LINEAR SYZYGIES OF CURVES WITH PRESCRIBED GONALITY

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Abstract. We prove two statements concerning the linear strand of the minimal free resolution of a k-gonal curve C of genus g. Firstly, we show that a general curve C of genus g of non-maximal gonality $k \leq \frac{g+1}{2}$ satisfies Schreyer's Conjecture, that is, $b_{g-k,1}(C, K_C) = g-k$. This is a statement going beyond Green's Conjecture and predicts that all highest order linear syzygies in the canonical embedding of C are determined by the syzygies of the $(k-1)$-dimensional scroll containing C. Secondly, we formulate an optimal effective version of the Gonality Conjecture and prove it for general k-gonal curves. This generalizes the asymptotic Gonality Conjecture proved by Ein-Lazarsfeld and improves results of Rathmann in the case where C is a general curve of fixed gonality.

0. Introduction

0.1. **The effective gonality conjecture.** Let C be a smooth complex algebraic curve and L a very ample line bundle on C inducing an embedding $\varphi_L : C \hookrightarrow \mathbf{P}H^0(C, L)$. In order to describe the equations of this embedding, after setting $r := r(L)$, we consider the finitely generated graded $S := \mathrm{Sym}\ H^0(C, L) \cong \mathbb{C}[x_0, \ldots, x_r]$-module $\Gamma_C(L) := \bigoplus_n H^0(C, L^{\otimes n})$. By the *Hilbert Syzygy Theorem*, one has a minimal free resolution

$$0 \longrightarrow F_{r+1} \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \Gamma_C(L) \longrightarrow 0,$$

where

$$F_p = \bigoplus_{q>0} K_{p,q}(C, L) \otimes S(-p-q),$$

with $K_{p,q}(C, L)$ being the Koszul cohomology group of p-th order syzygies of weight q. As usual, the *graded Betti numbers* of (C, L) are defined by $b_{p,q} := \dim K_{p,q}(C, L)$. If L is non-special, then $K_{p,q}(C, L) = 0$ for all $q \geq 3$. Accordingly, the graded Betti diagram of (C, L) consists only of two non-trivial rows: the linear strand ($q = 1$) and the quadratic strand ($q = 2$).

The quadratic strand of the resolution is the subject of the Green-Lazarsfeld Secant Conjecture [GL1] and has been studied extensively in [FK], [K2]. The linear row is the subject of the Gonality Conjecture formulated in the same paper [GL1].

Assume C is k-gonal and let L be a line bundle on C of degree $\deg(L) \geq 2g - 1 + k$. By the Green-Lazarsfeld Nonvanishing Theorem [G, Appendix], one has $K_{h^0(L)-k-1,1}(C, L) \neq 0$. In a major breakthrough, Ein and Lazarsfeld [EL] proved that for an *arbitrary* smooth curve C, if $\deg(L) \gg 0$, then

$$K_{h^0(L)-k,1}(C, L) = 0. \tag{1}$$

This result has been significantly improved by Rathmann [R], who showed that the vanishing (1) holds for every smooth curve C when $\deg(L) \geq 4g - 3$. Earlier, an asymptotic version of (1) had been established by Aprodu and Voisin [AV] in the case of general k-gonal curves. As already indicated in the original paper [GL1] Conjecture 3.7, one can ask for an effective version of the Gonality Conjecture. We put forward the following:

Conjecture 0.1. *We say that a smooth curve C of genus g and gonality k satisfies the Effective Gonality Conjecture if for each line bundle L on C of degree* $\deg(L) \geq 2g-1+k$*, one has*

$$K_{h^0(L)-k,1}(C,L)=0.$$

While the original Gonality Conjecture has been formulated as an asymptotic statement in $\deg(L)$, the Effective Gonality Conjecture is already raised as a possibility in [GL1, page 86]. Clearly Conjecture 0.1 implies $K_{p,1}(C,L)=0$, for all $p \geq h^0(C,L)-k$. The bound on $\deg(L)$ in Conjecture 0.1 is optimal. Indeed, if $A \in W^1_k(C)$ is a pencil of minimal degree, then

$$K_{g-1,1}(C,\omega_C \otimes A) \neq 0,$$

by the Green–Lazarsfeld Nonvanishing Theorem, that is, on every curve there exist line bundles of degree $2g-2+k$ which do not verify the Gonality Conjecture. In light of this fact, the Effective Gonality Conjecture admits the following equivalent reformulation: If on a curve C there exists a line bundle $L \in \mathrm{Pic}^{2g-1+k}(C)$ such that $K_{g,1}(C,L) \neq 0$, then $\mathrm{gon}(C) \leq k-1$.

