry, see [19] Theorem 1.4:

$$\psi : H^2(M_S(r, \ell, s), \mathbb{Z}) \xrightarrow{\cong} v^{\perp}/\mathbb{Z}v.$$

We observe that $\hat{\ell} := \psi^{-1}((0, \ell, 2s))$ is a nef primitive vector with $(\hat{\ell})^2 = 2g-2$, and in this way the pair $(\hat{S}, \hat{\ell})$ becomes a polarized $K3$ surface of genus g. The *Fourier–Mukai involution* is the morphism $FM : \mathcal{F}_g \to \mathcal{F}_g$ defined by $FM([S, \ell]) := [\hat{S}, \hat{\ell}]$.

We turn to the case $g = 11$, when we set $r = 2$ and $s = 5$. For a general curve $[C] \in \mathcal{M}_{11}$, the Lagrangian Brill–Noether locus

$$\mathcal{SU}_C(2, K_C, 7) := \{E \in \mathcal{U}_C(2, 20) : \det(E) = K_C,\ h^0(C, E) = 7\}$$

is a smooth $K3$ surface. The main result of [21] can be summarized as saying a general $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface which moreover can be realized as $\widehat{\mathcal{SU}_C(2, K_C}, 7)$. Furthermore, there is a birational isomorphism

$$\phi_{11} : \mathcal{M}_{11} \dashrightarrow \mathcal{P}_{11}, \quad \phi_{11}([C]) := [\widehat{\mathcal{SU}_C(2, K_C}, 7), C]$$

and we set $q_{11} := \pi \circ \phi_{11} : \mathcal{M}_{11} \dashrightarrow \mathcal{F}_{11}$. On the moduli space $\mathcal{M}_{11}$ there exist two distinct irreducible Brill–Noether divisors

$$\mathcal{M}^1_{11,6} := \{[C] \in \mathcal{M}_{11} : W^1_6(C) \neq \emptyset\} \quad \text{and} \quad \mathcal{M}^2_{11,9} := \{[C] \in \mathcal{M}_{11} : W^2_9(C) \neq \emptyset\}.$$

Via the residuation morphism $W^1_6(C) \ni L \mapsto K_C \otimes L^{\vee} \in W^5_{14}(C)$, the Hurwitz divisor is the pull-back of a Noether–Lefschetz divisor on $\mathcal{F}_{11}$, that is,

$\mathcal{M}^1_{11,6} = q^*_{11}(D^1_6)$ where

$$D^1_6 := \{[S,\ell] \in \mathcal{F}_{11} : \exists\, H \in \mathrm{Pic}(S),\ H^2 = 8, H\cdot\ell = 14\}.$$

Similarly, via the residuation map $W^2_9(C) \ni L \mapsto K_C \otimes L^\vee \in W^3_{11}(C)$, one has the equality of divisors $\mathcal{M}^2_{11,9} = q^*_{11}(D^2_9)$, where

$$D^2_9 := \{[S,\ell] \in \mathcal{F}_{11} : \exists\, H \in \mathrm{Pic}(S),\ H^2 = 4, H\cdot\ell = 11\}.$$

Next we establish Mercat's conjecture for general curves of genus 11.

Theorem 5.1. *The equality* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ *holds for a general curve* $[C] \in \mathcal{M}_{11}$.

Proof. We fix a curve $[C] \in \mathcal{M}_{11}$ such that (i) $W^1_7(C)$ is a smooth curve, (ii) $W^2_9(C) = \emptyset$ (in particular, any Petri general curve will satisfy these conditions) and (iii) the rank 2 Brill-Noether locus $\mathcal{SU}_C(2, K_C, 7)$ is a smooth $K3$ surface of Picard number 1. As discussed in both [12, Proposition 4.5; 5, Question 3.5], in order to verify (M_2), it suffices to show that C possesses no bundles $E \in \mathcal{U}_C(2, 13)$ with $h^0(C,E) = 4$. Suppose E is such a vector bundle. Then $L := \det(E) \in W^4_{13}(C)$ is a linear series such that the multiplication map $\nu_2(L) : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$ is not injective. For each extension class

