THE MODULI SPACE OF TWISTED CANONICAL DIVISORS

GAVRIL FARKAS AND RAHUL PANDHARIPANDE

ABSTRACT. The moduli space of canonical divisors (with prescribed zeros and poles) on nonsingular curves is not compact since the curve may degenerate. We define a proper moduli space of twisted canonical divisors in $\overline{\mathcal{M}}_{g,n}$ which includes the space of canonical divisors as an open subset. The theory leads to geometric/combinatorial constraints on the closures of the moduli spaces of canonical divisors.

In case the differentials have at least one pole (the strictly meromorphic case), the moduli spaces of twisted canonical divisors on genus g curves are of pure codimension g in $\overline{\mathcal{M}}_{g,n}$. In addition to the closure of the canonical divisors on nonsingular curves, the moduli spaces have virtual components. In the Appendix, a complete proposal relating the sum of the fundamental classes of all components (with intrinsic multiplicities) to a formula of Pixton is proposed. The result is a precise and explicit conjecture in the tautological ring for the weighted fundamental class of the moduli spaces of twisted canonical divisors.

As a consequence of the conjecture, the classes of the closures of the moduli spaces of canonical divisors on nonsingular curves are determined (both in the holomorphic and meromorphic cases).

CONTENTS

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0. Introduction

0.1. **Zeros and poles.** Let $\mathcal{M}_g$ be the moduli space of nonsingular curves of genus $g \geq 2$. The *Hodge bundle*,

$$\mathbb{E} \to \mathcal{M}_g \, ,$$

has fiber over the moduli point $[C] \in \mathcal{M}_g$ given by the space of *holomorphic differentials* $H^0(C, \omega_C)$. The projectivization

$$\mathcal{H}_g = \mathbb{P}(\mathbb{E}) \to \mathcal{M}_g$$

is a moduli space of *canonical divisors* on nonsingular curves. We may further stratify $\mathcal{H}_g$ by canonical divisors with multiplicities of zeros specified by a partition μ of $2g-2$. Neither $\mathcal{H}_g$ nor the strata with specified zero multiplicities are compact. We describe a natural compactification of the strata associated to a partition μ.

Meromorphic differentials arise naturally in the analysis of the boundary of the spaces of holomorphic differentials. We will consider meromorphic differentials on curves with prescribed zero and pole multiplicities. Let

$$\mu = (m_1, \ldots, m_n) \, , \quad m_i \in \mathbb{Z}$$

satisfy $\sum_{i=1}^n m_i = 2g-2$. The vector μ prescribes the zero multiplicities of a meromorphic differential via the positive parts $m_i > 0$ and the pole multiplicities via the negative parts $m_i < 0$. Parts with $m_i = 0$ are also permitted (and will correspond to points on the curve which are neither zeros nor poles and otherwise unconstrained).

Let g and n be in stable range $2g-2+n > 0$. For a vector μ of length n, we define the closed substack $\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$ by

$$\mathcal{H}_g(\mu) = \left\{ [C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} \, \middle| \, \mathcal{O}_C\Big(\sum_{i=1}^n m_i p_i\Big) = \omega_C \right\} .$$

Consider first the case where all parts of μ are non-negative. Since $\mathcal{H}_g(\mu)$ is the locus of points

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$$

for which the evaluation map

$$H^0(C, \omega_C) \to H^0\big(C, \omega_{C|m_1p_1+\cdots+m_np_n}\big)$$

is not injective, every component of $\mathcal{H}_g(\mu)$ has dimension at least $2g-2+n$ in $\mathcal{M}_{g,n}$ by degeneracy loci considerations [10]. Polishchuk [20] has shown that $\mathcal{H}_g(\mu)$ is a *nonsingular* substack of $\mathcal{M}_{g,n}$ of pure dimension $2g-2+n$. In fact, the arguments of [20] can be extended to the case where the vector μ has negative parts. Hence, in the strictly meromorphic case, $\mathcal{H}_g(\mu)$ is a nonsingular substack of dimension $2g-3+n$ in $\mathcal{M}_{g,n}$.

In the holomorphic case, the spaces $\mathcal{H}_g(\mu)$ have been intensely studied from the point of view of flat surfaces (which leads to natural coordinates and a volume form), see [7]. However, an algebro-geometric study of the strata has been neglected: basic questions concerning the birational geometry and the cohomology classes of the strata of differentials are open.

We define here a proper moduli space of *twisted canonical divisors*

$$\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

which contains $\mathcal{H}(\mu)$ as an open set

$$\mathcal{H}_g(\mu) \subset \widetilde{\mathcal{H}}_g(\mu)\,.$$

The space $\widetilde{\mathcal{H}}_g(\mu)$ will typically be *larger* than the closure

$$\overline{\mathcal{H}}_g(\mu) \subset \widetilde{\mathcal{H}}_g(\mu)\,.$$

We prove every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ supported entirely in the boundary of $\overline{\mathcal{M}}_{g,n}$ has codimension g in $\overline{\mathcal{M}}_{g,n}$.

In the *strictly meromorphic case* where there exists an $m_i < 0$, the moduli space $\widetilde{\mathcal{H}}_g(\mu)$ is of pure codimension g in $\overline{\mathcal{M}}_{g,n}$. The closure

$$\overline{\mathcal{H}}_g(\mu) \subset \widetilde{\mathcal{H}}_g(\mu)$$

is a union of irreducibe components. The components of $\widetilde{\mathcal{H}}_g(\mu)$ which do not lie in $\overline{\mathcal{H}}_g(\mu)$ are called *virtual*. The virtual components play a basic role in the study of canonical divisors.

0.2. **Pixton's formula.** A relation to a beautiful formula of Pixton for an associated cycle class in $R^g(\overline{\mathcal{M}}_{g,n})$ is explored in the Appendix: the contributions of the virtual components of $\widetilde{\mathcal{H}}_g(\mu)$ are required to match Pixton's formula. The result is a precise and explicit conjecture in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ for the sum of the fundamental classes (with intrinsic multiplicities) of all irreducible components of $\widetilde{\mathcal{H}}_g(\mu)$ in the strictly meromorphic case.

In the holomorphic case where all $m_i \geq 0$, the connection to Pixton's formula is more subtle since

$$\widetilde{\mathcal{H}}_g(\mu) \subset \overline{M}_{g,n}$$

is not of pure codimension g. The more refined approach to the moduli of canonical divisors proposed by Janda [15] via relative Gromov-Witten theory and virtual fundamental classes will likely be required to understand the holomorphic case (and to prove the conjecture of the Appendix in the strictly meromorphic case). However, the virtual components clearly also play a role in the holomorphic case.

Remarkably, the fundamental classes of the varieties of closures

$$\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n} \,,$$

in *both* the holomorphic and strictly meromorphic cases, are determined in the Appendix as a consequence of the conjectured link between Pixton's formula and the weighted fundamental class of the moduli space of twisted canonical divisors in the strictly meromorphic case.

0.3. **Twists.** Let $[C, p_1, \ldots, p_n]$ be a Deligne-Mumford stable n-pointed curve.[1] Let

$$\mathsf{N}(C) \subset C$$

be the nodal locus. A node $q \in \mathsf{N}(C)$ is *basic* if q lies in the intersection of two distinct irreducible components of C. Let

$$\mathsf{BN}(C) \subset \mathsf{N}(C)$$

be the set of basic nodes, and let

$$\widetilde{\mathsf{BN}}(C) \to \mathsf{BN}(C)$$

be the canonical double cover defined by

$$\widetilde{\mathsf{BN}}(C) = \big\{ \, (q, D) \,|\, q \in \mathsf{BN}(C) \,, \; q \in D \,, \text{ and } D \subset C \text{ an irreducible component} \big\} \,.$$

A *twist* I of the curve C is a function

$$I : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}$$

satisfying the *balancing*, *vanishing*, *sign*, and *transitivity* conditions defined as follows.

Balancing: *If a basic node q lies in the intersection of distinct irreducible components $D, D' \subset C$, then*

$$I(q, D) + I(q, D') = 0 \,.$$

Let $\mathsf{Irr}(C)$ be the set of irreducible components of C. If

$$D, D' \in \mathsf{Irr}(C)$$

[1]The definition of a twist given here is valid for any connected nodal curve.

and there exists $q \in D \cap D'$ with $I(q, D) = 0$, we write

$$D \approx D' .$$

By the balancing condition, $\approx$ is symmetric. Let $\sim$ be the minimal *equivalence relation* on $\mathsf{Irr}(C)$ generated by $\approx$.

Vanishing: *If $D, D' \in \mathsf{Irr}(C)$ are distinct irreducible components in the same $\sim$-equivalence class and $q \in D \cap D'$, then*

$$I(q, D) = I(q, D') = 0 .$$

Sign: *Let $q \in D \cap D'$ and $\widehat{q} \in \widehat{D} \cap \widehat{D}'$ be basic nodes of C. If $D \sim \widehat{D}$ and $D' \sim \widehat{D}'$, then*

$$I(q, D) > 0 \implies I(\widehat{q}, \widehat{D}) > 0 .$$

Let $\mathsf{Irr}(C)^{\sim}$ be the set of $\sim$-equivalence classes. We define a directed graph $\Gamma_I(C)$ with vertex set $\mathsf{Irr}(C)^{\sim}$ by the following construction. A directed edge

$$v \to v'$$

is placed between equivalences classes $v, v' \in \mathsf{Irr}(C)^{\sim}$ if there exist

$$D \in v\,, \quad D' \in v' \ \text{ and } \ q \in D \cap D' \tag{1}$$

satisfying $I(q, D) > 0$.

The vanishing condition prohibits self-edges at the vertices of $\Gamma_I(C)$. By the sign condition, vertices are *not* connected by directed edges in both directions: there is at most a single directed edge between vertices. The fourth defining condition for a twist I is easily stated in terms of the associated graph $\Gamma_I(C)$.

Transitivity: *The graph $\Gamma_I(C)$ has no directed loops.*

If C is curve of compact type, distinct irreducible components of C intersect in at most one point. The vanishing and sign conditions are therefore trivial. Since $\Gamma_I(C)$ is a tree, the transitivity condition is always satisfied. In the compact type case, only the balancing condition is required for the definition of a twist.

0.4. **Twisted canonical divisors.** Let $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$, and let

$$I : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}$$

be a twist. Let $\mathsf{N}_I \subset \mathsf{BN}(C)$ be the set of basic nodes at which I is non-zero,

$$\mathsf{N}_I = \left\{ q \in \mathsf{BN}(C) \,\middle|\, I(q, D) \neq 0 \ \text{ for } q \in D \right\} .$$

Associated to I is the partial normalization

$$\nu : C_I \to C$$

defined by normalizing exactly the nodes in N_I. The curve C_I may be disconnected.

For a node $q \in \mathsf{N}_I$ in the intersection of distinct components D' and D'' of C, we have $\nu^{-1}(q) = \{q', q''\}$. Let

$$D'_q \subset \nu^{-1}(D') \text{ and } D''_q \subset \nu^{-1}(D'')$$

denote the irreducible components of C_I such that $q' \in D'_q$ and $q'' \in D''_q$. By the definition of ν and the sign condition,

$$D'_q \cap D''_q = \emptyset \text{ in } C_I .$$

Let $\mu = (m_1, \ldots, m_n)$ be a vector satisfying $\sum_{i=1}^n m_i = 2g - 2$. To the stable curve $[C, p_1, \ldots, p_n]$, we associate the Cartier divisor $\sum_{i=1}^n m_i p_i$ on C.

Definition 1. The divisor $\sum_{i=1}^n m_i p_i$ associated to $[C, p_1, \ldots, p_n]$ is *twisted canonical* if there exists a twist I for which

$$\nu^* \mathcal{O}_C \left(\sum_{i=1}^n m_i p_i \right) \stackrel{\sim}{=} \nu^*(\omega_C) \otimes \mathcal{O}_{C_I} \left(\sum_{q \in \mathsf{N}_I} I(q, D'_q) \cdot q' + I(q, D''_q) \cdot q'' \right)$$

on the partial normalization C_I.

We define the subset $\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$ parameterizing twisted canonical divisors by

$$\widetilde{\mathcal{H}}_g(\mu) = \left\{ [C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n} \;\middle|\; \sum_{i=1}^n m_i p_i \text{ is a twisted canonical divisor} \right\} .$$

By definition, we have

$$\widetilde{\mathcal{H}}_g(\mu) \cap \mathcal{M}_{g,n} = \mathcal{H}_g(\mu) ,$$

so $\mathcal{H}_g(\mu) \subset \widetilde{\mathcal{H}}_g(\mu)$ is an open set.

If μ has a part equal to 0, we write $\mu = (\mu', 0)$. Let

$$\tau : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$$

be the map forgetting the 0 part (when permitted by stability). As a straightforward consequence of Definition 1, we obtain

$$\widetilde{\mathcal{H}}_g(\mu) = \tau^{-1}\Big(\widetilde{\mathcal{H}}_g(\mu')\Big) .$$

Theorem 2. *If all parts of μ are non-negative, $\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$ is a closed substack with irreducible components of dimension either $2g-2+n$ or $2g-3+n$. The substack*

$$\overline{\mathcal{H}}_g(\mu) \subset \widetilde{\mathcal{H}}(\mu)$$

is the union of the components of dimension $2g-2+n$.

Theorem 3. *If μ has a negative part, $\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$ is a closed substack of pure dimension $2g-3+n$ which contains*

$$\overline{\mathcal{H}}_g(\mu) \subset \widetilde{\mathcal{H}}(\mu)$$

as a union of components.

Theorems 2 and 3 constrain the closures of the strata of holomorphic and meromorphic differentinals in geometric and combinatorial terms depending solely on the partial normalization C_I of C. By degree considerations on the curve $[C, p_1, \ldots, p_n]$, there are only finitely many twists I which are relevant to determining whether $\sum_{i=1}^n m_i p_i$ is a twisted canonical divisor.

Because of the virtual components in the boundary, Theorems 2 and 3 do not characterize the closure $\overline{\mathcal{H}}_g(\mu)$. Further constraints can easily be found on the closure via residue conditions. Our perspective is not to exclude the virtual components, but rather to include them. In the strictly meromorphic case, the virtual components in the boundary are described in Lemma 13 via star graphs defined in Section 2.2. The virtual components may be seen as a shadow of the virtual fundamental class of an approach to the moduli space of differentials via relative stable maps proposed by Janda [15].

At the end the paper, we discuss the relationship between twisted canonical divisors and spin curves via theta characteristics in case all the parts of μ are positive and even. A basic criterion, proven in Proposition 16, for the smoothability of a twisted canonical divisor in the holomorphic case is used in the discussion of these classical examples.

0.5. **Related work.** There are several approaches to compactifying the spaces of canonical divisors:

- Janda's approach (which has been discussed briefly above) is motivated by relative Gromov-Witten theory and and the proof of Pixton's conjecture for the double ramification cycle [14]. Our moduli of twisted canonical divisors predicts the image of Janda's space under the map to $\overline{\mathcal{M}}_{g,n}$.

Because of the pure dimension result of Theorem 3 in the strictly meromorphic case, the push-forward of the virtual class can also be predicted (see the Appendix).

The purity of dimension in the strictly meromorphic case allows for the approach to Pixton's formula taken in the Appendix based entirely on classical geometry. The development is unexpected. For example, both the definition and the calculation of the double ramification cycle in [14] make essential use of obstruction theories and virtual fundamental classes.

- Sauvaget and Zvonkine [21] propose a different approach to the compactification of canonical divisors which stays closer to the projectivization of the Hodge bundle
$$\mathbb{P}(\mathbb{E}) \to \overline{\mathcal{M}}_g\,.$$
An advantage is the existence of the tautological line $\mathcal{O}(1)$ of the projectivization which is hoped eventually to provide a link to the volume calculations of [7, 8]. There should also be connections between the recursions for fundamental classes found by Sauvaget and Zvonkine and the conjecture of the Appendix.

- Twisting related (but not equal) to Definition 1 was studied by Gendron [11]. Chen [3] considered twists in the compact type case motivated by the study of limit linear series. In further work of Bainbridge, Chen, and Grushevsky [1], the authors study twists and residue conditions for arbitrary stable curves via analytic methods with the goal of characterizing the closure $\overline{\mathcal{H}}_g(\mu)$.

For the study of the moduli of canonical divisors from the point of complex dynamics, we refer the reader to [7].

Our paper takes a more naive perspective than the work discussed above. We propose a precise definition of a twisted canonical divisor and take the full associated moduli space seriously as a mathematical object.

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1. Twists of degenerating canonical bundles

1.1. **Valuative criterion.** Let $\mu = (m_1, \ldots, m_n)$ be a vector of zero and pole multiplicities satisfying

$$\sum_{i=1}^{n} m_i = 2g - 2\,.$$

By Definition 1,

$$\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

is easily seen to be a construcible subset.

Proposition 4. *The locus $\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$ is Zariski closed.*

To prove Proposition 4, we will use the valuative criterion. Consider a 1-parameter family of stable n-pointed curves over a disk Δ,

$$\pi : \mathcal{C} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \mathcal{C}\,,$$

where the punctured disk $\Delta^\star = \Delta \setminus 0$ maps to $\widetilde{\mathcal{H}}_g(\mu)$. The sections p_i correspond to the markings. We must show the special fiber over $0 \in \Delta$,

$$[C_0, p_1, \ldots, p_n]\,,$$

also lies in $\widetilde{\mathcal{H}}_g(\mu)$.