Our first result is then:

Theorem 0.2. *The Effective Gonality Conjecture holds for a general k-gonal of genus $g \geq 4$.*

The statement fails for $g=3$. Indeed, in this case the general curve is trigonal and it is easy to see that $K_{3,1}(C,\omega_C^{\otimes 2}) \neq 0$, using the fact that the canonical linear system embeds C in the plane. It remains an important question to determine precisely which curves satisfy the Effective Gonality Conjecture. For hyperelliptic curves, or when $k=3$ and $g \geq 4$, an arbitrary curves satisfies the Effective Gonality Conjecture, by Green's $K_{p,1}$-theorem, see [G, Theorem 3.c.1]. Similarly, Conjecture 0.1 holds for each 4-gonal curve of genus $g \geq 7$, see [Te, Proposition 3.8] or [AS]. It seems plausible that for each k, there is a constant $g(k)$ such that the Effective Gonality Conjecture is true for every curve of genus $g \geq g(k)$. Furthermore, extrapolating [G, Theorem 3.c.1], all exceptions to the Effective Gonality Conjecture should correspond to curves $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^r$ lying on a variety of low degree.

For curves of maximal gonality of odd genus $g \geq 5$, our results are complete:

Theorem 0.3. *The Effective Gonality Conjecture is valid for every smooth curve of odd genus $g \geq 5$ and maximal gonality.*

Theorem 0.3, which plays an essential role in the proof of Theorem 0.2 turns out to be intimately related to the divisorial case of the Green-Lazarsfeld Secant Conjecture proved in full generality [FK, Theorem 1.4]. We observe that using [FK], if C is a smooth curve of genus $g=2n+1$ and gonality $n+2$, the following equivalence holds for a line bundle $M \in \mathrm{Pic}^{2g}(C)$:

$$K_{n,1}(C,M) \neq 0 \Longleftrightarrow M-K_C \in C_{n+1}-C_{n-1}. \tag{2}$$

The right hand side denotes the divisorial difference variety $C_{n+1}-C_{n-1} \subseteq \mathrm{Pic}^2(C)$. An argument involving the geometry of secant varieties for line bundles on C then shows that (2) implies the vanishing $K_{g,1}(C,L)=0$, for *every* line bundle $L \in \mathrm{Pic}^{5n+3}(C)$, thus establishing Theorem 0.3. In order to deduce Theorem 0.2, we fix a value for the gonality $k \leq \frac{g+3}{2}$ and perform induction on the genus g; the initial step is Theorem 0.3. By induction, assume that the general smooth curve C of genus g and gonality k satisfies the Effective Gonality Conjecture. The stable curve X of genus $g+1$ obtained by adding an elliptic curve E at a point of ramification of a degree k pencil on C lies in the limit in $\overline{\mathcal{M}}_{g+1}$ of the locus of smooth k-gonal curves of genus $g+1$. An analysis of syzygies of line bundles of bidegree $(2g+k,1)$ on X allows us to deduce the Effective Gonality Conjecture for a smooth deformation of X having gonality k.

0.2. **Schreyer's Conjecture.** Consider a general k-gonal curve canonically embedded curve $C \hookrightarrow \mathbf{P}^{g-1}$ of gonality k. Green's Conjecture, which is known in this case, see [V1], [V2], [Ap2] and asserts that

$$K_{p,1}(C, K_C) = 0 \quad \text{if and only if} \quad p \geq g - k + 1,$$

determines the length of the linear (as well as that of the quadratic) strand of the resolution of C. Schreyer's Conjecture [Sch1, §6] and [SSW] addresses the more refined question of what actually is the Betti diagram of C, that is, determine the values $b_{p,1}(C, K_C)$ for $p \leq g - k$. Note that in the case when C has the same gonality as a general curve of genus g, i.e. $\mathrm{gon}(C) = \lfloor \frac{g+3}{2} \rfloor$, and only in this case, Green's Conjecture determines the entire resolution of C. Indeed, in this case Green's Conjecture is equivalent to the statement that the resolution of $C \subseteq \mathbf{P}^{g-1}$ is *natural*, that is,

$$b_{p,2}(C, K_C) \cdot b_{p+1,1}(C, K_C) = 0$$

for all p. Since the differences $b_{p+1,1}(C, K_C) - b_{p,2}(C, K_C)$ are known and independent of C, knowing which Betti numbers vanish amounts to knowing the entire Betti diagram.