$$e \in \mathbf{P}_L := \mathbf{P}(\mathrm{Coker}\ \nu_2(L))^\vee \subset \mathbf{P}(H^0(C, L^{\otimes 2}))^\vee = \mathbf{P}\mathrm{Ext}^1(L, K_C \otimes L^\vee),$$

one obtains a rank 2 vector bundle F on C sitting in an exact sequence

$$0 \to K_C \otimes L^\vee \to F \to L \to 0, \tag{5.1}$$

such that $h^0(C,F) = h^0(C,L) + h^0(C, K_C \otimes L^\vee) = 7$. We claim that any non-split vector bundle F with $h^0(C,F) = 7$ and which sits in an exact sequence (5.1), is semistable. Indeed, let us assume by contradiction that $M \subset F$ is a destabilizing line subbundle with $\deg(M) \geq 11$. Since $\deg(M) > \deg(K_C \otimes L^\vee)$, the composite morphism $M \to L$ is nonzero, hence we can write that $M = L(-D)$, where D is an effective divisor of degree 1 or 2. Because $W^2_9(C) = \emptyset$, one finds that $h^0(C, K_C \otimes L^\vee(D)) = 2$ and L must be very ample, that is, $h^0(C, L(-D)) = h^0(C,L) - \deg(D)$. We obtain that

$$\begin{aligned} h^0(L) + h^0(K_C \otimes L^\vee) = h^0(F) &\leq h^0(M) + h^0(K_C \otimes M^\vee) \\ &= h^0(L) - \deg(D) + h^0(K_C \otimes L^\vee), \end{aligned}$$

a contradiction. Thus one obtains an induced morphism $u : \mathbf{P}_L \to \mathcal{SU}_C(2, K_C, 7)$. Since $\mathcal{SU}_C(2, K_C, 7)$ is a $K3$ surface, this also implies that Coker $\nu_2(L)$ is two-dimensional, hence $\mathbf{P}_L = \mathbf{P}^1$.

We claim that u is an embedding. Setting $A := K_C \otimes L^\vee \in W^1_7(C)$, we write the exact sequence $0 \to H^0(C, \mathcal{O}_C) \to H^0(C, F^\vee \otimes L) \to H^0(C, K_C \otimes A^{\otimes(-2)})$, and note that the last vector space is the kernel of the Petri map $H^0(C,A) \otimes H^0(C,L) \to H^0(C, K_C)$, which is injective, hence $h^0(C, F^\vee \otimes L) = 1$. This implies that u is an embedding. But this contradicts the fact that Pic $\mathcal{SU}_C(2,K_C,7)=\mathbb{Z}$, in particular $\mathcal{SU}_C(2,K_C,7)$ contains no (-2)-curves. We conclude that $\nu_2(L)$ is injective for *every* $L\in W^4_{13}(C)$. □

This proof also shows that the failure locus of statement (M_2) on $\mathcal{M}_{11}$ is equal to the Koszul divisor

$$\mathfrak{Syz}^4_{11,13}:=\{[C]\in\mathcal{M}_{11}:\exists\ L\in W^4_{13}(C) \text{ such that } K_{1,1}(C,L)\neq 0\}.$$

Suppose now that $[C]\in\mathfrak{Syz}^4_{11,13}$ is a general point corresponding to an embedding $C\overset{|L|}{\hookrightarrow}\mathbf{P}^4$ such that C lies on a $(2,3)$ complete intersection $K3$ surface $S\subset\mathbf{P}^4$. Then $S=\widehat{\mathcal{SU}_C(2,K_C},7)$ and $\rho(S)=2$ and furthermore $\mathrm{Pic}(S)=\mathbb{Z}\cdot C\oplus\mathbb{Z}\cdot H$, where $H^2=6, C\cdot H=13$ and $C^2=20$. In particular we note that S contains no (-2)-curves, hence S and $\hat{S}$ are not isomorphic.