After possibly shrinking Δ, the topological type of the fibers over $\Delta^\star$ may be assumed to be constant. After base change, we may assume there is no monodromy in the components of the fibers over $\Delta^\star$. Finally, after further shrinking, we may assume the twist I guaranteed by Definition 1 for each fiber over $\Delta^\star$ is the *same*.

Since the topological type and the twist I is the same over $\Delta^\star$, the structures

$$\mathrm{Irr}(C_\zeta)\,, \quad \mathrm{Irr}(C_\zeta)^\sim\,, \quad \Gamma_I(C_\zeta)$$

do *not* vary as the fiber C_ζ varies over $\zeta \in \Delta^\star$.

The basic nodes of the special fiber C_0 are of two types: basic nodes smoothed by the family π and basic nodes *not* smoothed by the family. The unsmoothed basic nodes correspond to basic nodes of $C_{\zeta\neq 0}$, so the twist I already assigns integers to the unsmoothed basic nodes of C_0. However, we must assign twists to the basic nodes of C_0 which are smoothed by π.

For $\zeta \in \Delta^\star$, consider a vertex $v \in \Gamma_I(C_\zeta)$ which corresponds (by taking the union of the irreducible components in $\sim$-equivalence class v) to a connected subcurve C^v_ζ. As ζ varies, an associated family

$$\mathcal{C}^v \to \Delta^\star$$

is defined with closure in $\mathcal{C}$ given by

$$\pi^v : \overline{\mathcal{C}}^v \to \Delta\,.$$

The markings which lie on $\overline{\mathcal{C}}^v$ yield sections

$$p_{1_v}, \ldots, p_{x_v} : \Delta \to \overline{\mathcal{C}}^v\,.$$

The nodes connecting C^v_ζ to the complement in C_ζ yield further sections

$$q_{1_v}, \ldots, q_{y_v} : \Delta \to \overline{\mathcal{C}}^v$$

at which the twist I is *not* zero.

By the definition of a twisted canonical divisor, we have

$$\omega_{C^v_\zeta} \cong \mathcal{O}_{C^v_\zeta}\Big(\sum_{i=1}^{x} m_{i_v} p_{i_v} - \sum_{j=1}^{y} (I(q_{j_v}, v)+1)\, q_{j,v}\Big)\,.$$

Hence, the curve

$$[C^v_\zeta, p_{1_v}, \ldots, p_{x_v}, q_{1_v}, \ldots, q_{y_v}]$$

is a twisted canonical divisor for the vector

$$(m_{1_v}, \ldots, m_{x_v}, -I(q_{1_v}, v)-1, \ldots, -I(q_{y_v}, v)-1)\,.$$

In Section 1.2 below, we will show the fiber of

$$\pi^v : \overline{\mathcal{C}}^v \to \Delta\,, \qquad p_{1_v}, \ldots, p_{x_v}, q_{1_v}, \ldots, q_{y_v} : \Delta \to \mathcal{C}\,,$$

over $0 \in \Delta$ is a twisted canonical divisor (with nonzero twists only at basic nodes of $\overline{C}^v_0$ which are smoothed by the family π^v).

By considering all the vertices $v \in \Gamma_I(C_{\zeta\neq 0})$, we define twists at all basic nodes of the special fiber C_0 which are smoothed by π. We now have defined twists at *all* basic nodes of C_0,

$$I_0 : \widetilde{\mathsf{BN}}(C_0) \to \mathbb{Z}\,.$$

Via these twists, we easily see

$$[C_0, p_1, \ldots, p_n]$$

is a twisted canonical divisor. Checking the balancing, vanishing, sign, and transitivity conditions is straightforward:

- Balancing holds for I_0 by construction.
- The vanishing, sign, and transitivity for I_0 are all implied by the respective conditions for I and for the twists constructed on $\overline{\mathcal{C}}^v$.
- The required isomorphism of line bundles on the partial normalization

  $$\nu : C_{0,I_0} \to C_0$$

  determined by I_0 is a consequence of corresponding isomorphisms on the partial normalizations of $\overline{\mathcal{C}}^v_0$ determined by the twists.

1.2. **Generically untwisted families.** In Section 1.1, we have reduced the analysis of the valuative criterion to the generically untwisted case.

Let $\mu = (m_1, \ldots, m_n)$ be a vector of zero and pole multiplicities satisfying $\sum_{i=1}^n m_i = 2g-2$. Let

$$\pi : \mathcal{C} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \mathcal{C}$$

be a 1-parameter family of stable n-pointed curves for which we have an isomorphism

$$\omega_{C_\zeta} \stackrel{\sim}{=} \mathcal{O}_{C_\zeta}\Big(\sum_{i=1}^n m_i p_i\Big) \tag{2}$$

for all $\zeta \in \Delta^\star$. We will show the special fiber over $0 \in \Delta$,

$$[C_0, p_1, \ldots, p_n]\,,$$

is a twisted canonical divisor with respect to μ via a twist

$$I_0 : \widetilde{\mathsf{BN}}(C_0) \to \mathbb{Z}\,.$$

supported only on the basic nodes of C_0 which are smoothed by the family π.

The total space of the family $\mathcal{C}$ has (at worst) A_r-singularities at the nodes of C_0 which are smoothed by π. Let

$$\widetilde{\mathcal{C}} \xrightarrow{\epsilon} \mathcal{C} \xrightarrow{\pi} \Delta$$

be the crepant resolution (via chains of (-2)-curves) of all singularities occuring at the non-smoothed nodes of C_0. The line bundles

$$\epsilon^*\omega_\pi \quad \text{and} \quad \epsilon^*\mathcal{O}_{\mathcal{C}}\Big(\sum_{i=1}^m m_i p_i\Big) \tag{3}$$

on $\widetilde{\mathcal{C}}$ are isomorphic over $\Delta^\star$ by (2). Therefore, the bundles (3) differ by a Cartier divisor $\mathcal{O}_{\widetilde{\mathcal{C}}}(T)$ on $\widetilde{\mathcal{C}}$ satisfying the following properties:

(i) $\mathcal{O}_{\widetilde{\mathcal{C}}}(T)$ is supported over $0 \in \Delta$,

(ii) $\mathcal{O}_{\widetilde{\mathcal{C}}}(T)$ restricts to the trivial line bundle on every exceptional (-2)-curve of the resolution ϵ.

By property (i), the Cartier divisor $\mathcal{O}_{\widetilde{\mathcal{C}}}(T)$ must be a sum of irreducible components of the fiber $\widetilde{C}_0$ of $\widetilde{\mathcal{C}}$ over $0 \in \Delta$, that is,

$$T = \sum_{D \in \mathsf{Irr}(\widetilde{C}_0)} \gamma_D \cdot D\,, \quad \text{with } \gamma_D \in \mathbb{Z}.$$

The irreducible components of $\widetilde{C}_0$ correspond[2] to the irreducible components of C_0 together with all the exceptional (-2)-curves,

$$\mathsf{Irr}(\widetilde{C}_0) \ = \ \mathsf{Irr}(C_0) \cup \{E_i\}\ .$$

The Cartier property implies that

$$\gamma_D = \gamma_{D'}$$

for distinct components $D, D' \in \mathsf{Irr}(C_0)$ which intersect in at least 1 node of C_0 which is *not* smoothed by π.

Let $E_1 \cup \ldots \cup E_r \subset \widetilde{C}_0$ be the full exceptional chain of (-2)-curves for the resolution of an A_r-singularity of $\mathcal{C}$ corresponding to a node $q \in C_0$ which is not smoothed by π. We have the following data:

- E_1 intersects the irreducible component $D \subset \widetilde{C}_0$,
- E_r intersects the irreducible component $D' \subset \widetilde{C}_0$,

see Figure 1. By property (ii), we find r equations obtained from the triviality of

[2] The irreducible components of C_0 maybe be partially normalized in $\widetilde{C}_0$. We denote the strict transform of $D \in \mathsf{Irr}(C_0)$ also by $D \in \mathsf{Irr}(\widetilde{C}_0)$.

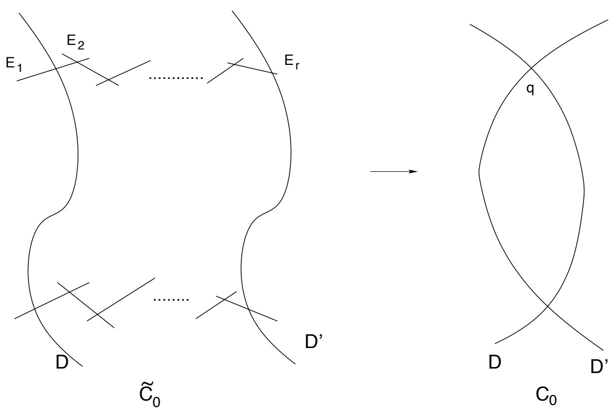


FIGURE 1. The curve $\widetilde{C}_0$

$\mathcal{O}_{\widetilde{\mathcal{C}}}(T)$ on each subcurve E_i:

(4) $$2\gamma_{E_1} = \gamma_D + \gamma_{E_2}\,,\ \ \ldots\,,\ \ 2\gamma_{E_i} = \gamma_{E_{i-1}} + \gamma_{E_{i+1}}\,,\ \ \ldots\,,\ \ 2\gamma_{E_r} = \gamma_{E_{r-1}} + \gamma_{D'}\,.$$

The equations (4) are uniquely solvable[3] in the variables $\gamma_{E_1}, \ldots, \gamma_{E_r}$ in terms of γ_D and $\gamma_{D'}$. If $\gamma_D = \gamma_{D'}$, then the unique solution is

(5) $$\gamma_{E_i} = \gamma_D \quad \text{for all } i.$$

The solution (5) is always the case when q is a non-basic node (since then $D = D'$).

If q is a basic node and $\gamma_D \neq \gamma_{D'}$, then equations (4) imply the values of γ_{E_i} are uniformly spaced and lie strictly between γ_D and $\gamma_{D'}$. For example if γ_D=3 and $\gamma_{D'} = 9$ and $r = 2$, we have

$$3 \,<\, \gamma_{E_1} = 5 \,<\, \gamma_{E_2} = 7 \,<\, 9\,.$$

The equations imply

(6) $$-\gamma_D + \gamma_{E_1} = -\gamma_{E_r} + \gamma_{D'}\,.$$

If $q \in \mathsf{BN}(C_0)$ is basic node, then we assign the twists

$$I_0(q, D) = -\gamma_D + \gamma_{E_1}\,,\quad I_0(q, D') = -\gamma_{D'} + \gamma_{E_r}\,.$$

If there are no (-2)-curves over $q \in C_0$, then

$$I_0(q, D) = -\gamma_D + \gamma_{D'}\,,\quad I_0(q, D') = -\gamma_{D'} + \gamma_D\,.$$

Lemma 5. *The assignment* $I_0 : \mathsf{BN}(C_0) \to \mathbb{Z}$ *defined above satisfies the balancing, vanishing, sign, and transitivity conditions.*

[3]The unique solution yields $\gamma_{E_i} \in \mathbb{Q}$. For solution $\gamma_{E_i} \in \mathbb{Z}$, $r+1$ must divide $-\gamma_D + \gamma_{D'}$.

Proof. As usual, the balancing condition holds by construction (6). We index the components $D \in \mathsf{Irr}(C_0)$ by the integer γ_D,

$$\gamma : \mathsf{Irr}(C_0) \to \mathbb{Z}\,, \quad D \mapsto \gamma_D.$$

The equivalence relation $\sim$ defined by I_0 is easily understood: the equivalence classes are exactly maximal connected subcurves of C_0 for which the value of γ is constant. The basic nodes of C_0 lying entirely within such a subcurve have twist 0 by (5). The basic nodes of C_0 connecting different equivalence classes have non-zero twist by the uniform spacing property of the solutions to (4). The vanishing condition is therefore established. The sign condition follows again from the uniform spacing property. A directed edge in $\mathsf{Irr}(C_0)^\sim$ points in the direction of higher γ values, so transitivity is immediate. □

Finally, in order to show that the special fiber of π,

$$[C_0, p_1, \ldots, p_n]\,,$$

is a twisted canonical divisor (with respect to μ and I_0), we must check the isomorphism

$$\nu^*\mathcal{O}_{C_0}\left(\sum_{i=1}^n m_i p_i\right) \stackrel{\sim}{=} \nu^*(\omega_{C_0}) \otimes \mathcal{O}_{C_{0,I_0}}\left(\sum_{q\in \mathsf{N}_I} I(q, D'_q)\cdot q' + I(q,D''_q)\cdot q''\right) \tag{7}$$

on the partial normalization

$$C_{0,I_0} \to C_0\,.$$

The isomorphism (7) follows directly from the isomorphism

$$\epsilon^*\mathcal{O}_{\mathcal{C}}\Big(\sum_{i=1}^m m_i p_i\Big) \stackrel{\sim}{=} \epsilon^*\omega_\pi \otimes \mathcal{O}_{\widetilde{\mathcal{C}}}(T)$$

on $\widetilde{C}$. The proof of Proposition 4 is complete. □

1.3. **Smoothings.** Let $\mu = (m_1, \ldots, m_n)$ be a vector of zero and pole multiplicities satisfying $\sum_{i=1}^n m_i = 2g-2$. Let

$$[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}\,,$$

be a twisted canonical divisor via a twist

$$I : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}\,.$$

Lemma 6. *There exists a 1-parameter family*

$$\pi : \mathcal{C} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \mathcal{C}$$

of stable n-pointed curves and a line bundle

$$\mathcal{L} \to \mathcal{C}$$

satisfying the following properties:

(i) *There is an isomorphism of n-pointed curves*

$$\pi^{-1}(0) \stackrel{\sim}{=} [C, p_1, \ldots, p_n]$$

under which

$$\mathcal{L}_0 \stackrel{\sim}{=} \mathcal{O}_C\left(\sum_{i=1}^{n} m_i p_i\right)\,.$$

(ii) *The generic fiber*

$$C_\zeta = \pi^{-1}(\zeta)\,, \quad \zeta \in \Delta^\star$$

is a nonsingular curve and

$$\mathcal{L}_\zeta \stackrel{\sim}{=} \omega_{C_\zeta}\,.$$

Lemma 6 shows the *line bundle* associated to a twisted canonical divisor can always be smoothed to a canonical line bundle on a nonsingular curve.

Proof. The twist I determines connected subcurves $C_v \subset C$ associated to equivalence classes $v \in \mathsf{Irr}(C)^\sim$. Let

$$\pi : \mathcal{C} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \mathcal{C}$$

be a smoothing of the special fiber $[C, p_1, \ldots, p_n]$. We can construct a line bundle

$$\mathcal{L} \stackrel{\sim}{=} \omega_\pi\left(\sum_{v \in \mathsf{Irr}(C)^\sim} \gamma_v \cdot [C_v]\right)$$

for integers γ_v. The restriction of $\mathcal{L}$ determines a twist

$$I^\gamma : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}$$

by the following rule: if $q \in C_v \cap C_w$, then

$$I^\gamma(q, C_v) = -\gamma_v + \gamma_w\,.$$

In all other cases, I^γ vanishes. An immediate issue is whether

$$\gamma : \mathsf{Irr}(C)^\sim \to \mathbb{Z} \quad v \mapsto \gamma_v$$

can be chosen so

$$I^\gamma = I \ . \tag{8}$$

Unfortunately, equality (8) may be impossible to satisfy. A simple obstruction is the following: if

$$q, q' \in \mathsf{BN}(C)$$

both lie in the intersection of the subcurves C_v and C_w, then

$$I^\gamma(q, C_v) = I^\gamma(q', C_w) \ . \tag{9}$$

So if $I(q, C_v) \neq I(q', C_w)$, then $I^\gamma \neq I$ for all γ. In order to satisfy (8), we will destabilize the special fiber C by adding chains of rational components at the nodes of $\mathsf{BN}(C)$ at which I is supported.

Let $\Gamma_I(C)$ be the directed graph associated to I with vertex set $\mathsf{Irr}(C)^\sim$. For $v \in \mathsf{Irr}(C)^\sim$, we define depth$(v)$ to be the length of the longest chain of directed edges which ends in v. By the transitivity condition for the twist I, depth(v) is finite and non-negative.[4] Let

$$M_I = \prod_{(q,D) \in \widetilde{\mathsf{BN}}(C),\, I(q,D)>0} I(q, D) \ .$$

We define $\gamma_v \in \mathbb{Z}$ by

$$\gamma_v = \text{depth}(v) \cdot M_I \ .$$

Let $q \in \mathsf{BN}(C)$ be a node lying in the intersection

$$q \in C_v \cap C_w \quad \text{with} \quad I(q, C_v) > 0 \, .$$

Since depth$(v) <$ depth(w) and

$$M_I \,|\, -\gamma_v + \gamma_w \quad \text{we have} \quad I(q, C_v) \,|\, -\gamma_v + \gamma_w \ .$$

We add a chain of

$$\frac{-\gamma_v + \gamma_w}{I(q, C_v)} - 1$$

destabilizing rational curve at each such node q. For each rational curve P_i^q in the chain

$$P_1^q \cup \ldots \cup P^q_{\frac{-\gamma_v+\gamma_w}{I(q,C_v)}-1} \, ,$$

we define $\gamma_{P_i^q} = \gamma_v + i \cdot I(q, C_v)$.