Assume now $\mathrm{gon}(C) \leq \frac{g+1}{2}$, that is, C has non-maximal gonality. In this case, Green's Conjecture predicts the following resolution, where we observe that $b_{p,1}(C, K_C) \cdot b_{p,2}(C, K_C) \neq 0$ for $k - 2 \leq p \leq g - k$.

1	2	...	$k-3$	$k-2$	...	$g-k$	$g-k+1$	...	$g-2$
$b_{1,1}$	$b_{2,1}$	...	$b_{k-3,1}$	$b_{k-2,1}$	...	$b_{g-k,1}$	0	...	0
0	0	...	0	$b_{k-2,2}$	...	$b_{g-k,2}$	$b_{g-k+1,2}$	...	$b_{g-2,2}$

TABLE 1. The Betti table of a general canonical k-gonal curve of genus g.

It is known [AC] that such a curve C carries a unique pencil $A \in W^1_k(C)$ of minimal degree, inducing a $(k-1)$-dimensional scroll $X \subseteq \mathbf{P}^{g-1}$ swept out by the fibres of $|A|$. The Betti numbers of $(X, \mathcal{O}_X(1))$ are determined by the Eagon-Northcott complex, see [Sch1]. Since $C \subseteq X \subseteq \mathbf{P}^{g-1}$, one has the following inequality (see also Section 4)

$$b_{p,1}(C, K_C) \geq b_{p,1}\big(X, \mathcal{O}_X(1)\big) = p \cdot \binom{g-k+1}{p+1}. \tag{3}$$

It was originally expected that the inequality (3) is always an equality for $p \geq \lceil \frac{g-1}{2} \rceil$. This, however, is now known to fail. Indeed, Bopp [B] showed that the for a general 5-gonal curve of sufficiently high genus, if $m := \lceil \frac{g-1}{2} \rceil$, then $b_{m,1}(C, K_C) > b_{m,1}(X, \mathcal{O}_X(1))$. Schreyer's Conjecture [SSW] concerns the value of the highest non-zero Betti number in the linear strand and predicts that in this case, under suitable generality assumptions, inequality (3) is an equality.

Conjecture 0.4 (Schreyer's Conjecture)**.** *Let C be a curve of genus g and non-maximal gonality $3 \leq k \leq \frac{g+1}{2}$. Assume $W^1_k(C) = \{A\}$ is a reduced single point and A is the unique line bundle of degree at most $g-1$ achieving the Clifford index. Then*

$$b_{g-k,1}(C, K_C) = g - k \text{ and } b_{p,1}(C, K_C) = 0, \text{ for } p > g - k.$$

One direction of this conjecture is straightforward. Indeed, if $W^1_k(C)$ does *not* consist of a reduced single point, then $b_{g-k,1}(C, K_C) > g - k$, see [SSW, Proposition 4.10]. As already pointed out, Green's conjecture is known for general curves in each gonality stratum. Thus Schreyer's Conjecture in the case of generic k-gonal curves, purely concerns the condition

$$b_{g-k,1}(C, K_C) = g - k.$$

In fact, Schreyer further conjectures that unless C is isomorphic to a smooth plane quintic, the condition $b_{g-k,1}(C, K_C) = g - k$ automatically implies the vanishing statements $b_{p,1}(C, K_C) = 0$, for $p > g-k$, see [SSW, Conjecture 4.3]. Conjecture 0.2 is known to hold for a *general* k-gonal curve provided $(k-1)^2 < g$, see [Sch2]. An important piece of evidence for the conjecture is the case of general k-gonal curves of odd genus $2k-1$. Such curves form a divisor $\mathfrak{Hur}$ in the moduli space $\mathcal{M}_{2k-1}$, much studied by Harris and Mumford in [HM]. Combining results in [HR] and those in [V2], it follows that Conjecture 0.2 holds in this case. Outside this divisorial range, little has been known. The main result of this paper is the following:

Theorem 0.5. *Schreyer's Conjecture holds for a general k-gonal curve C of genus $g \geq 2k-1$:*

$$b_{g-k,1}(C, K_C) = g-k.$$

Part of Theorem 0.5 is that there is a canonical identification

$$K_{g-k,1}(C,K_C) \cong \bigwedge^{g-k+1} H^0(C, K_C \otimes A^{\vee})^{\vee} \otimes \mathrm{Sym}^{g-k-1} H^0(C,A) \otimes \bigwedge^{2} H^0(C,A),$$

where A is the unique degree k pencil on C. *All* the $(g-k)$-th syzygies linear syzygies of the canonical curve $C \subseteq \mathbf{P}^{g-1}$ are of *Eagon-Nothcott type* and can be written down explictely. Precisely, if $(\tau_0, \ldots, \tau_{g-k})$ is a basis of $H^0(C, K_C \otimes A^{\vee})$ and $\sigma \in H^0(C,A)$, then the syzygy corresponding to the power $\sigma^{g-k+1} \in \mathrm{Sym}^{g-k+1} H^0(C,A)$ has the form

$$\sum_{j=1}^{g-k} (-1)^j (\sigma\tau_1) \wedge \ldots \widehat{(\sigma\tau_j)} \wedge \ldots (\sigma\tau_{g-k}) \wedge \big\{(\sigma\tau_0) \otimes (\sigma'\tau_j) - (\sigma'\tau_0) \otimes (\sigma\tau_j)\big\} \in \bigwedge^{g-k} H^0(K_C) \otimes H^0(K_C),$$

where $\sigma' \in H^0(C,A)$ is another section such that (σ, σ') form a basis of $H^0(C,A)$.

The proof of Theorem 0.5 begins in Section 3 with the already mentioned observation that via [HR] and [V2], a smooth curve C of genus $2k-1$ and gonality k satisfies $b_{k-1,1}(C,K_C) = k-1$, provided $W^1_k(C)$ is integral of dimension zero. Consider the Hurwitz space $\mathcal{H}_{2k-1,k}$ of smooth curves of genus g which are k-fold covers of $\mathbf{P}^1$. We define the *Eagon-Northcott* divisor $\mathcal{EN}$ on $\mathcal{H}_{2k-1,k}$ parametrizing moduli points $[f : C \to \mathbf{P}^1] \in \mathcal{H}_{2k-1,k}$ with $b_{k-1,1}(C,K_C) > k-1$. In other words, points of $\mathcal{EN}$ correspond to canonical curves $C \subseteq \mathbf{P}^{g-1}$ having a $(g-k)$-th order linear syzygy which is *not* of Eagon-Northcott type. We also consider the Brill-Noether type divisor $\mathfrak{BN}$ on $\mathcal{H}_{2k-1,k}$ consisting of points $[f : C \to \mathbf{P}^1]$, such that C has an extra pencil of degree k. By the above discussion these two divisors coincide set-theoretically, that is,

$$\mathcal{EN} = \mathfrak{BN}.$$

Now suppose we are no longer in the divisorial case and choose $k \leq \frac{g+1}{2}$. We follow a strategy reminiscent of [Ap2]. Starting with a general k-gonal curve C of genus g, we form the irreducible nodal curve $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ obtained by identifying $g-2k+1$ general pairs of points on C. Clearly $p_a(D) = 2g-2k+1$ and $\mathrm{gon}(D) \leq g-k+1$, that is, $[D]$ belongs to the closure $\overline{\mathfrak{Hur}} = \overline{\mathcal{M}}^1_{2g-2k+1,g-k+1}$ of the Hurwitz divisor, already considered in [HM], [HR] and [FK]. Let

$$\pi : \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$

denote the forgetful map from the space of admissible covers of degree $g-k+1$ compactifying the Hurwitz space $\mathcal{H}_{2g-2k+1,g-k+1}$. Assuming the curve C we started with is sufficiently general, one checks directly that *set-theoretically* $W^1_{g-k+1}(D)$ consists of one point (that is, $\pi^{-1}([D])$ consists of one admissible cover $[f]$). This point corresponds to the torsion free sheaf on D given by pushing forward the unique degree k pencil on C. By an argument inspired by limit linear series, we show that $[f] \notin \overline{\mathfrak{BN}}$. To conclude $b_{g-k,1}(C,K_C) = g-k$, we extend in Section 4 the Eagon-Northcott divisor $\mathcal{EN}$ over a partial compactification of $\mathcal{H}_{2g-2k+1,g-k+1}$ containing the moduli point of $[f]$. In the short Section 5, we then use $K3$ surfaces to show that this extended Eagon-Northcott divisor $\widetilde{\mathcal{EN}}$ does not contain the unique boundary component of

$\overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ containing $[f]$. Since always $K_{g-k,1}(C,K_C) \hookrightarrow K_{g-k,1}(D,\omega_D)$, this completes the proof of Theorem 0.5.[1]

The organisation of the paper is as follows: We first review some background on syzygies of curves in Section 1. In Section 2, we prove Theorem 0.2. We prove Theorem 0.5 in Sections 3, 4 and 5.