Let us define the Noether–Lefschetz divisor

$$D^4_{13}:=\{[S,\ell]\in\mathcal{F}_{11}:\exists\ H\in\mathrm{Pic}(S),\ H^2=6, H\cdot\ell=13\},$$

therefore $\mathfrak{Syz}^4_{11,13}=q^*_{11}(D^4_{13})$.

Proposition 5.2. *The action of the Fourier–Mukai involution $FM:\mathcal{F}_{11}\to\mathcal{F}_{11}$ on the three distinguished Noether–Lefschetz divisors is described as follows*:

(i) $FM(D^1_6)=D^1_6$.
(ii) $FM(D^2_9)=\{[S,\ell]\in\mathcal{F}_{11}:\exists\ R\in\mathrm{Pic}(S)$ *such that* $R^2=-2,\ R\cdot\ell=1\}$.
(iii) $FM(D^4_{13})=\{[S,\ell]\in\mathcal{F}_{11}:\exists\ R\in\mathrm{Pic}(S)$ *such that* $R^2=-2,\ R\cdot\ell=3\}$.

Proof. For $[S,\ell]\in\mathcal{F}_{11}$, we set $v:=(2,\ell,5)\in\widetilde{H}(S,\mathbb{Z})$ and $\hat{\ell}:=(0,\ell,10)\in\widetilde{H}(S,\mathbb{Z})$ for the class giving the genus 11 polarization. We describe the lattice $\psi(NS(\hat{S}))\subset\widetilde{NS}(S)$.

In the case of a general point of D^1_6 with lattice $NS(S)=\mathbb{Z}\cdot\ell\oplus\mathbb{Z}\cdot H$, by direct calculation we find that $\psi(NS(\hat{S}))$ is generated by the vectors $\hat{\ell}$ and $(2,\ell+H,12)$. Furthermore, $(2,\ell+H,12)^2=8$ and $(2,H+\ell,12)\cdot\hat{\ell}=14$, that is, $\mathrm{Pic}(\hat{S})\cong\mathrm{Pic}(S)$, hence D^1_6 is a fixed divisor for the automorphism FM.

A similar reasoning for a general point of the divisor D^2_9 shows that the Neron–Severi groups $\psi\big(NS(\hat{S})\big)$ is generated by $\hat{\ell}$ and $(-1,H-\ell,-2)$, where $(-1,H-\ell,-2)^2=-2$ and $(-1,H-\ell,-2)\cdot\hat{\ell}=1$. In other words, the class $(-1,H-\ell,-2)$ corresponds to a line in the embedding $\hat{S}\overset{|\hat{\ell}|}{\hookrightarrow}\mathbf{P}^{11}$. Finally, for a general point of D^4_{13} corresponding to a lattice $\mathbb{Z}\cdot\ell\oplus\mathbb{Z}\cdot H$, the Picard lattice of the Fourier–Mukai partner is spanned by the vectors $\hat{\ell}$ and $(-1,H-\ell,-1)$, where $(-1,H-\ell,-1)^2=-2$ and $(-1,H-\ell,-1)\cdot\hat{\ell}=3$. □

Remark 5.3. The fact that the divisor D^1_6 is fixed by the automorphism FM is already observed and proved with geometric methods in [21, Theorem 3].

Remark 5.4. It is instructive to point out the difference between a general element of D^4_{13} and its Fourier–Mukai partner. As a polarized $K3$ surface, $\mathcal{SU}_C(2, K_C, 7)$ is characterized by the existence of a degree 3 rational curve $u(\mathbf{P}_L) \subset \mathcal{SU}_C(2, K_C, 7)$. On the other hand, the complete intersection surface $S \subset \mathbf{P}^4$ containing $C \overset{|L|}{\hookrightarrow} \mathbf{P}^4$, where $L \in W^4_{13}(C)$, carries no smooth rational curves. It contains however elliptic curves in the linear system $|\mathcal{O}_S(C - H)|$. Thus the involution FM assigns to a $K3$ surface with a degree 7 elliptic pencil, a $K3$ surface containing a (-2)-curve. Since $S = \widehat{\mathcal{SU}_C(2, K_C}, 7)$, it also follows that the complete intersection S is a smooth $K3$ surface, which *a priori* is not at all obvious.

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