The result is a new curve $\widetilde{C}$ with a map

$$\widetilde{C} \to C$$

[4] The depth of v may be 0.

contracting the added chains. Moreover, for a 1-parameter family

$$\widetilde{\pi} : \widetilde{\mathcal{C}} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \widetilde{\mathcal{C}} \tag{10}$$

smoothing the special fiber $[\widetilde{C}, p_1, \ldots, p_n]$, we construct a line bundle[5]

$$\widetilde{\mathcal{L}} \cong \omega_{\widetilde{\pi}} \left(\sum_{v \in \mathsf{Irr}(C)^{\sim}} \gamma_v \cdot [C_v] + \sum_{q \in \mathsf{BN}(C)} \sum_{i=1}^{\frac{\gamma_w - \gamma_v}{I(q,C_v)} - 1} \gamma_{P_i^q} \cdot [P_i^q] \right) .$$

The line bundle $\widetilde{\mathcal{L}}$ satisfies several properties:

(i) $\widetilde{\mathcal{L}}|_{P_i^q}$ is trivial,

(ii) for every $v \in \mathsf{Irr}(C)^{\sim}$,

$$\mathcal{O}_{C_v}\left(\sum_{i=1}^n m_i p_i\right) \cong \omega_C|_{C_v} \otimes \mathcal{O}_{C_v}\left(\sum_{q \in \mathsf{N}_I} I(q, C_v) \cdot q\right)$$

(iii) for $\zeta \in \Delta^\star$, $\widetilde{\mathcal{L}}_\zeta \cong \omega_{\widetilde{C}_\zeta}$.

We can contract the extra rational components P_i^q in the special fiber of $\widetilde{\mathcal{C}}$ to obtain 1-parameter family of stable n-pointed curves

$$\pi : \mathcal{C} \to \Delta\,, \qquad p_1, \ldots, p_n : \Delta \to \mathcal{C}$$

which smooths $[C, p_1, \ldots, p_n]$. By (i), the line line bundle $\widetilde{\mathcal{L}}$ descends to

$$\mathcal{L} \to \mathcal{C}\ .$$

By (ii), for every $v \in \mathsf{Irr}(C)^{\sim}$,

$$\mathcal{L}|_{C_v} \cong \mathcal{O}_{C_v}\left(\sum_{p_i \in C_v} m_i p_i\right) . \tag{11}$$

The isomorphisms (11) after restriction to the subcurves do *not* quite imply the required isomorphism

$$\mathcal{L}_0 = \mathcal{L}|_C \cong \mathcal{O}_C\left(\sum_{i=1}^n m_i p_i\right) \tag{12}$$

because there are additional $h^1(\Gamma_I)$ factors of $\mathbb{C}^*$ in the Picard variety of C. However, these $\mathbb{C}^*$-factors for

$$\mathcal{L}|_C \quad \text{and} \quad \mathcal{O}_{C_v}\left(\sum_{i=1}^n m_i p_i\right)$$

[5]The subcurve $C_v \subset C$ corresponds to a subcurve $C_v \subset \widetilde{C}$ by strict transformation.

can be matched by correctly choosing the smoothing parameters at the the nodes of $\widetilde{C}$ in the original family (10). $\square$

2. Dimension estimates

2.1. **Estimates from above.** Let $\mu = (m_1, \ldots, m_n)$ be a vector[6] of zero and pole multiplicities satisfying $\sum_{i=1}^n m_i = 2g-2$. The *boundary*

$$\partial\overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$$

is the divisor parameterizing stable n-pointed curves with at least one node. We estimate from above the dimension of irreducible components of $\widetilde{\mathcal{H}}_g(\mu)$ supported in the boundary.

Proposition 7. *Every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ supported entirely in the boundary of $\overline{\mathcal{M}}_{g,n}$ has dimension at most $2g-3+n$.*

Proof. Let $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ be an irreducible component supported entirely in $\partial\overline{\mathcal{M}}_{g,n}$. Let Γ_Z be the dual graph of C for the generic element

$$[C, p_1, \ldots, p_n] \in Z\,.$$

The dual graph[7] consists of vertices (with genus assignment), edges, and legs (corresponding to the markings):

$$\Gamma_Z = (\mathsf{V},\ \mathsf{E}\,, \mathsf{L}\,, \mathrm{g}: \mathsf{V} \to \mathbb{Z}_{\geq 0})\,.$$

Each vertex $v \in \mathsf{V}$ corresponds to an irreducible component of C. The valence $\mathrm{n}(v)$ counts both half-edges and legs incident to v. The genus formula is

$$g - 1 = \sum_{v \in \mathsf{V}} (\mathrm{g}(v) - 1) + |\mathsf{E}|\,.$$

Since Z is supported in the boundary, Γ_Z must have at least one edge.

We estimate from above the dimension of the moduli of twisted canonical divisors in $\overline{\mathcal{M}}_{g,n}$ with dual graph exactly Γ_Z. Let $v \in \mathsf{V}$ be a vertex corresponding to the moduli space $\mathcal{M}_{\mathrm{g}(v),\mathrm{n}(v)}$. The dimension of the moduli of the canonical divisors *on the the component corresponding to* v is bounded from above by

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v)\,. \tag{13}$$

[6]For the dimension estimates here, we consider all μ. No assumptions on the parts are made.

[7]A refined discussion of dual graphs via half-edges is required in the Appendix. Here, a simpler treatment is sufficient.

Here, we have used the dimension formula for the locus of canonical divisors with prescribed zero multiplicities [20]. The strictly meromorphic case has lower dimension (see Section 0.1).

Summing over the vertices yields a dimension bound for Z:

$$\begin{aligned} \dim Z &\leq \sum_{v\in\mathsf{V}} 2\mathrm{g}(v)-2+\mathrm{n}(v) \\ &= 2\sum_{v\in\mathsf{V}}(\mathrm{g}(v)-1)+2|\mathsf{E}|+n \\ &= 2g-2+n\,. \end{aligned} \tag{14}$$

The result falls short of the claim of the Proposition by 1.

We improve the bound (14) by the following observation. The dual graph Γ_Z must have at least one edge e since Z is supported in the boundary:

- If the edge e corresponds to a non-basic node, then all vertices incident to e are allowed a pole at the associated nodal point.
- If the edge e corresponds to a basic node, there must be at least one vertex $v\in\mathsf{V}$ incident to e which carries meromorphic differentials.

In either case, we can apply the stronger bound

$$2\mathrm{g}(v)-3+\mathrm{n}(v)\,. \tag{15}$$

Since we are guaranteed at least one application of (15), we conclude

$$\dim Z \leq 2g-3+n$$

which is better than (14) by 1. □

2.2. **Star graphs.** Let $Z\subset\widetilde{\mathcal{H}}_g(\mu)$ be an irreducible component supported entirely in $\partial\overline{\mathcal{M}}_{g,n}$. By the proof of Proposition 7, if Γ_Z has for every twist I at least *two* vertices corresponding to meromorphic differentials, then

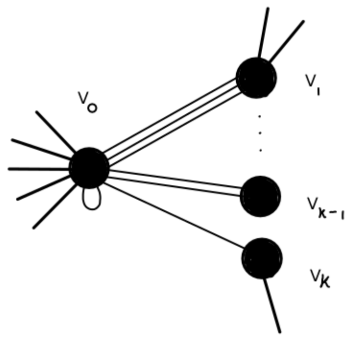
$$\dim Z \leq 2g-4+n\,.$$

The dual graphs Γ_Z with a least one edge (since Z is supported in the boundary) and carrying a twist I with only one vertex corresponding to meromorphic differentials are easy to classify. The dual graph Γ_Z must be a *star*:

- The vertices $\{v_0,v_1,\ldots,v_k\}$ of Γ_Z consist of a *center* vertex v_0 and *outlying* vertices $\{v_1,\ldots,v_k\}$.
- The edges of Γ_Z are of two types:
  - (i) self-edges at v_0,

(ii) edges (possibly multiple) connecting the center v_0 to the outlying vertices.

- The parts[8] of μ are distributed to the vertices with *all* negative parts distributed to the center v_0.



A twist I which orients every edge of type (ii) as outgoing from v_0 yields meromorphic differentials at v_0 only. The self-edges at v_0 correspond to non-basic nodes and can not be twisted.

2.3. **Twisted canonical bundles.** We have defined a twisted canonical *divisor* in Section 0.4. A twisted canonical divisor determines a section (up to scale) of a twisted canonical *bundle* on C.

Definition 8. A line bundle L on a connected nodal curve C is *twisted canonical* if there exists a twist

$$I : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}$$

for which there is an isomorphism

$$\nu^* L \;\stackrel{\sim}{=}\; \nu^*(\omega_C) \otimes \mathcal{O}_{C_I}\left(\sum_{q \in \mathsf{N}_I} I(q, D'_q) \cdot q' + I(q, D''_q) \cdot q'' \right)$$

on the partial normalization $\mu : C_I \to C$.

The limits of twisted canonical line bundles are also twisted canonical. The proof has already been given as a step in the proof of the properness of $\widetilde{\mathcal{H}}_g(\mu)$ in Section 1. We state the result as a Lemma.

Lemma 9. *Let $\pi : \mathcal{C} \to \Delta$ be a flat family of connected nodal curves, and let*

$$\mathcal{L} \to \mathcal{C}$$

[8]The parts of μ correspond to the markings.

be a line bundle. If $L_\zeta \to C_\zeta$ is a twisted canonical line bundle for all $\zeta \in \Delta^\star$, then

$$L_0 \to C_0$$

is also a twisted canonical line bundle.

The definition of a twisted canonical line bundle does *not* specify a twist. Only the existence of a twist is required. However, there are only finitely many possible twists.

Lemma 10. *If $L \to C$ is a twisted canonical line bundle, there are only* finitely *many twists*

$$I : \widetilde{\mathsf{BN}}(C) \to \mathbb{Z}$$

for which there exists an isomorphism

$$\nu^* L \stackrel{\sim}{=} \nu^*(\omega_C) \otimes \mathcal{O}_{C_I}\left(\sum_{q \in \mathsf{N}_I} I(q, D'_q) \cdot q' + I(q, D''_q) \cdot q'' \right)$$

on the partial normalization $\mu : C_I \to C$.

Proof. Let I be a twist for which the above isomorphism holds. By definition, I must satisfy the balancing, vanishing, sign, and transitivity conditions of Section 0.3. Since the are no directed loops in the graph $\Gamma_I(C)$, there must be a vertex v with only outgoing arrows. Let

$$C_v \subset C$$

be the associated subcurve. The twist I is always positive on the nodes N_I incident to the subcurve C_v (on the side of C_v). Since the degrees of $\nu^*(L)|_{C_v}$ and $\nu^*(\omega_C)|_{C_v}$ are determined, we obtain a bound on the twists of I on the nodes N_I incident to C_v: only finitely many values for I on these nodes are permitted.

Next, we find a vertex $v' \in \Gamma_I(C)$ which has only only outging arrows (except from possibly v). Repeating the above argument easily yields the required finiteness statement for I. □

2.4. **Estimates from below.** Let $\mu = (m_1, \ldots, m_n)$ be a vector of zero and pole multiplicities satisfying $\sum_{i=1}^n m_i = 2g - 2$. We now prove a lower bound on the dimension of irreducible components of $\widetilde{\mathcal{H}}_g(\mu)$.

Proposition 11. *Every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ has dimension at least $2g-3+n$.*

Proof. Let $[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(\mu)$ be a stable n-pointed curve. Let

$$L \stackrel{\sim}{=} \mathcal{O}_C\left(\sum_{i=1}^n m_i p_i\right) \to C$$

be the associated twisted canonical bundle. Let

$$m_1, \dots, m_k$$

be the negative parts of μ (if there are any). Consider the k-pointed nodal curve

$$[C, p_1, \dots, p_k]$$

obtained by dropping[9] the markings $p_{k+1}, \dots, p_n$. Let r_1 and r_2 denote the number of rational components of $[C, p_1, \dots, p_k]$ with exactly 1 and 2 special points respectively. To kill the automorphisms of these unstable component, we add new markings

$$q_1, \dots, q_{2r_1+r_2}$$

to the curve C (away from $p_1, \dots, p_k$ and away from the nodes):

- we add two $q's$ on each components with 1 special point,
- we add one q on each components with 2 special points.

The result

$$[C, p_1, \dots, p_k, q_1, \dots, q_{2r_1+r_2}] \tag{16}$$

is a stable pointed curve.

Let $\mathcal{V}$ be the nonsingular versal deformation space of the $k+2r_1+r_2$-pointed curve (16),

$$\dim \mathcal{V} = \dim \mathrm{Def}([C, p_1 \dots, p_k, q_1, \dots, q_{2r_1+r_2}]) = 3g - 3 + k + 2r_1 + r_2.$$

There is a universal curve[10]

$$\pi : \mathcal{C} \to \mathcal{V}\,, \quad p_1, \dots, p_k : \mathcal{V} \to \mathcal{C}\,.$$

We consider the relative moduli space[11] of degree $2g-2$ line bundles on the fibers of π,

$$\epsilon : \mathcal{B} \to \mathcal{V}\,.$$

Let $\mathcal{V}^\star \subset \mathcal{V}$ be the locus of nonsingular curves in the versal deformation space, and let

$$\mathcal{B}^\star \to \mathcal{V}^\star$$

the relative Jacobian of degree $2g-2$. Let

$$\mathcal{W}^\star \subset \mathcal{B}^\star$$

[9]We do *not* contract unstable components.

[10]We will not use the q-sections.

[11]The Quot scheme parameterization of line bundles can be used to avoid the separation issues of the moduli of line bundles. The dimension calculus is parallel.

be the codimension g locus in the universal Jacobian defined fibrewise by the relative canonical bundle ω_π. Let $\mathcal{W}$ be the closure of $\mathcal{W}_0$ in $\mathcal{B}$. By Lemma 9, every point of $\mathcal{W}$ parameterizes a twisted canonical bundle. By Lemmas 6 and 9, the line bundle L lies in the closure $\mathcal{W}$ over the special fiber $[C] \in \mathcal{V}$. The dimension of $\mathcal{W}$ is

$$\dim \mathcal{W} \,=\, \dim \mathcal{B}^\star - g \,=\, 2g-3+k+2r_1+r_2\,.$$

Let $[C', L'] \in \mathcal{W}$ be a pair where $[C'] \in \mathcal{V}$ and

$$L' \to C'$$

is a twisted canonical bundle. A *good section* of $L'(-\sum_{i=1}^k m_i p_i)$ on C' is section s satisfying the following properties:

- s does *not* vanish at $p_1, \ldots, p_k$,
- s does *not* vanish at any node of C',
- s does *not* vanish identically on any irreducible component of C',
- the points $p_1, \ldots, p_k$ and $\mathrm{Div}(s)$ together stabilize C'.

Good sections are open in the space of sections of $[C', L']$. The zeros of a good section define a twisted canonical divisor.

Since we started with a twisted canonical divisor, $[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(\mu)$. We have a good section

$$L\left(-\sum_{i=1}^k m_i p_i\right) \,\widetilde{=}\, \mathcal{O}_C\left(\sum_{i=k+1}^n m_i p_i\right)\,. \tag{17}$$

associated to the pair $[C, L]$.

We now can estimate the dimension of the space of good sections of

$$L'\left(-\sum_{i=1}^k m_i p_i\right)$$

as $[C', L']$ varies in $\mathcal{W}$ near $[C, L]$ using Proposition 14 of Section 4.1. The local dimension of the space of sections near is at least

$$\begin{aligned} \dim \mathcal{W} + \chi(C, L) \;&=\; 2g-3+k+2r_1+r_2+g-1+\sum_{i=1}^k m_i \\ &=\; 3g-4+k+2r_1+r_2+\sum_{i=1}^k m_i\,. \end{aligned}$$

Hence, dimension of the space $\mathcal{T}$ of twisted canonical divisors on the fibers of

$$\pi : \mathcal{C} \to \mathcal{W}$$

near $[C, p_1, \ldots, p_n]$ is *at least* $3g - 5 + k + 2r_1 + r_2 + \sum_{i=1}^{k} m_i$.

We must further impose conditions on the twisted canonical divisors parameterized by $\mathcal{T}$ to obtain the shape $\sum_{i=k+1}^{n} m_i p_i$ for the positive part. These conditions impose at most

$$2g - 2 + \sum_{i=1}^{k} m_i - (n-k) = 2g - 2 + k - n + \sum_{i=1}^{k} m_i$$

constraints. Hence, we conclude the dimension of space $\mathcal{T}(\mu)$ of twisted canonical divisors on the fibers of

$$\pi : \mathcal{C} \to \mathcal{W} \tag{18}$$

near $[C, p_1, \ldots, p_n]$ of shape μ is *at least*

$$3g - 5 + k + 2r_1 + r_2 + \sum_{i=1}^{k} m_i - \left(2g - 2 + k - n + \sum_{i=1}^{k} m_i\right)$$
$$= 2g - 3 + n + 2r_1 + r_2 \,.$$

Suppose $[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(\mu)$ is a generic element of an irreducible component $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ of dimension strictly less than $2g - 3 + n$. In the versal deformation space $\mathcal{V}$ above, we consider the dimension of the sublocus

$$\mathcal{Z} \subset \mathcal{T}(\mu)$$

corresponding to twisted canonical divisors on the fibers of (18) which have the *same* dual graph as C. The dimension of the sublocus $\mathcal{Z}$ is equal to

$$\dim Z + 2r_1 + r_2$$

which is less than $2g - 3 + 2r_1 + r_2$. The summands $2r_1$ and r_2 appear here since we do not now quotient by the automorphism of the unstable components of the fibers.

We conclude at least one node of the curve C can be smoothed in $\widetilde{\mathcal{H}}_g(\mu)$. Therefore, there does not exist a component $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ of dimension strictly less than $2g - 3 + n$. $\square$

3. Theorems 2 and 3

3.1. **Proof of Theorem 2.** By Proposition 4,

$$\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

is a closed subvariety. In case all parts of $\mu = (m_1, \ldots, m_n)$ are non-negative, every irreducible component of

$$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$$

has dimension $2g-2+n$. Hence, every irreducible component of

$$\overline{\mathcal{H}}_g(\mu) \subset \mathcal{M}_{g,n}$$

has dimension $2g-2+n$.

By Proposition 7, every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ which is supported in the boundary $\partial\overline{\mathcal{M}}_{g,n}$ has dimension at most $2g-3+n$. By Proposition 11, every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ has dimension at least $2g-3+n$. Hence, the boundary components of $\mathcal{H}_g(\mu)$ all have dimension $2g-3+n$. □

3.2. **Locus of irreducible curves.** From the point of view of twists, the locus of stable pointed curves

$$\mathcal{M}^{\mathsf{Irr}}_{g,n} \subset \mathcal{M}_{g,n}$$

with irreducible domains is very natural to consider,

$$[C, p_1, \ldots, p_n] \in \mathcal{M}^{\mathsf{Irr}}_{g,n} \quad \leftrightarrow \quad C \text{ is irreducible.}$$

Since an irreducible curve has no basic nodes, a twisted canonical divisor

$$[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(\mu) \,\cap\, \mathcal{M}^{\mathsf{Irr}}_{g,n}$$

is a usual canonical divisor

$$\mathcal{O}_C\left(\sum_{i=1}^{n} m_i p_i\right) \stackrel{\sim}{=} \omega_C\ .$$

We define $\mathcal{H}^{\mathsf{Irr}}_g(\mu)$ by the intersection

$$\mathcal{H}^{\mathsf{Irr}}_g(\mu) = \widetilde{\mathcal{H}}_g(\mu) \cap \mathcal{M}^{\mathsf{Irr}}_{g,n}\, .$$

Lemma 12. *If all parts of μ are non-negative,*

$$\overline{\mathcal{H}}_g(\mu) \cap \mathcal{M}^{\mathsf{Irr}}_{g,n} \;=\; \mathcal{H}^{\mathsf{Irr}}_g(\mu)\, .$$

Proof. We must prove there does not exist a component of $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ generically supported in the boundary $\partial\mathcal{M}^{\mathsf{Irr}}_{g,n}$. We have seen that the dimension of Z must be $2g-3+n$. However, every irreducible component of the locus of canonical divisors in $\mathcal{M}^{\mathsf{Irr}}_{g,n}$ has dimension at least $2g-2+n$ since the canonical bundle always has g sections. □

In the strictly meromorphic case, the equality

$$\overline{\mathcal{H}}_g(\mu) \cap \mathcal{M}_{g,n}^{\mathsf{Irr}} \;=\; \mathcal{H}_g^{\mathsf{Irr}}(\mu)\,.$$

also holds, but the result does not follow from elementary dimension arguments. Instead, the analytic perspective of flat surface is required, see [21].

By Lemma 12 in the holomorphic case and Theorem 3 in the strictly meromorphic case, we conclude the following two dimension results:

(i) If all parts of μ are non-negative, $\mathcal{H}_g^{\mathsf{Irr}}(\mu)$ has pure dimension $2g-2+n$.
(ii) If μ has a negative part, $\mathcal{H}_g^{\mathsf{Irr}}(\mu)$ has pure dimension $2g-3+n$.

The dimensions (i) and (ii) are identical to the dimensions of $\mathcal{H}_g(\mu)$ in the corresponding cases.

Since $\overline{\mathcal{H}}_g(\mu) = \overline{\mathcal{H}}_g^{\mathsf{Irr}}(\mu)$ for all μ, the star graphs of Section 2.2 with self-edges at the center vertex do *not* correspond to the dual graph Γ_Z of a generic element of an irreducible component $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ supported in the boundary $\partial\overline{\mathcal{M}}_{g,n}$. We state the result as a Lemma.

Lemma 13. *The dual graph Γ_Z of a generic element of an irreducible component*

$$Z \subset \widetilde{\mathcal{H}}_g(\mu)$$

supported in the boundary $\partial\overline{\mathcal{M}}_{g,n}$ is a star graph with no self-edges at the center.

Such star graphs will be called *simple* star graphs. In the Appendix, in the strictly meromorphic case, we will include in the set of simple star graphs the trivial star graph with a center vertex carrying all the parts of μ and *no* edges or outlying vertices.

3.3. **Proof of Theorem 3.** By Proposition 4,

$$\widetilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

is a closed subvariety. In case μ has at least one negative part, every irreducible component of

$$\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$$

has dimension $2g-3+n$, and every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ which is supported in the boundary $\partial\overline{\mathcal{M}}_{g,n}$ has dimension at most $2g-3+n$ by Proposition 7. By Proposition 11 every irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ has dimension at least $2g-3+n$. Hence, $\widetilde{\mathcal{H}}_g(\mu)$ is pure of dimension $2g-3+n$. □

4. SECTIONS OF LINE BUNDLES

4.1. **Dimension estimates.** Let $\mathcal{X}$ be a variety of pure dimension d, and let

$$\pi : \mathcal{C} \to \mathcal{X}$$

be a flat family of nodal curves. Let

$$\mathcal{L} \to \mathcal{C}$$

be a line bundle with Euler characteristic $\chi(\mathcal{L})$ on the fibers of π. Let

$$\phi : \mathcal{M} \to \mathcal{X}$$

be the variety parameterizing *non-trivial* sections of $\mathcal{L}$ on the fibers of π,

$$\phi^{-1}(\xi) = H^0(\mathcal{C}_\xi, \mathcal{L}_\xi) \setminus \{0\}$$

for $\xi \in \mathcal{X}$.

Proposition 14. *Every irreducible component of $\mathcal{M}$ has dimension at least $d + \chi(\mathcal{L})$.*

The result is immediate from the point of view of obstruction theory. For the convenience of the reader, we include an elementary proof based on the construction of the parameter space $\mathcal{M}$.

Proof. Using a π-relatively ample bundle $\mathcal{N} \to \mathcal{C}$, we form a quotient

$$\oplus_{i=1}^r \mathcal{N}^{-k_i} \to \mathcal{L} \to 0\,, \qquad k_i > 0.$$

Let $\mathcal{A} = \oplus_{i=1}^r \mathcal{N}^{-k_i}$ and consider the associated short exact sequence

$$0 \to \mathcal{B} \to \mathcal{A} \to \mathcal{L} \to 0 \tag{19}$$

of bundles on $\mathcal{C}$. Since $\mathcal{A}$ is π-relatively negative, $\mathcal{A}$ and $\mathcal{B}$ have no sections on the fibers of π. Hence,

$$V_{\mathcal{A}} = R^1\pi_*(\mathcal{A}) \quad \text{and} \quad V_{\mathcal{B}} = R^1\pi_*(\mathcal{B})$$

are vector bundles of ranks a and b on $\mathcal{X}$. For $\xi \in \mathcal{X}$, we have

$$0 \to H^0(\mathcal{C}_\xi, \mathcal{L}_\xi) \to V_{\mathcal{B},\xi} \to V_{\mathcal{A},\xi} \to H^0(\mathcal{C}_\xi, \mathcal{L}_\xi) \to 0\,. \tag{20}$$

Therefore, the ranks satisfy $b - a = \chi(L)$.

The sequence in cohomology associated to (19) yields a bundle map

$$\gamma : V_{\mathcal{B}} \to V_{\mathcal{A}}$$

on $\mathcal{X}$. Let $V_{\mathcal{B}}^*$ be the total space of $V_{\mathcal{B}}$ with the zero section removed,

$$q : V_{\mathcal{B}}^* \to \mathcal{X}\,.$$

The pull-back to $V_{\mathcal{B}}^*$ of $V_{\mathcal{B}}$ carries a tautological line bundle

$$0 \to Q \to q^*(V_{\mathcal{B}})\,.$$

By (20), the parameter space $\mathcal{M}$ sits in $V_{\mathcal{B}}^*$ as the zero locus of the canonical map

$$Q \to q^*(V_{\mathcal{A}})$$

obtained from the composition

$$Q \ \to \ q^*(V_{\mathcal{B}}) \ \stackrel{q^*(\gamma)}{\longrightarrow} \ q^*(V_{\mathcal{A}})\,.$$

Since $\mathcal{X}$ is of pure dimension d, $V_{\mathcal{B}}^*$ is of pure dimension $d+b$. Finally, since the rank of $q^*(V_{\mathcal{A}})$ is a, every irreducible component of $\mathcal{M}$ is of dimension at least

$$d+b-a = d+\chi(\mathcal{L})\ ,$$

by standard dimension theory. □

The following result is a consequence of the existence of a reduced obstruction theory (see the cosection method of [16]). Again, we give an elementary proof. An application will appear in Section 4.2 below.

Proposition 15. *If there exists a trivial quotient on $\mathcal{X}$,*

$$R^1\pi_*\mathcal{L} \to \mathbb{C} \to 0\,,$$

then every irreducible component of $\mathcal{M}$ has dimension at least $d+\chi(\mathcal{L})+1$.

Proof. We must improve the dimension estimate in Proposition 14 by 1. The long exact sequence obtained from (19) yields a quotient

$$V_{\mathcal{A}} \to R^1\pi_*(L) \to 0$$

on $\mathcal{X}$. Composing with the given quotient

$$R^1\pi_*(\mathcal{L}) \to \mathbb{C} \to 0$$

yields a quotient

$$V_{\mathcal{A}} \to \mathbb{C} \to 0\,.$$

Let $K = \mathrm{Ker}(V_{\mathcal{A}} \to \mathbb{C})$. Then,

$$K \to \mathcal{X}$$

is a vector bundle of rank $a-1$. Since the image of

$$\gamma : V_{\mathcal{B}} \to V_{\mathcal{A}}$$

lies in K, $\mathcal{M}$ is in fact the zero locus of

$$Q \to q^*(K)$$

on $V^*_{\mathcal{B}}$. The dimension of every irreducible component of M is at least

$$d + b - a + 1 = d + \chi(\mathcal{L}) + 1$$

by dimension theory as before. □

4.2. **Irreducible components in the boundary.** We consider here the holomorphic case: $\mu = (m_1, \ldots, m_n)$ is a vector of zero multiplicities satisfying

$$m_i \geq 0 \, , \quad \sum_{i=1}^{n} m_i = 2g - 2 \, .$$

Let $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ be an irreducible component of dimension $2g - 3 + n$ supported in the boundary $\partial\overline{\mathcal{M}}_{g,n}$, and let

$$[C, p_1, \ldots, p_n] \in Z$$

be a generic element with associated twist I. The dual graph Γ_C of C must be of the form described in Section 2.1 and Lemma 13: Γ_C is a simple star with a center vertex v_0, edges connecting v_0 to outlying vertices $v_1, \ldots, v_k$, and a distribution of the parts of μ (with all negative parts distributed to the center v_0).

By Lemma 12, there are no irreducible components $Z \subset \widetilde{\mathcal{H}}_g(\mu)$ supported in the boundary with generic dual graphs with *no* outlying vertices. All such loci are in the closure of $\mathcal{H}_g(\mu)$. A similar result holds in case there is a single outlying vertex.

Proposition 16. *If all parts of μ are non-negative, there are no irreducible components*

$$Z \subset \widetilde{\mathcal{H}}_g(\mu)$$

supported in the boundary with generic dual graph having exactly 1 outlying vertex. All such loci are in the closure of $\mathcal{H}_g(\mu)$.

Proof. As in the proof of Proposition 11, we will study the versal deformation space of C where

$$[C, p_1, \ldots, p_n] \in Z$$

is a generic element. However, a more delicate argument is required here. We assume the generic twist I is non-trivial. If the twist I is trivial, then $[C, p_1, \ldots, p_n]$ is a usual canonical divisor and the proof of Proposition 11 directly applies. The nodes of C corresponding to self-edges of v_0 cannot be twisted by I.

We assume[12] the vertices v_0 and v_1 both have genus $g_i > 0$. Let $\mathcal{V}$ be the versal deformation space of the unpointed nodal curve C,

$$\dim \mathcal{V} = 3g - 3 \,.$$

let e_1 be the number of edges connecting v_0 and v_1. The subvariety $\mathcal{S} \subset \mathcal{V}$ of curves preserving the nodes of C corresponding to the edges connecting v_0 to v_1 is of codimension e_1.

We blow-up $\mathcal{V}$ in the subvariety $\mathcal{S}$,

$$\nu : \mathcal{V}_1 \to \mathcal{V} \,.$$

The fiber of $\mathcal{V}_1$ above the original point $[C] \in \mathcal{V}$ is

$$\nu^{-1}([C]) = \mathbb{P}^{e_1 - 1} \,.$$

Via pull-back, we have a universal family of curves

$$\pi : \mathcal{C} \to \mathcal{V}_1 \,.$$

The blow-up yields a nonsingular exceptional divisor

$$E_1 \subset \mathcal{V}_1$$

which contains $\nu^{-1}([C])$. By locally base-changing $\mathcal{V}_1$ via the r^{th}-power of the equation of E_1, we can introduce transverse A_{r-1}-singularities along the nodes of the fibers of $\mathcal{C}$ corresponding to the edges e_1. The crepant resolution introduces a chain of $r - 1$ rational curves.

The outcome after such a base change is a family

$$\widetilde{\pi} : \widetilde{C} \to \mathcal{V}_1$$

with the following properties over the generic point $\xi \in \nu^{-1}([C])$:

(i) $\widetilde{C}_\xi$ is a semistable curve obtained C by inserting chains of $r - 1$ rational components at the nodes,

(ii) the total space of $\widetilde{C}$ is nonsingular near $\widetilde{C}_\xi$.

(iii) the components of $\widetilde{C}_\xi$ correspond to *divisors* in the total space of $\widetilde{C}$.

After selecting r appropriately, we can twist the total space $\widetilde{C}$ as specified by I. Such a geometric twisting is possible because of condition (iii) above.

[12] In case $g = 0$ vertices occur, we can rigidify with q markings as in the proof of Proposition 11. We leave the routine modification to the reader.

We follow the construction of Section 1.3. There, a non-negative twist γ_D is associated to each component $D \subset \widetilde{C}_\xi$. More precisely, the construction assigns

$$\gamma_{v_0} = 0 \,, \qquad \gamma_{v_1} = \prod_{i=1}^{e_1} I(q_i, v_0) > 0$$

and strictly positive twists less than γ_{v_1} for the additional rational components. By twisting by the divisor $-\gamma_{v_1} \cdot E_1$ near $\xi \in \mathcal{V}_1$, we can change the assignments to

$$\widehat{\gamma}_{v_0} = -\prod_{i=1}^{e_1} I(q_i, v_0) < 0 \,, \qquad \widehat{\gamma}_{v_1} = 0 \tag{21}$$

and strictly negative twists greater than $\widehat{\gamma}_{v_0}$ for the additional rational components.

Using the twists (21), we obtain a line bundle

$$\mathcal{L} = \omega_{\widetilde{\pi}} \left(\sum_D \widehat{\gamma}_D \cdot [D] \right) \tag{22}$$

where the sum is over the irreducible components of $\widetilde{C}_\xi$ for generic $\xi \in \nu^{-1}([C])$. The line bundle (22) is defined over an open set

$$\mathcal{U} \subset \mathcal{V}_1$$

which contains the generic element $\xi \in \nu^{-1}([C])$.

On the fibers of $\widetilde{\pi}$ over $\mathcal{U}$,

$$H^1\Big(\omega_{\widetilde{\pi}}\Big(\sum_D \widehat{\gamma}_D \cdot [D]\Big)\Big) \cong H^0\Big(\mathcal{O}\Big(\sum_D -\widehat{\gamma}_D \cdot [D]\Big)\Big)^\vee$$

by Serre duality. Since $-\widehat{\gamma}_D \geq 0$, we obtain a canonical section

$$\mathbb{C} \to H^0\Big(\mathcal{O}\Big(\sum_D -\widehat{\gamma}_D \cdot [D]\Big)\Big)^\vee .$$

Since $-\widehat{\gamma}_{v_1} = 0$, the canonical section does not vanish identically on *any* fiber of $\widetilde{\pi}$ over $\mathcal{U}$. Hence, we obtain a canonical quotient

$$H^1\Big(\omega_{\widetilde{\pi}}\Big(\sum_D \widehat{\gamma}_D \cdot [D]\Big)\Big) \to \mathbb{C} \to 0 \,.$$

We now can estimate the dimension of the space of good sections of $\mathcal{L}$ on the fibers of $\widetilde{\pi}$ over $\mathcal{U}$ near the generic element of $\nu^{-1}([C])$ using Proposition 15 of Section 4.1. The dimension of the space of sections is at least

$$\dim \mathcal{U} + \chi(C, L) + 1 \;=\; 3g - 3 + g - 1 + 1 \;=\; 4g - 3 \,.$$

Hence, dimension of the space $\mathcal{T}$ of twisted canonical divisors on the fibers of $\widetilde{\pi}$ over $\mathcal{U}$ corresponding to twisted canonical line bundle $\mathcal{L}$ near the generic element of $\nu^{-1}([C])$ is at least $4g-4$.

We must further impose conditions on the twisted canonical divisors parameterized by $\mathcal{T}$ to obtain the shape $\sum_{i=1}^n m_i p_i$. These conditions impose at most

$$2g-2-n$$

constraints. Hence, we conclude the dimension of space $\mathcal{T}(\mu)$ of twisted canonical divisors on the fibers of

$$\widetilde{\pi}: \widetilde{\mathcal{C}} \to \mathcal{U}$$

near $[C, p_1, \ldots, p_n]$ of shape μ is *at least*

$$4g-4-(2g-2-n) = 2g-2+n\,.$$

The result contradicts the dimension $2g-3+n$ of Z. □

By Lemma 12 and Proposition 16 in the holomorphic case, the loci

$$Z \subset \widetilde{\mathcal{H}}_g(\mu)$$

corresponding to star graphs with 0 or 1 outlying vertices are contained in the closure of $\mathcal{H}_g(\mu)$. However, there are virtual components of $\widetilde{\mathcal{H}}_g(\mu)$ with more outlying vertices.

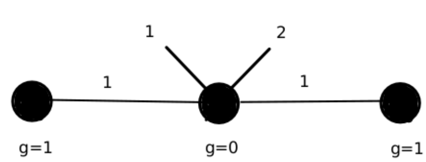
A simple example occurs in genus $g = 2$ with $\mu = (1,1)$. A nonsingular pointed curve

$$[C, p_1, p_2] \in \mathcal{M}_{2,2}$$

lies in $\mathcal{H}_2(1,1)$ if and only if $\{p_1, p_2\} \subset C$ is a fiber of the unique degree 2 map

$$C \to \mathbb{P}^1$$

determined by the canonical series. An easy argument using admissible covers shows that the generic element of the virtual component corresponding to the star graph below is not in the closure of $\mathcal{H}_2(1,1)$.



Both parts of μ are distributed to the center. The genera of the vertices and the twists of the edges are specified in the diagram. The above example (which occurs even in compact type) was discussed earlier by Chen [3].

5. Twisted canonical divisors and limits of theta characteristics

5.1. **Theta characteristics.** We illustrate the results of the paper (particularly Theorem 2 and Proposition 16) in two much studied cases linked to the classical theory of theta characteristics. Assume $\mu = (m_1, \ldots, m_n)$ consists of *even* non-negative parts,

$$m_i = 2a_i\,, \qquad \sum_{i=1}^{n} a_i = g-1\,.$$

We write $\mu = 2\alpha$ where $\alpha \vdash g-1$.

If $[C, p_1, \ldots, p_n] \in \mathcal{H}_g(2\alpha)$ is a canonical divisor, then

$$\eta \stackrel{\sim}{=} \mathcal{O}_C(a_1p_1 + \cdots + a_np_n)$$

is a theta characteristic. The study of the closure $\overline{\mathcal{H}}_g(2\alpha)$ is therefore intimately related to the geometry of the moduli space $\overline{\mathcal{S}}_g$ of spin curves of genus g, see [6].

Two cases are of particular interest. If $\mu = (2g-2)$ is of length one, $\mathcal{H}_g(2g-2)$ is the locus of *subcanonical* pointed curves:

$$[C,p] \in \mathcal{M}_{g,1} \quad \Leftrightarrow \quad \omega_C \ \stackrel{\sim}{=} \ \mathcal{O}_C((2g-2)p)\,.$$

Subcanonical points are extremal Weierstrass points on the curve. For $g \geq 4$, Kontsevich and Zorich [17] have shown that the space $\mathcal{H}_g(2g-2)$ has three connected components (of dimension $2g-1$):

- the locus $\mathcal{H}_g(2g-2)^+$ of curves $[C,p]$ for which $\mathcal{O}_C((g-1)p)$ is an even theta characteristic,
- the locus $\mathcal{H}_g(2g-2)^-$ of curves $[C,p]$ for which $\mathcal{O}_C((g-1)p)$ is an odd theta characteristic,
- the locus $\mathcal{H}_g(2g-2)^{\mathrm{hyp}}$ of curves $[C,p]$ where C is hyperelliptic and $p \in C$ is a Weierstrass point.

For $g = 3$, the first and the third coincide

$$\mathcal{H}_3(4)^+ = \mathcal{H}_3(4)^{\mathrm{hyp}}\,,$$

thus $\mathcal{H}_3(4)$ has only two connected components. For $g = 2$, the space $\mathcal{H}_2(2)$ is irreducible. The geometry of the compactifications of these loci in small genera has been studied in [4].

The second case we consider is $\mu = (2, \ldots, 2)$ of length $g-1$. The space $\mathcal{H}_g(\underline{2}) = \mathcal{H}_g(2, \ldots, 2)$ splits into two connected components:

- the locus $\mathcal{H}_g(\underline{2})^-$ of curves $[C, p_1, \ldots, p_{g-1}] \in \mathcal{M}_{g,g-1}$ for which

$$\eta \stackrel{\sim}{=} \mathcal{O}_C(p_1 + \cdots + p_{g-1})$$

is an odd theta characteristic,

- the locus $\mathcal{H}_g(\underline{2})^+$ of curves $[C, p_1, \dots, p_{g-1}] \in \mathcal{M}_{g,g-1}$ for which
$$\eta \cong \mathcal{O}_C(p_1 + \cdots + p_{g-1})$$
is a vanishing theta-null (even and $h^0(C, \eta) \geq 2$).

The component $\mathcal{H}_g(\underline{2})^-$ is a generically finite cover of $\mathcal{M}_g$. For example, $\mathcal{H}_3(\underline{2})^-$ is birationally isomorphic to the canonical double cover over the moduli space of bitangents of nonsingular quartic curves. On the other hand, the component $\mathcal{H}_g(\underline{2})^+$ maps with 1 dimensional fibres over the divisor Θ_{null} of curves with a vanishing theta-null, see [9].

5.2. **Spin curves.** We require a few basic definitions concerning spin curves. A connected nodal curve X is *quasi-stable* if, for every component
$$E \cong \mathbb{P}^1 \subset X\,,$$
the following two properties are satisfied:

- $k_E = |E \cap \overline{(X - E)}| \geq 2$,
- rational components $E, E' \subset X$ with $k_E = k_{E'} = 2$ are always disjoint.

The irreducible components $E \subset X$ are *exceptional*.

Definition 17 (Cornalba [6]). A stable *spin curve* of genus g consists of a triple (X, η, β) where

(i) X is a genus g quasi-stable curve,
(ii) $\eta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle of total degree $g-1$ with $\eta_E = \mathcal{O}_E(1)$ for all exceptional components $E \subset X$,
(iii) $\beta : \eta^{\otimes 2} \to \omega_X$ is a homomorphism of sheaves which is generically non-zero along each non-exceptional component of X.

If (X, η, β) is a spin curve with exceptional components $E_1, \dots, E_r \subset X$, then $\beta_{E_i} = 0$ for $i = 1, \dots, r$ by degree considerations. Furthermore, if
$$\widetilde{X} = \overline{X - \bigcup_{i=1}^r E_i}$$
is viewed as a subcurve of X, then we have an isomorphism
$$\eta_{\widetilde{X}}^{\otimes 2} \xrightarrow{\sim} \omega_{\widetilde{X}}.$$

The moduli space $\overline{\mathcal{S}}_g$ of stable spin curves of genus g has been constructed by Cornalba [6]. There is a proper morphism
$$\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$$

associating to a spin curve $[X, \eta, \beta]$ the curve obtained from X by contracting all exceptional components. The parity $h^0(X, \eta)$ mod 2 of a spin curve is invariant under deformations. The moduli space $\overline{\mathcal{S}}_g$ splits into two connected components $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$ of relative degree $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$ over $\mathcal{M}_g$ respectively. The birational geometry of $\overline{\mathcal{S}}_g$ has been studied in [9].

5.3. **Twisted canonical divisors on 2-component curves.** We will describe the limits of supports of theta characteristics in case the underlying curve

$$C = C_1 \cup C_2$$

is a union of two nonsingular curves C_1 and C_2 of genera i and $g-i-\ell+1$ meeting transversally in a set of ℓ distinct points

$$\Delta = \{x_1, \ldots, x_\ell\} \subset C\,.$$

Let $2a_1p_1 + \cdots + 2a_np_n$ be a twisted canonical divisor on C,

$$[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(2\alpha)\,.$$

By definition, there exist twists

$$I(x_j, C_1) = -I(x_j, C_2)\,, \quad j = 1, \ldots, \ell$$

for which the following linear equivalences on C_1 and C_2 hold:

$$\omega_{C_1} \equiv \sum_{p_i \in C_1} 2a_ip_i - \sum_{j=1}^{\ell} \Big(I(x_j, C_1)+1\Big)x_j \text{ and } \omega_{C_2} \equiv \sum_{p_i \in C_2} 2a_ip_i - \sum_{j=1}^{\ell} \Big(I(x_j, C_2)+1\Big)x_j.$$

By Proposition 16, all twisted holomorphic differentials on C are smoothable,

$$[C, p_1, \ldots, p_n] \in \overline{\mathcal{H}}_g(\mu)\,.$$

Following [6], we describe all spin structures having C as underlying stable curve. If $[X, \eta, \beta] \in \pi^{-1}([C])$, then the number of nodes of C where no exceptional component is inserted must be *even*. Hence, X is obtained from C by *blowing-up* $\ell - 2h$ nodes.[13] Let $\{x_1, \ldots, x_{2h}\} \subset C$ be the non blown-up nodes. We have

$$X = C_1 \cup C_2 \cup R_{2h+1} \cup \ldots \cup R_\ell\,,$$

where $C_1 \cap R_i = \{x_i'\}$ and $C_2 \cap R_i = \{x_i''\}$ for $i = 2h+1, \ldots, \ell$, see Figure 2. Furthermore,

$$\eta_{R_i} \cong \mathcal{O}_{R_i}(1)\,, \ \ \eta_{C_1} \in \mathrm{Pic}^{i-1+h}(C_1)\,, \ \ \eta_{C_2} \in \mathrm{Pic}^{g-i-\ell+h}(C_2)$$

[13] The term *blowing-up* here just means inserting an exceptional $\mathbb{P}^1$.

with the latter satisfying

$$\eta_{C_1}^{\otimes 2} \cong \omega_{C_1}(x_1 + \cdots + x_{2h}) \quad \text{and} \quad \eta_{C_2}^{\otimes 2} \cong \omega_{C_2}(x_1 + \cdots + x_{2h}).$$

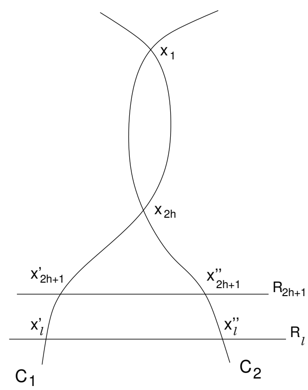


FIGURE 2. The quasi-stable curve X

There is a precise correspondence between the spin structures of Cornalba [6] and the system of twists associated to a point in $\widetilde{\mathcal{H}}_g(2\alpha)$ and

Proposition 18. *Let $C = C_1 \cup C_2$ as above. The spin structure in $\overline{\mathcal{S}}_g$ corresponding to $[C, p_1, \ldots, p_n] \in \widetilde{\mathcal{H}}_g(2\alpha)$ with twist I is obtained by blowing-up C at all the nodes x_j where the twist*

$$I(x_j, C_1) = -I(x_j, C_2)$$

is odd.

Proof. Since $[C, p_1, \ldots, p_n] \in \overline{\mathcal{H}}_g(2\alpha)$ by Proposition 16, there exists an n-pointed family of stable spin curves over a disc

$$\mathcal{X} \to \Delta\,, \quad \eta \in \mathrm{Pic}(\mathcal{X})\,, \quad p_1, \ldots, p_n : \Delta \to \mathcal{X},$$

satisfying

- for $t \in \Delta^\star$, the fibre X_t is nonsingular and

$$\eta_{X_t} \cong \mathcal{O}_{X_t}\big(a_1 p_1(t) + \cdots + a_n p_n(t)\big)\,,$$

- the central fibre X_0 is obtained from C by inserting nonsingular rational components $R_{2h+1}, \ldots, R_\ell$ at some of the nodes of C which we assume to be $x_{2h+1}, \ldots, x_\ell$.
- $\eta_{R_j} \cong \mathcal{O}_{R_j}(1)$ for $j = 2h+1, \ldots, \ell$.

By carrying out an argument following Section 1.2, after a finite base change and after resolution of singularities, we arrive at a new family

$$\widetilde{\pi} : \widetilde{\mathcal{X}} \to \Delta \,,$$

with central fibre

$$\mu : \widetilde{\pi}^{-1}(0) \to X_0$$

obtained from X_0 by inserting chains of nonsingular rational curves at the nodes of X_0 for which the line bundle

$$\mu^*(\eta_X)(-a_1 p_1 - \cdots - a_n p_n)$$

is the restriction to the central fibre of a line bundle on $\widetilde{\mathcal{X}}$ supported *only* on the irreducible components of $\widetilde{\pi}^{-1}(0)$.

Just as in Section 1.2, we then obtain integral twists

$$\tau_j = J(x_j, C_1) = -J(x_j, C_2)$$

for $j = 1, \ldots, 2h$ and

$$\tau_j' = J(x_j', C_1) = -J(x_j', R_j) \quad \text{and} \quad \tau_j'' = J(x_j'', R_j) = -J(x_j'', C_2)$$

for $j = 2h{+}1, \ldots, \ell$, for which the restrictions of $\eta_X\big(-\sum_{i=1}^n a_i p_i\big)$ to the irreducible components of X_0 are isomorphic to the bundles given by the twists J at the incident nodes. In particular,

$$\eta_{R_j} \cong \mathcal{O}_{R_j}(1) = \mathcal{O}_{R_j}(-\tau_j' + \tau_j''),$$

hence $\tau_j'' = \tau_j' + 1$ for $j = 2h+1, \ldots, \ell$. Furthermore, we have

$$\eta_{C_1}\Big(-\sum_{p_i \in C_1} a_i p_i\Big) \cong \mathcal{O}_{C_1}\Big(\tau_1 x_1 + \cdots + \tau_{2h} x_{2h} + \tau_{2h+1}' x_{2h} + \cdots + \tau_\ell' x_\ell'\Big)\,,$$

$$\eta_{C_2}\Big(-\sum_{p_i \in C_2} a_i p_i\Big) \cong \mathcal{O}_{C_2}\Big(-\tau_1 x_1 - \cdots - \tau_{2h} x_{2h} - \tau_{2h+1}'' x_{2h}'' - \cdots - \tau_\ell'' x_\ell''\Big).$$

By squaring these relations and comparing to the original I twists, we conclude

$$I(x_j, C_1) = -2\tau_j$$

for $j = 1, \ldots, 2h$ and

$$I(x_j, C_1) = -2\tau_j' - 1$$

for $j = 2h+1, \ldots, \ell$ respectively. □

5.4. **Subcanonical points in genus** 3. We illustrate Proposition 18 in case $g = 3$ where

$$\mathcal{H}_3(4) = \mathcal{H}_3(4)^- \sqcup \mathcal{H}_3(4)^{\mathrm{hyp}} \,.$$

Assume C_1 and C_2 are both nonsingular elliptic curves and $p \in C_1 \setminus \{x_1, x_2\}$. Suppose

$$[C_1 \cup C_2, p] \in \overline{\mathcal{H}}_3(4) \,.$$

The only possible twists are

$$I(x_1, C_1) = I(x_2, C_1) = 1 \,.$$

The curve $[C_1 \cup C_2, p] \in \overline{\mathcal{M}}_{3,1}$ lies in $\overline{\mathcal{H}}_3(4)$ if and only if the linear equivalence

$$4p \equiv 2x_1 + 2x_2 \tag{23}$$

holds on C_1.

The associated spin structure $[X, \eta]$ is obtained by inserting at both x_1 and x_2 rational components R_1 and R_2,

$$X = C_1 \cup C_2 \cup R_1 \cup R_2$$

with the intersections

$$C_1 \cap R_1 = \{x_1'\} \,,\ \ C_1 \cap R_2 = \{x_2'\} \,,\ \ C_2 \cap R_1 = \{x_1''\} \,,\ \ C_2 \cap R_2 = \{x_2''\} \,.$$

The line bundle η satisfies

$$\eta_{R_i} \cong \mathcal{O}_{R_i}(1) \,,\ \ \eta_{C_1} \cong \mathcal{O}_{C_1}(2p - x_1 - x_2) \,,\ \ \eta_{C_2} \cong \mathcal{O}_{C_2} \,.$$

An argument via admissible covers implies

$$[C, p] \in \overline{\mathcal{H}}_3(4)^{\mathrm{hyp}} \quad \Leftrightarrow \quad \mathcal{O}_{C_1}(2p - x_1 - x_2) \cong \mathcal{O}_C \,.$$

In the remaining case, where

$$\eta_{C_1} \cong \mathcal{O}_{C_1}(2p - x_1 - x_2) \in JC_1[2] - \{\mathcal{O}_{C_1}\}$$

is a non-trivial 2-torsion point, η_{C_1} is an even and η_{C_2} is an odd spin structure, so $[X, \eta] \in \overline{\mathcal{S}}_3^-$, and accordingly $[C, p] \in \overline{\mathcal{H}}_3(4)^-$.

5.5. **Subcanonical points in genus** 4**.** Consider next genus 4 where

$$\mathcal{H}_4(6) = \mathcal{H}_4(6)^+ \sqcup \mathcal{H}_4(6)^- \sqcup \mathcal{H}_4(6)^{\mathrm{hyp}}$$

has three connected components of dimension 7. Let

$$C = C_1 \cup C_2\,, \quad g(C_1) = 1\,, \ \ g(C_2) = 2$$

with $C_1 \cap C_2 = \{x_1, x_2\}$. Suppose $p \in C_1$ and

$$[C_1 \cup C_2, p] \in \overline{\mathcal{H}}_4(6)\,.$$

There are two possible twists.

(i) $I(x_1, C_1) = I(x_2, C_1) = 2$.

We then obtain the following linear equivalences on C_1 and C_2:

$$\mathcal{O}_{C_1}(6p - 3x_1 - 3x_2) \stackrel{\sim}{=} \mathcal{O}_{C_1} \ \text{ and } \ \omega_{C_2} \stackrel{\sim}{=} \mathcal{O}_{C_2}(x_1 + x_2). \tag{24}$$

Each equation appearing in (24) imposes a codimension 1 condition on the respective Jacobian. The corresponding spin curve in $[X, \eta] \in \overline{\mathcal{S}}_4$ has underlying model $X = C$ (no nodes are blown-up), and the spin line bundle $\eta \in \mathrm{Pic}(X)$ satisfies

$$\eta_1 \stackrel{\sim}{=} \mathcal{O}_{C_1}(3p - x_1 - x_2) \ \text{ and } \ \eta_2 \stackrel{\sim}{=} \mathcal{O}_{C_2}(x_1 + x_2)\,.$$

As expected, $\eta_i^{\otimes 2} \stackrel{\sim}{=} \omega_{C_i}(x_1 + x_2)$ for $i = 1, 2$.

We can further distinguish between the components of the closure $\overline{\mathcal{H}}_4(6)$. We have $[C, p] \in \overline{\mathcal{H}}_4(6)^{\mathrm{hyp}}$ if and only if

$$\mathcal{O}_{C_1}(2p - x_1 - x_2) \stackrel{\sim}{=} \mathcal{O}_{C_1} \text{ and } \ \omega_{C_2} \stackrel{\sim}{=} \mathcal{O}_{C_2}(x_1 + x_2)\,,$$

whereas $[C, p] \in \overline{\mathcal{H}}_4(6)^-$ if and only if

$$\mathcal{O}_{C_1}(2p - x_1 - x_2) \in JC_1[3] - \{\mathcal{O}_{C_1}\} \ \text{ and } \ \omega_{C_2} \stackrel{\sim}{=} \mathcal{O}_{C_2}(x_1 + x_2)\,.$$

(ii) $I(x_1, C_1) = 3$ and $I(x_2, C_1) = 1$.

We then obtain the following linear equivalences on C_1 and C_2:

$$\mathcal{O}_{C_1}(6p - 4x_1 - 2x_2) \stackrel{\sim}{=} \mathcal{O}_{C_1} \ \text{ and } \ \omega_{C_2} \stackrel{\sim}{=} \mathcal{O}_{C_2}(2x_1)\,. \tag{25}$$

As before, the conditions in (25) are both codimension 1 in the respective Jacobians. The corresponding spin curve $[X, \eta]$ is obtained by blowing-up both nodes x_1 and x_2,

$$X = C_1 \cup C_2 \cup R_1 \cup R_2,$$

with the intersections

$$C_1 \cap R_1 = \{x_1'\}\,,\ \ C_1 \cap R_2 = \{x_2'\}\,,\ \ C_2 \cap R_1 = \{x_1''\}\,,\ \ C_2 \cap R_2 = \{x_2''\}\,.$$

The line bundle η satisfies

$$\eta_{R_i} \cong \mathcal{O}_{R_i}(1)\,,\ \ \eta_{C_1} \cong \mathcal{O}_{C_1}(3p - 2x_1 - x_2)\,,\ \ \eta_{C_2} \cong \mathcal{O}_{C_2}(x_1)\,.$$

We have $[C,p] \in \overline{\mathcal{H}}_4(6)^+$ if and only if $\eta_{C_1} \cong \mathcal{O}_{C_1}$. If η_{C_1} is a non-trivial 2-torsion point, then $[C,p] \in \overline{\mathcal{H}}_4(6)^-$.

If $p \in C_2$ and $[C_1 \cup C_2, p] \in \overline{\mathcal{H}}_4(6)$, then $I(x_1, C_1) = I(x_2, C_1) = -1$ is the only possible twist, and

$$\omega_{C_2} \cong \mathcal{O}_{C_2}(6p - 2x_1 - 2x_2)\,. \tag{26}$$

The corresponding spin structure is realized on the curve X obtained by inserting rational componets at both $x_1, x_2 \in C$ and

$$\eta_{C_1} \cong \mathcal{O}_{C_1}\,, \quad \eta_{C_2} \cong \mathcal{O}_{C_2}(3p - x_1 - x_2)\,.$$

We have $[C,p] \in \overline{\mathcal{H}}_4(6)^+$ (respectively $\overline{\mathcal{H}}_4(6)^-$) if and only if η_{C_2} is an odd (respectively even) theta characteristic on C_2.

5.6. **Limits of bitangent lines for singular curves of genus** 3**.** Recall the decomposition into components in genus 3:

$$\mathcal{H}_3(2,2) = \mathcal{H}_3(2,2)^- \sqcup \mathcal{H}_3(2,2)^+,$$

where the general point $[C, p_1, p_2]$ of $\mathcal{H}_3(2,2)^-$ corresponds to a bitangent line

$$\langle p_1, p_2 \rangle \in (\mathbb{P}^2)^\vee$$

of the canonical model $C \subset \mathbb{P}^2$ and the general point of $\mathcal{H}_3(2,2)^+$ corresponds to a hyperelliptic curve C together with two points lying a fibre of the hyperelliptic pencil.

As before, let $C = C_1 \cup C_2$ where C_1 and C_2 are nonsingular elliptic curves with

$$C_1 \cap C_2 = \{x_1, x_2\}\,.$$

Suppose $[C, p_1, p_2] \in \overline{\mathcal{H}}_3(2,2)$. There are two cases.

(i) $p_1, p_2 \in C_1 \setminus \{x_1, x_2\}$.

Then $I(x_1, C_1) = I(x_2, C_2) = 1$, and the following equation holds:

$$\mathcal{O}_{C_1}(2x_1 + 2x_2 - 2p_1 - 2p_2) \cong \mathcal{O}_{C_1}\,. \tag{27}$$

The induced spin structure is obtained by blowing-up both nodes x_1 and x_2, and the corresponding spin bundle η restricts as

$$\eta_{C_1} \cong \mathcal{O}_{C_1}(p_1 + p_2 - x_1 - x_2) \quad \text{and} \quad \eta_{C_2} \cong \mathcal{O}_{C_2}.$$

If the linear equivalence $p_1 + p_2 \equiv x_1 + x_2$ holds on C_1, then $[C, p_1, p_2] \in \overline{\mathcal{H}}_3(2,2)^+$ as can be seen directly via admissible covers. If

$$\eta_{C_1} \in JC_1[2] - \{\mathcal{O}_{C_2}\}\,,$$

then, since η_{C_2} is odd, we have $[C, p_1, p_2] \in \overline{\mathcal{H}}_3(2,2)^-$. There is a 1-dimensional family of limiting bitangents to C, which is to be expected since C is hyperelliptic.

(ii) $p_i \in C_i \setminus \{x_1, x_2\}$.

Then $I(x_1, C_1) = I(x_2, C_1) = 0$. No node is twisted, so no rational components are inserted for the induced spin structure. The spin bundle $\eta \in \mathrm{Pic}^2(C)$ has restrictions

$$\eta_{C_i} \cong \mathcal{O}_{C_i}(p_i)$$

for $i = 1, 2$. Since the resulting spin curve is odd,

$$[C, p_1, p_2] \in \overline{\mathcal{H}}_3(2,2)^-\,.$$

We obtain $16 = 4 \times 4$ limits of bitangents corresponding to the choice of points $p_i \in C_i$ satisfying

$$\mathcal{O}_{C_i}(2p_i - x_1 - x_2) \cong \mathcal{O}_{C_i}\,.$$

We now study three further degenerate cases to offer an illustration of the transitivity condition in the definition of twists.

(iii) The case $[C_1 \cup C_2 \cup R, p_1, p_2] \in \widetilde{\mathcal{H}}_3(2,2)$ with $p_1 \in R \cong \mathbb{P}^1$, $p_2 \in C_1$.

We have the following component intersections:

$$C_1 \cap C_2 = \{x_2\}\,, \;\; C_1 \cap R = \{x_1'\}\,, \;\; C_2 \cap R = \{x_1''\}\,.$$

The twists are subject to three dependent equations. Let $a = I(x_1', C_1)$. We find

$$I(x_1'', C_2) = -I(x_1'', R) = -2 - a \quad \text{and} \quad I(x_2, C_1) = -I(x_2, C_2) = -a\,.$$

If $a \leq -2$, the ordering on components defined by I,

$$C_1 < R \leq C_2 < C_1\;\;,$$

contradicts transitivity. Similarly, if $a \geq 0$, we obtain

$$R \leq C_1 \leq C_2 < R\,,$$

which is also contradiction. Therefore $a = -1$, and we have

$$\mathcal{O}_{C_1}(2p_2 - 2x_2) \cong \mathcal{O}_{C_1}.$$

The result is a 4-dimensional subvariety of $\overline{\mathcal{M}}_{3,2}$ and therefore cannot be a component of $\widetilde{\mathcal{H}}_3(2,2)$.

(iv) The case $\left[C_1 \cup C_2 \cup R_1 \cup R_2, p_1, p_2\right] \in \widetilde{\mathcal{H}}_3(2,2)$ with $p_i \in R_i \cong \mathbb{P}^1$.

We have the following component intersections:

$$C_1 \cap R_1 = \{x_1'\}\,, \;\; C_1 \cap R_2 = \{x_2'\}\,, \;\; C_2 \cap R_1 = \{x_1''\}\,, \;\; C_2 \cap R_2 = \{x_2''\}\,.$$

Let $a = I(x_1', C_1) = -I(x_1', R_1)$. We obtain, after solving a system of equations,

$$I(x_2', C_1) = -I(x_2', R_2) = -2 - a\,, \quad I(x_2'', R_2) = -I(x_2'', C_2) = -a\,,$$
$$I(x_1'', R_1) = -I(x_1'', C_2) = a + 2\,.$$

When $a \leq -2$, we obtain

$$C_1 < R_1 \leq C_2 \leq R_2 \leq C_1\,,$$

a contradiction. The case $a \geq 0$ is similarly ruled out, which forces $a = -1$. Then the twists impose no further constraints on the point in $\overline{\mathcal{M}}_{3,2}$, so we obtain again a 4-dimensional subvariety of $\widetilde{\mathcal{H}}_3(2,2)$ which cannot be an irreducible component.

(v) The case $[C_1 \cup C_2 \cup R, p_1, p_2] \in \widetilde{\mathcal{H}}_3(2,2)$ with $p_1 \neq p_2 \in R \cong \mathbb{P}^1$.

We have the following component intersections:

$$C_1 \cap C_2 = \{x_2\}\,, \;\; C_1 \cap R = \{x_1'\}\,, \;\; C_2 \cap R = \{x_1''\}\,.$$

Arguments as above yield the conditions

$$\mathcal{O}_{C_1}(2x_1' - 2x_2) \cong \mathcal{O}_{C_1} \quad \text{and} \quad \mathcal{O}_{C_2}(2x_1'' - 2x_2) \cong \mathcal{O}_{C_2}$$

which constrains the locus to 3 dimensions in $\overline{\mathcal{M}}_{3,2}$.

Appendix: The weighted fundamental class of $\widetilde{\mathcal{H}}_g(\mu)$

by F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine

A.1 **Stable graphs and strata.**

In the strictly meromorphic case, the fundamental class of $\widetilde{\mathcal{H}}_g(\mu)$ weighted by intrinsic multiplicities is conjectured here to be an explicit cycle in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ found by Pixton in 2014. The formula is written in term of a summation over stable graphs Γ indexing the strata of $\overline{\mathcal{M}}_{g,n}$. The summand in Pixton's formula corresponding to Γ is a product over vertex, marking, and edge factors. We review here the standard indexing of the strata of $\overline{\mathcal{M}}_{g,n}$ by stable graphs.

The strata of the moduli space of curves correspond to *stable graphs*

$$\Gamma = (\mathrm{V}, \mathrm{H}, \mathrm{L},\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0},\ v : \mathrm{H} \to \mathrm{V},\ \iota : \mathrm{H} \to \mathrm{H})$$

satisfying the following properties:

(i) V is a vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : \mathrm{H} \to \mathrm{V}$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of ι and is endowed with a bijective correspondence with a set of markings,

(v) the pair (V, E) defines a *connected* graph,

(vi) for each vertex v, the stability condition holds:

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

where $\mathrm{n}(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave invariant the structures L, g, v, and ι. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by:

$$\mathrm{g}(\Gamma) = \sum_{v \in V} \mathrm{g}(v) + h^1(\Gamma).$$

A stratum of the moduli space $\overline{\mathcal{M}}_{g,n}$ of Deligne-Mumford stable curves naturally determines a stable graph of genus g with n legs by considering the dual graph of a generic pointed curve parametrized by the stratum. Let $\mathsf{G}_{g,n}$ be the set of isomorphism classes of stable graphs of genus g with n legs. The set $\mathsf{G}_{g,n}$ is finite.

To each stable graph Γ, we associate the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}.$$

Let π_v denote the projection from $\overline{\mathcal{M}}_\Gamma$ to $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ associated to the vertex v. There is a canonical morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \tag{28}$$

with image[14] equal to the stratum associated to the graph Γ. To construct ξ_Γ, a family of stable pointed curves over $\overline{\mathcal{M}}_\Gamma$ is required. Such a family is easily defined by attaching the pull-backs of the universal families over each of the $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ along the sections corresponding to half-edges.

A.2 **Additive generators of the tautological ring.**

Let Γ be a stable graph. A *basic class* on $\overline{\mathcal{M}}_\Gamma$ is defined to be a product of monomials in κ classes[15] at each vertex of the graph and powers of ψ classes at each half-edge (including the legs),

$$\gamma = \prod_{v \in \mathrm{V}} \prod_{i>0} \kappa_i[v]^{x_i[v]} \cdot \prod_{h \in \mathrm{H}} \psi_h^{y[h]} \in A^*(\overline{\mathcal{M}}_\Gamma) ,$$

where $\kappa_i[v]$ is the i^{th} kappa class on $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$. We impose the condition

$$\sum_{i>0} i x_i[v] + \sum_{h \in \mathrm{H}[v]} y[h] \le \dim_{\mathbb{C}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)} = 3\mathrm{g}(v) - 3 + \mathrm{n}(v)$$

at each vertex to avoid the trivial vanishing of γ. Here, $\mathrm{H}[v] \subset \mathrm{H}$ is the set of legs (including the half-edges) incident to v.

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n}$ whose basis is given by the isomorphism classes of pairs $[\Gamma, \gamma]$, where Γ is a stable graph of genus g with n legs and γ is a basic class on $\overline{\mathcal{M}}_\Gamma$. Since there are only finitely many pairs $[\Gamma, \gamma]$ up to isomorphism, $\mathcal{S}_{g,n}$ is finite dimensional. The canonical map

$$q : \mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n})$$

[14] The degree of ξ_Γ is $|\mathrm{Aut}(\Gamma)|$.

[15] Our convention is $\kappa_i = \pi_*(\psi_{n+1}^{i+1}) \in A^i(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ where

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

is the map forgetting the marking $n+1$. For a review of κ and and cotangent ψ classes, see [12].

is surjective [12] and provides additive generators of the tautological ring. The kernel of q is the ideal of tautological relations, see [18, Section 0.3].

A.3 Pixton's formula.

A.3.1 Weighting.

Let $\mu = (m_1, \ldots, m_n)$ be a vector of zero and pole multiplicities satisyfing

$$\sum_{i=1}^{n} m_i = 2g - 2\,.$$

It will be convenient to work with the shifted vector defined by

$$\widetilde{\mu} = (m_1 + 1, \ldots, m_n + 1)\,, \quad \widetilde{m}_i = m_i + 1\,.$$

Let Γ be a stable graph of genus g with n legs. An *admissible weighting* of Γ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma) \to \mathbb{Z},$$

which satisfies the following three properties:

(i) $\forall h_i \in \mathrm{L}(\Gamma)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(h_i) = \widetilde{m}_i\,,$$

(ii) $\forall e \in \mathrm{E}(\Gamma)$, corresponding to two half-edges $h(e), h'(e) \in \mathrm{H}(\Gamma)$,

$$w(h) + w(h') = 0$$

(iii) $\forall v \in \mathrm{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = 2\mathrm{g}(v) - 2 + \mathrm{n}(v)\,,$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident v.

Let r be a positive integer. An *admissible weighting mod* r of Γ is a function,

$$w : \mathrm{H}(\Gamma) \to \{0, \ldots, r-1\},$$

which satisfies exactly properties (i-iii) above, but with the equalities replaced, in each case, by the condition of *congruence mod* r. For example, for (i), we require

$$w(h_i) = \widetilde{m}_i \mod r\,.$$

Let $\mathsf{W}_{\Gamma,r}$ be the set of admissible weightings mod r of Γ. The set $\mathsf{W}_{\Gamma,r}$ is finite.

A.3.2 **Pixton's cycle.** Let r be a positive integer. We denote by $\mathsf{P}_{g,\mu}^{d,r} \in R^d(\overline{\mathcal{M}}_{g,n})$ the degree d component of the tautological class

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \Bigg[\prod_{v \in \mathrm{V}(\Gamma)} \exp(-\kappa_1[v]) \prod_{i=1}^{n} \exp(\widetilde{m}_i^2 \psi_{h_i}) \cdot \prod_{e=(h,h') \in \mathrm{V}(\Gamma)} \frac{1-\exp(-w(h)w(h')(\psi_h+\psi_{h'}))}{\psi_h+\psi_{h'}} \Bigg].$$

The following fundamental polynomiality property of $\mathsf{P}_{g,A}^{d,r}$ has been proven by Pixton.

Proposition (Pixton [19]). *For fixed g, μ, and d, the class*

$$\mathsf{P}_{g,\mu}^{d,r} \in R^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r for sufficiently large r.

We denote by $\mathsf{P}_{g,\mu}^{d}$ the value at $r = 0$ of the polynomial associated to $\mathsf{P}_{g,\mu}^{d,r}$ by Proposition 5.6. In other words, $\mathsf{P}_{g,\mu}^{d}$ is the *constant* term of the associated polynomial in r. The cycle $\mathsf{P}_{g,\mu}^{d}$ was constructed by Pixton [19] in 2014.

If $d > g$, Pixton conjectured (and Clader and Janda have proven [5]) the vanishing

$$\mathsf{P}_{g,\mu}^{d} = 0 \ \in R^*(\overline{\mathcal{M}}_{g,n}).$$

If $d = g$, the class $\mathsf{P}_{g,\mu}^{g}$ is non-trivial. When restricted to the moduli of curves of compact type, $\mathsf{P}_{g,\mu}^{g}$ is related to canonical divisors via the Abel-Jacob map by earlier work of Hain [13]. We propose a precise relationship between $\mathsf{P}_{g,\mu}^{g}$ and a weighted fundamental classes of the moduli space of twisted canonical divisors on $\overline{\mathcal{M}}_{g,n}$.

A.4 **The moduli of twisted canonical divisors.**

We consider here the strictly meromorphic case where $\mu = (m_1, \ldots, m_n)$ has at least one negative part. We construct a cycle

$$\mathsf{H}_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})$$

by summing over the irreducible components of the moduli space $\widetilde{\mathcal{H}}_g(\mu)$ of twisted canonical divisors defined in the paper.

Let $\mathsf{S}_{g,\mu} \subset \mathsf{G}_{g,n}$ be the set of *simple* star graphs defined by the following properties:

- $\Gamma \in \mathsf{S}_{g,\mu}$ has a single center vertex v_0,

- edges (possibly multiple) of Γ connect v_0 to outlying vertices $v_1, \ldots, v_k$.
- the negative parts of μ are distributed to the center v_0 of Γ.

A simple star graph Γ has no self-edges. Let $\mathrm{V}_{\mathrm{out}}(\Gamma)$ denote the set of outlying vertices. The simplest star graph consists of the center alone with no outlying vertices.

A twist I of a simple star graph is a function

$$I : \mathsf{E}(\Gamma) \to \mathbb{Z}_{>0}$$

satisfying:

- for the center v_0 of Γ,

$$2\mathrm{g}(v_0) - 2 + \sum_{e \mapsto v_0} (I(e) + 1) = \sum_{i \mapsto v_0} m_i \,,$$

- for each outlying vertex v_j of Γ,

$$2\mathrm{g}(v_j) - 2 + \sum_{e \mapsto v_j} (-I(e) + 1) = \sum_{i \mapsto v_j} m_i \,,$$

Let $\mathrm{Tw}(\Gamma)$ denote the finite set of possible twists.

The notation $i \mapsto v$ in above equations denotes markings (corresponding to the parts of μ) which are incident to the vertex v. We denote by $\mu[v]$ the vector consisting of the parts of μ incident to v. Similarly, $e \mapsto v$ denotes edges incident to the vertex v. We denote by $-I[v_0] - 1$ the vector of values $-I(e) - 1$ indexed by all edges incident to v_0, and by $I[v_j] - 1$ the vector of values $I(e) - 1$ indexed by all edges incident to an outlying vertex v_j.

We define the weighted fundamental class $\mathsf{H}_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})$ of the moduli space $\widetilde{H}_g(\mu)$ of twisted canonical divisors in the strictly meromorphic case by

$$\mathsf{H}_{g,\mu} = \sum_{\Gamma \in \mathsf{S}_{g,\mu}} \sum_{I \in \mathrm{Tw}(\Gamma)} \frac{\prod_{e \in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|} \, \xi_{\Gamma *} \Bigg[\Big[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu[v_0], -I[v_0] - 1\big) \Big] \cdot \prod_{v \in \mathrm{V}_{\mathrm{out}}(\Gamma)} \Big[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu[v], I[v] - 1\big) \Big] \Bigg] .$$

The right side of the definition of $\mathsf{H}_{g,\mu}$ may be viewed as a sum over all irreducible components of

$$Z \subset \widetilde{\mathcal{H}}_g(\mu) \,.$$

If the curves of Z generically do not have a separating node,

$$Z \subset \overline{\mathcal{H}}_g^{\mathrm{Irr}}(\mu) \subset \overline{\mathcal{M}}_{g,n} \,,$$

see Section 3.2. Since there is an equality of closures $\overline{\mathcal{H}}_g(\mu) = \overline{\mathcal{H}}_g^{\mathrm{Irr}}(\mu)$, we obtain

$$Z \subset \overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n} .$$

Hence, Z contributes to the term corresponding to the trivial star graph

$$\Gamma = \{v_0\}$$

of genus g carrying all the parts of μ.

The trivial star graph Γ has no edges, nothing to twist, and no automorphisms. Since ξ_Γ is the identity map here, the term corresponding to Γ is

$$\left[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu\big)\right] \in A^{\mathrm{g}(v_0)}(\overline{\mathcal{M}}_{g,n}) .$$

If Z has a separating node, then, by Section 2.2 of the paper, Z contributes to the term corresponding to the associated simple star graph. Every outlying vertex contributes the class

$$\left[\overline{\mathcal{H}}_{\mathrm{g}(v_j)}\big(\mu[v_j], I[v_j]-1\big)\right] \in A^{\mathrm{g}(v_j)-1}(\overline{\mathcal{M}}_{g,n})$$

corresponding to a canonical divisor in the holomorphic case (since the parts of $\mu[v_j]$ and $I[v_j]-1$ are non-negative).

The formula for $\mathrm{H}_{g,\mu}$ differs from the usual fundamental class of $\widetilde{\mathcal{H}}_g(\mu)$ by the weighting factor $\prod_{e\in \mathsf{E}(\Gamma)} I(e)$ which is motivated by relative Gromov-Witten theory.

Conjecture A. *In the strictly meromorphic case,* $\mathsf{H}_{g,\mu} = 2^{-g}\, \mathsf{P}_{g,\mu}^{g}$.

Our conjecture provides a completely geometric representative of Pixton's cycle in terms of twisted canonical divisors. The geometric situation here is parallel, but much better behaved, than the corresponding result of [14] proving Pixton's conjecture for the double ramification cycle (as the latter carries virtual contributions from contracted components of excess dimension).

Finally, we speculate that the study of volumes of the moduli spaces of meromorphic differentials may have a much simpler answer if summed over all the components of $\widetilde{\mathcal{H}}_g(\mu)$. How to properly define such volumes here is a question left for the future.

A.5 **Examples.**

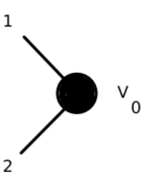
A.5.1 **Genus at most 1.** Conjecture A is trivial in case $g = 0$ since both sides are the the identity in $R^0(\overline{\mathcal{M}}_{0,n})$. In case $g = 1$, Conjecture A is nontrivial and true. The proof is indirect. We prove

- $\mathsf{H}_{1,\mu}$ equals the double ramification cycle in genus 1 associated to μ via geometric arguments.
- $2^{-1}\mathsf{P}^1_{1,\mu}$ equals Pixton's prediction for the double ramification cycle via direct analysis of the formulas.

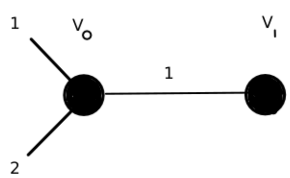
Then, by the main result of [14], Conjecture A holds in $g = 1$. An alternate approach via direct calculation is also available in $g = 1$.

A.5.2 **Genus 2.** A more interesting example is $g = 2$ with $\mu = (3, -1)$. We first enumerate all simple star graphs (together with their possible twists) which contribute to the cycle $\mathsf{H}_{2,(3,-1)}$:

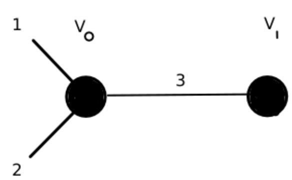
(i) $\Gamma = \{v_0\}$, $\mathrm{g}(v_0) = 2$, $|\mathrm{Aut}(\Gamma)| = 1$.



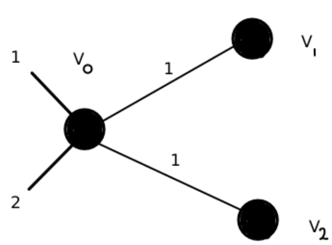
(ii) $\Gamma = \{v_0, v_1\}$, $\mathrm{g}(v_0) = 1$, $\mathrm{g}(v_1) = 1$, $|\mathrm{Aut}(\Gamma)| = 1$.



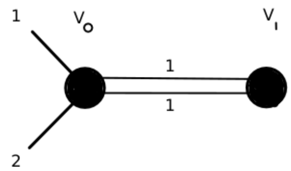
(iii) $\Gamma = \{v_0, v_1\}$, $\mathrm{g}(v_0) = 0$, $\mathrm{g}(v_1) = 2$, $|\mathrm{Aut}(\Gamma)| = 1$.



(iv) $\Gamma = \{v_0, v_1, v_2\}$, $\mathrm{g}(v_0) = 0$, $\mathrm{g}(v_1) = 1$, $\mathrm{g}(v_2) = 1$, $|\mathrm{Aut}(\Gamma)| = 2$.



(v) $\Gamma = \{v_0, v_1\}$, $\mathrm{g}(v_0) = 0$, $\mathrm{g}(v_1) = 1$, $|\mathrm{Aut}(\Gamma)| = 2$.



In the diagrams, the legs are assigned a marking, and the edges are assigned a twist. In all the cases Γ above, the twist $I \in \mathrm{Tw}(\Gamma)$ is unique.

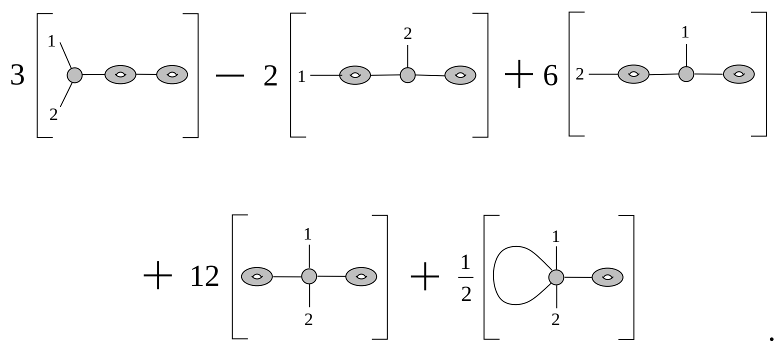
The second step is to calculate, for each star graph Γ and associated twist $I \in \mathrm{Tw}(\Gamma)$, the contribution

$$\frac{\prod_{e\in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|}\, \xi_{\Gamma*}\left[\left[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu[v_0], -I[v_0]-1\big)\right] \cdot \prod_{v\in \mathrm{V}_{\mathrm{out}}(\Gamma)} \left[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu[v], I[v]-1\big)\right]\right].$$

The contributions in cases (i)-(v) above are:

(i) The moduli space $\overline{\mathcal{H}}_2(3,-1)$ is empty since there are no meromorophic differentials with a simple pole. The contribution is 0.

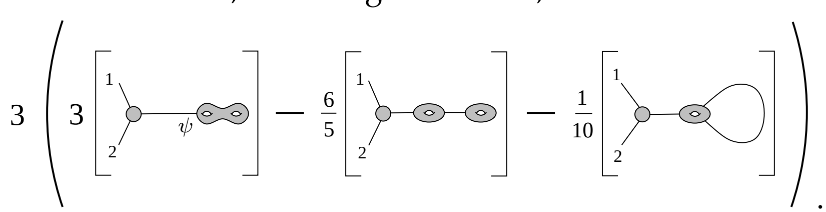
(ii) We must calculate here $\overline{\mathcal{H}}_1(3,-1,-2)$ which is easily obtained from the method of test curves (or by applying Conjecture A in the proven genus 1 case). The contribution is



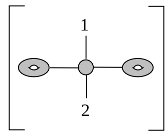
(iii) We require here the well-known formula for the Weierstrass locus,

$$\overline{\mathcal{H}}_2(2) \subset \overline{\mathcal{M}}_{2,1}\,,$$

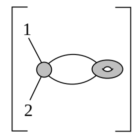
studied by Eisenbud and Harris (Lemma 5 of [2] follows our notation here). The contribution, including the twist 3, is



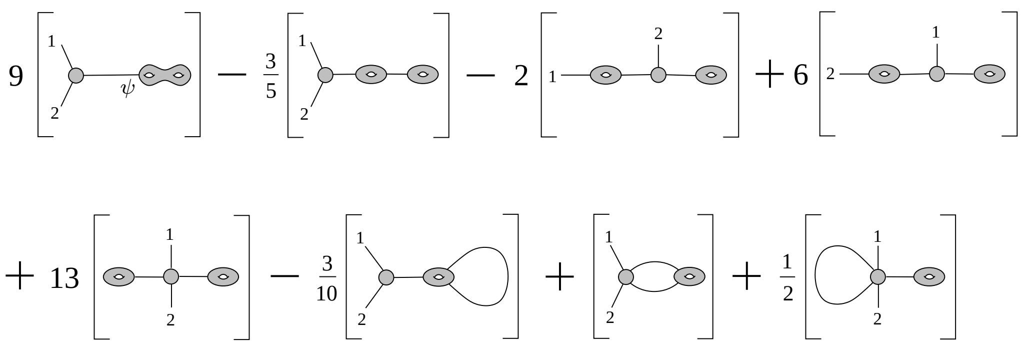
(iv) The locus is already codimension 2, so the contribution (including the automorphism factor) is



(v) The locus is again codimension 2, so the contribution (including the automorphism factor) is



After summing over the cases (i)-(v), we obtain a formula in $R^2(\overline{\mathcal{M}}_{2,2})$ for the weighted fundamental class $\mathsf{H}_{2,(3,-1)}$ of the moduli space of twisted canonical divisors:



The result exactly equals $2^{-2}\,\mathsf{P}^2_{2,(3,-1)} \in R^2(\overline{\mathcal{M}}_{2,2})$.

The match is obtained by expanding $2^{-2}\,\mathsf{P}^2_{2,(3,-1)}$ via the definition in Section A.3.2 and then applying relations in the tautological ring. Conjecture A is therefore true for $g = 2$ with $\mu = (3,-1)$.

We have also checked Conjecture A for $g = 2$ with $\mu = (2,1,-1)$. The calculation of $\mathsf{H}_{2,(2,1,-1)}$ involves 7 simple star graphs, Lemmas 5 and 6 of [2], and 23 strata classes of $R^2(\overline{\mathcal{M}}_{2,3})$. The matching with

$$2^{-2}\,\mathsf{P}^2_{2,(2,1,-1)} \in R^2(\overline{\mathcal{M}}_{2,3})$$

as predicted by Conjecture A exactly holds.

A.6 The cycle $\overline{\mathcal{H}}_g(\mu)$

A.6.1 **Meromorphic differentials.** In the strictly meromorphic case, where

$$\mu = (m_1, \ldots, m_n)$$

has at least one negative part, Conjecture A provides a formula in the tautological ring for the weighted fundamental class

$$\mathsf{H}_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})\,.$$

Can Conjecture A also be used to determine the class

$$[\overline{\mathcal{H}}_g(\mu)] \in A^g(\overline{\mathcal{M}}_{g,n})$$

of closure of the moduli of canonical divisors $\mathcal{H}_g(\mu)$ on nonsingular curves ? We will prove that the answer is *yes*.

We have seen $[\overline{\mathcal{H}}_g(\mu)]$ appears exactly as the contribution to $\mathsf{H}_{g,\mu}$ of the trivial simple star graph. Rewriting the definition of $\mathsf{H}_{g,\mu}$ yields the formula:

$$[\overline{\mathcal{H}}_g(\mu)] = \mathsf{H}_{g,\mu} - \sum_{\Gamma \in \mathsf{S}^\star_{g,\mu}} \sum_{I \in \mathrm{Tw}(\Gamma)} \frac{\prod_{e\in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|}\, \xi_{\Gamma *}\left[\left[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu[v_0], -I[v_0]-1\big)\right] \cdot \prod_{v \in \mathrm{V}_{\mathrm{out}}(\Gamma)} \left[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu[v], I[v]-1\big)\right]\right] \tag{29}$$

where $\mathsf{S}^\star_{g,\mu}$ is the set of *nontrivial* simple star graphs.

A nontrivial simple star graph $\Gamma \in \mathsf{S}^\star_{g,\mu}$ must have at least one outlying vertex. An outlying vertex $v \in \mathrm{V}_{\mathrm{out}}(\Gamma)$ contributes the factor

$$\left[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu[v], I[v]-1\big)\right]$$

which concerns the *holomorphic* case. Hence if $\mathrm{g}(v) = 0$, the contribution of Γ vanishes (as there are no holomorphic differential in genus 0).

To state the first induction result, we require the standard partial order on the pairs (g,n). We define

$$(\widehat{g}, \widehat{n}) \overset{\circ}{<} (g,n) \tag{30}$$

if $\widehat{g} < g$ holds or if

$$\widehat{g} = g \quad \text{and} \quad \widehat{n} < n$$

both hold.

Lemma A1. *Let $\mu = (m_1, \ldots, m_n)$ be strictly meromorphic. The class*

$$[\overline{\mathcal{H}}_g(\mu)] \in A^g(\overline{\mathcal{M}}_{g,n})$$

is determined via formula (29) *by classes of the following three types:*

(i) $\mathsf{H}_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})$,

(ii) $[\overline{\mathcal{H}}_{g'}(\mu')] \in A^{g'}(\overline{\mathcal{M}}_{g',n'})$ *for* $(g', n') \overset{\circ}{<} (g, n)$ *and* $\mu' = (m'_1, \ldots, m'_{n'})$ *strictly meromorphic,*

(iii) $[\overline{\mathcal{H}}_{g''}(\mu'')] \in A^{g''-1}(\overline{\mathcal{M}}_{g'',n''})$ *for* $(g'', n'') \overset{\circ}{<} (g, n)$ *and* $\mu'' = (m''_1, \ldots, m''_{n''})$ *holomorphic.*

Proof. The class (i) appears as the leading term on the right side of (29). Nontrivial simple star graphs Γ contribute classes of type (ii) via the center vertex v_0 (with

$$g' = \mathrm{g}(v_0) < g$$

or else the contribution of Γ vanishes). The outlying vertices v of Γ contribute classes of type (iii). If

$$g'' = \mathrm{g}(v) = g\,,$$

then $\mathrm{g}(v_0) = 0$ and there are no outlying vertices other than v (or else the contribution of Γ vanishes) and only a single edge. By stability, at least two parts of μ must be distributed to v_0, so $n'' < n$. □

Lemma A.1 alone does *not* let us recursively calculate $[\overline{\mathcal{H}}_g(\mu)]$ in terms of $\mathsf{H}_{g,\mu}$ because of occurances of the holomorphic cases (iii).

A.6.2 **Holomorphic differentials.** Consider the holomorphic case where

$$\mu = (m_1, \ldots, m_n)$$

has only non-negative parts. Since

$$\sum_{i=1}^{n} m_i = 2g - 2$$

we must have $g \geq 1$. If $g = 1$, then all parts of μ must be 0 and

$$[\overline{\mathcal{H}}_1(0, \ldots, 0)] = 1 \in A^0(\overline{\mathcal{M}}_{1,n})\,.$$

We assume $g \geq 2$.

Let $\mu' = (m'_1, \ldots, m'_{n'})$ be obtained from μ by removing the parts equal to 0. Then $[\overline{\mathcal{H}}_g(\mu)]$ is obtained by pull-back[16] via the map

$$\tau : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n'}$$

forgetting the markings associated to the 0 parts of μ,

$$[\overline{\mathcal{H}}_g(\mu)] = \tau^*[\overline{\mathcal{H}}_g(\mu')] \in A^{g-1}(\overline{\mathcal{M}}_{g,n}) \,. \tag{31}$$

We assume μ has no 0 parts, so all parts m_i are positive. We place the parts of μ in increasing order

$$m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq m_n \,,$$

so m_n is the largest part. Let μ^+ be the partition defined by

$$\mu^+ = (m_1, \ldots, m_{n-1}, m_n + 1, -1) \,.$$

We have increased the largest part of μ, added a negative part, and preserved the sum

$$|\mu| = |\mu^+| \,.$$

For notational convenience, we will write

$$\mu^+ = (m_1^+, \ldots, m_n^+, -1) \,, \quad m_{i<n}^+ = m_i \,, \quad m_n^+ = m_n + 1 \,.$$

The length of μ^+ is $n + 1$.

Since μ^+ is strictly meromorphic, we are permitted to apply formula (29). We obtain

$$[\overline{\mathcal{H}}_g(\mu^+)] = \mathsf{H}_{g,\mu^+} - \sum_{\Gamma \in \mathsf{S}^\star_{g,\mu^+}} \sum_{I \in \mathrm{Tw}(\Gamma)} \frac{\prod_{e \in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|} \, \xi_{\Gamma *} \Bigg[\Big[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu^+[v_0], -I[v_0] - 1\big)\Big] \cdot \prod_{v \in \mathrm{V}_{\mathrm{out}}(\Gamma)} \Big[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu^+[v], I[v] - 1\big)\Big] \Bigg]$$

where $\mathsf{S}^\star_{g,\mu^+}$ is the set of nontrivial simple star graphs.

Since a meromorphic differential on a nonsigular curve *can not* have just one simple pole, $\mathcal{H}_g(\mu^+)$ is empty and $[\overline{\mathcal{H}}_g(\mu^+)] = 0$. We rewrite the above equation

[16]The pull-back is true on the level of subvarieties, $\overline{\mathcal{H}}_g(\mu) = \tau^{-1}\Big(\overline{\mathcal{H}}_g(\mu')\Big) \subset \overline{\mathcal{M}}_{g,n}$.

as

$$\mathsf{H}_{g,\mu^+} = \sum_{\Gamma\in \mathsf{S}^\star_{g,\mu^+}} \sum_{I\in \mathrm{Tw}(\Gamma)} \frac{\prod_{e\in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|}\, \xi_{\Gamma*}\Bigg[\Big[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu^+[v_0], -I[v_0]-1\big)\Big] \cdot \prod_{v\in \mathrm{V}_{\mathrm{out}}(\Gamma)} \Big[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu^+[v], I[v]-1\big)\Big]\Bigg]. \tag{32}$$

A nonvanishing term on right side of (32) corresponding to $\Gamma \in \mathsf{S}^\star_{g,\mu^+}$ has a center vertex factor

$$\Big[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu^+[v_0], -I[v_0]-1\big)\Big] \quad \text{satisfying} \quad \mathrm{g}(v_0) < g$$

and outlying vertex factors

$$\Big[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu^+[v], I[v]-1\big)\Big] \tag{33}$$

satisfying either $\mathrm{g}(v) < g$ or $\mathrm{g}(v) = g$. If $\mathrm{g}(v) = g$, then the entire genus of Γ is concentrated on v and there can be no other outlying vertices (or else the contribution of Γ vanishes) and only a single edge. By stability, at least two parts of μ^+ must be distributed to v_0, so $\mathrm{n}(v) \leq n$. Hence, the outlying vertex factors (33) satisfy either $\mathrm{g}(v) < g$ or

$$\mathrm{g}(v) = g \quad \text{and} \quad \mathrm{n}(v) \leq n\,. \tag{34}$$

We study now all the contributions of graphs $\Gamma \in \mathsf{S}^\star_{g,\mu^+}$ which carry an outlying vertex $v \in \mathrm{V}_{\mathrm{out}}(\Gamma)$ satisfying

$$\mathrm{g}(v) = g \quad \text{and} \quad \mathrm{n}(v) = n\,,$$

the extremal case of (34). We have seen

$$\Gamma = \{v_0, v_1\} \textit{ with a single edge } e \textit{ and } \mathrm{g}(v_0) = 0,\ \mathrm{g}(v_1) = g \tag{35}$$

is the only possibility for a nonvanishing contribution. By definition, the negative part of μ^+ must be distributed to v_0. In order for $\mathrm{n}(v_1) = n$, exactly one positive part m_i^+ of μ^+ must also be distributed to v_0. Let $\Gamma_i \in \mathsf{S}^\star_{g,\mu^+}$ be the simple star graph of type (35) with marking distribution

$$\{m_i^+, -1\} \mapsto v_0\,, \quad \{m_1^+, \ldots, \widehat{m_i^+}, \ldots, m_n^+\} \mapsto v_1\,. \tag{36}$$

The graphs $\Gamma_1,\ \Gamma_2,\ \ldots,\ \Gamma_n$ are the only elements of $\mathsf{S}^\star_{g,\mu^+}$ which saturate the bounds (34) and have possibly nonvanishing contributions to the right side of (32).

The negative part of μ^+ corresponds to the last marking of the associated moduli space $\overline{\mathcal{M}}_{g,n+1}$. Let

$$\epsilon : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

be the map forgetting the last marking. We push-forward formula (32) under ϵ,

$$\epsilon_* \mathsf{H}_{g,\mu^+} = \sum_{\Gamma \in \mathsf{S}^\star_{g,\mu^+}} \sum_{I \in \mathrm{Tw}(\Gamma)} \frac{\prod_{e \in \mathsf{E}(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)|} \epsilon_* \xi_{\Gamma *} \Bigg[\Big[\overline{\mathcal{H}}_{\mathrm{g}(v_0)}\big(\mu^+[v_0], -I[v_0]-1\big) \Big] \cdot \prod_{v \in \mathrm{V}_{\mathrm{out}}(\Gamma)} \Big[\overline{\mathcal{H}}_{\mathrm{g}(v)}\big(\mu^+[v], I[v]-1\big) \Big] \Bigg] \tag{37}$$

to obtain an equation in $A^{g-1}(\overline{\mathcal{M}}_{g,n})$.

We study the precise contribution to the right side of (37) of the graphs Γ_i characterized by (35) and (36). The graph Γ_i has a unique possible twist[17]

$$I(e) = m_i^+ \ .$$

The contribution of Γ_i is

$$m_i^+ \cdot \epsilon_* \xi_{\Gamma_i *} \Bigg[\Big[\overline{\mathcal{H}}_0\big(m_i^+, -1, -m_i^+ - 1\big) \Big] \cdot \Big[\overline{\mathcal{H}}_g\big(m_1^+, \ldots, m_{i-1}^+, m_i^+ - 1, m_{i+1}^+, \ldots m_n^+\big) \Big] \Bigg]$$

where we have

$$\epsilon \circ \xi_{\Gamma_i} : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{g,n} \xrightarrow{\sim} \overline{\mathcal{M}}_{g,n} \ . \tag{38}$$

Using the isomorphism (38), the contribution of Γ_i is simply

$$\mathrm{Cont}(\Gamma_i) = m_i^+ \cdot \Big[\overline{\mathcal{H}}_g\big(m_1^+, \ldots, m_{i-1}^+, m_i^+ - 1, m_{i+1}^+, \ldots m_n^+\big) \Big] \in A^{g-1}(\overline{\mathcal{M}}_{g,n}) \ .$$

The contribution of Γ_n to the right side of (37) is special. Since by construction $m_n^+ - 1 = m_n$, we see

$$\mathrm{Cont}(\Gamma_n) = (m_n + 1) \cdot \Big[\overline{\mathcal{H}}_g\big(m_1, \ldots, m_n\big) \Big] = (m_n + 1) \cdot [\overline{\mathcal{H}}_g\big(\mu\big)] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$$

with $m_n + 1 \neq 0$. The contributions of the graphs $\Gamma_1, \ldots, \Gamma_{n-1}$ are proportional to classes

$$[\overline{\mathcal{H}}_g\big(\mu'\big)] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$$

for non-negative partitions μ' with largest part *larger* than the largest part of μ.

Lemma A2. *Let $\mu = (m_1, \ldots, m_n)$ be holomorphic with no 0 parts. The class*

$$[\overline{\mathcal{H}}_g(\mu)] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$$

[17] We use the condition $m_i^+ > 0$ here.

is determined via formula (37) *by classes of the following four types:*

(i) $\mathsf{H}_{g,\mu^+} \in A^g(\overline{\mathcal{M}}_{g,n})$,

(ii) $[\overline{\mathcal{H}}_{g'}(\mu')] \in A^{g'}(\overline{\mathcal{M}}_{g',n'})$ *for* $(g',n') \overset{\circ}{<} (g,n)$ *and* $\mu' = (m'_1,\ldots,m'_{n'})$ *strictly meromorphic,*

(iii) $[\overline{\mathcal{H}}_{g''}(\mu'')] \in A^{g''-1}(\overline{\mathcal{M}}_{g'',n''})$ *for* $(g'',n'') \overset{\circ}{<} (g,n)$ *and* $\mu'' = (m''_1,\ldots,m''_{n''})$ *holomorphic.*

(iv) $[\overline{\mathcal{H}}_g(\mu'')] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$ *for* $\mu'' = (m''_1,\ldots,m''_n)$ *holomorphic with the largest part of* μ'' larger *than the largest part of* μ*.*

Proof. The ϵ push-forward of the class (i) appears on the left of (37). Nontrivial simple star graphs not equal to $\Gamma_1,\ldots,\Gamma_n$ contribute (ii) via the center vertex and (iii) via the outlying vertices. The graphs $\Gamma_1,\ldots,\Gamma_{n-1}$ contribute (iv). We then solve (37) for the contribution $(m_n+1)\cdot[\overline{\mathcal{H}}_g(\mu)]$ of Γ_n . □

A.6.3 **Determination.** We now combine Lemmas A1 and A2 to obtain the following basic result.

Theorem A3. *Conjecture A effectively determines the classes*

$$[\overline{\mathcal{H}}_g(\mu)] \in A^*(\overline{\mathcal{M}}_{g,n})$$

both in the holomorphic and the strictly meromorphic cases.

Proof. If $g=0$, the class $[\overline{\mathcal{H}}_g(\mu)]$ vanishes in the holomorphic case and is the indentity $1 \in A^*(\overline{\mathcal{M}}_{0,n})$ in the strictly meromorphic case. We proceed by induction with respect to the partial ordering $\overset{\circ}{<}$ on pairs defined by (30).

If μ is strictly meromorphic, we apply Lemma A1. Conjecture A determines the class

$$\mathsf{H}_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})\,.$$

The rest of the classes specified by Lemma A1 are *strictly lower* in the partial ordering $\overset{\circ}{<}$.

If μ is holomorphic and μ has parts equal to 0, then either

$$g=1\,,\ \ \mu=(0,\ldots,0)\,,\ \text{ and } \ [\overline{\mathcal{H}}_g(0,\ldots,0)] = 1 \in A^0(\overline{\mathcal{M}}_{1,n})$$

or μ' (obtained by removing the 0 parts) yields a class

$$[\overline{\mathcal{H}}_g(\mu')] \in A^{g-1}(\overline{\mathcal{M}}_{g,n'})$$

which is *strictly lower* in the partial ordering $\overset{\circ}{<}$. We then apply the pull-back (31) to determine $[\overline{\mathcal{H}}_g(\mu)] \in \overline{\mathcal{M}}_{g,n}$.

We may therefore assume μ is holomorphic with no 0 parts. We then apply Lemma A2. Conjecture A determines the class

$$\epsilon_* \mathsf{H}_{g,\mu^+} \in A^{g-1}(\overline{\mathcal{M}}_{g,n}) \, .$$

The rest of the classes specified by Lemma A2 either are *strictly lower* in the partial ordering $\overset{\circ}{<}$ or are

(iv) $[\overline{\mathcal{H}}_g(\mu'')] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$ for $\mu'' = (m''_1, \ldots, m''_n)$ holomorphic with the largest part of μ'' *larger* than the largest part of μ.

To handle (iv), we apply descending induction on the largest part of μ in the holomorphic case. The base for the descending induction occurs when the largest part is $2g-2$, then there are *no partitions with larger largest part*. □

We have presented a calculation of the classes of the closures

$$\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

of the moduli spaces of canonical divisors on nonsingular curves via Conjecture A *and* the virtual components of the moduli space of twisted canonical divisors. Theorem A3 yields an effective method with result

$$[\overline{\mathcal{H}}_g(\mu)] \in R^*(\overline{\mathcal{M}}_{g,n}) \, .$$

The $g = 2$ cases with $\mu = (3, -1)$ and $\mu = (2, 1, -1)$ discussed in Section A.5 may also be viewed as steps in the recursion of Theorem A3 to calculate

$$\mathsf{H}_{2,(2)} \in R^1(\overline{\mathcal{M}}_{2,1}) \quad \text{and} \quad \mathsf{H}_{2,(1,1)} \in R^1(\overline{\mathcal{M}}_{2,2}) \tag{39}$$

respectively. Since we already have formulas for the classes (39), the calculations serve to check Conjecture A.

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HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK
UNTER DEN LINDEN 6, 10099 BERLIN, GERMANY
E-mail address: farkas@math.hu-berlin.de

ETH ZÜRICH, DEPARTMENT OF MATHEMATICS, RAEMISTRASSE 101
8092 ZÜRICH, SWITZERLAND
E-mail address: rahul@math.ethz.ch