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1. Background on Syzygies

We recall a few definitions and collect some basic results on syzygies that will be used throughout the paper. Let X be a (possibly singular) projective variety and let $L, M \in \operatorname{Pic}(X)$ be line bundles. Consider the graded $S := \operatorname{Sym} H^0(X,L)$-module

$$\Gamma_X(M,L) := \bigoplus_{n\in\mathbb{Z}} H^0(X, L^{\otimes n} \otimes M).$$

One defines the Koszul cohomology groups $K_{p,q}(X,M;L)$ of p-th syzygies of weight q by resolving the module $\Gamma_X(M,L)$ and computes them via the Koszul complex, see [G]. When $M = \mathcal{O}_X$, we write $K_{p,q}(X,L) := K_{p,q}(X,\mathcal{O}_X;L)$. The following fact is surely well-known, we include a proof for the sake of completeness:

Lemma 1.1 (Semicontinuity). *Let $\pi: \mathcal{X} \to S$ be a flat, projective morphism of schemes over an integral base. Let $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ be a line bundle such that $h^0(X_s,\mathcal{L}_s) = c$, for each $s \in S$. Let $\mathcal{M} \in \operatorname{Pic}(\mathcal{X})$ be a second line bundle, and assume*

$$h^0(X_s, \mathcal{L}_s^{\otimes(q-1)} \otimes \mathcal{M}_s) = r_1,\ h^0(X_s, \mathcal{L}_s^{\otimes q} \otimes \mathcal{M}_s) = r_2,\ h^0(X_s, \mathcal{L}_s^{\otimes(q+1)} \otimes \mathcal{M}_s) = r_3$$

are also independent of $s \in S$. Then the function

$$\psi : s \mapsto \dim K_{p,q}(X_s, \mathcal{M}_s; \mathcal{L}_s)$$

is upper semicontinuous on S.

Proof. By Grauert's Theorem, the sheaves $\mathcal{E} := \pi_*\mathcal{L}$, $\mathcal{F}_1 := \pi_*(\mathcal{L}^{\otimes(q-1)} \otimes \mathcal{M})$, $\mathcal{F}_2 := \pi_*(\mathcal{L}^{\otimes q} \otimes \mathcal{M})$ and $\mathcal{F}_3 := \pi_*(\mathcal{L}^{\otimes(q+1)} \otimes \mathcal{M})$ are all locally free. It is clearly enough to assume that $S = \operatorname{Spec}(R)$ is affine. We have a Koszul complex

$$\bigwedge^{p+1} \mathcal{E} \otimes \mathcal{F}_1 \xrightarrow{\delta_1} \bigwedge^{p} \mathcal{E} \otimes \mathcal{F}_2 \xrightarrow{\delta_2} \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F}_3,$$

where both maps are R-module morphisms. For any $\mathfrak{p} \in \operatorname{Spec}(R)$, the group $K_{p,q}(X_{\mathfrak{p}}, \mathcal{M}_{\mathfrak{p}}; \mathcal{L}_{\mathfrak{p}})$ is given by the middle cohomology of the complex

$$\Big(\bigwedge^{p+1} \mathcal{E} \otimes \mathcal{F}_1\Big) \otimes k(\mathfrak{p}) \xrightarrow{\delta_1 \otimes k(\mathfrak{p})} \Big(\bigwedge^{p} \mathcal{E} \otimes \mathcal{F}_2\Big) \otimes k(\mathfrak{p}) \xrightarrow{\delta_2 \otimes k(\mathfrak{p})} \Big(\bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F}_3\Big) \otimes k(\mathfrak{p}).$$

[1]It might be tempting to carry out this argument at the level of $\overline{\mathcal{M}}_{2g-2k+1}$ rather than pass to the Hurwitz space. However, the scheme structure of $W^1_{g-k+1}(D)$ is difficult to analyse, in particular $[D]$ is a singular point of $\operatorname{Im}(\pi) = \overline{\mathfrak{Hur}}$. Thus a degenerate version of results in [HR], does not quite lead to a proof of Conjecture 0.2.

Hence

$$\dim K_{p,q}(X_{\mathfrak{p}}, \mathcal{M}_{\mathfrak{p}}; \mathcal{L}_{\mathfrak{p}}) = \dim \mathrm{Ker}(\delta_2 \otimes k(\mathfrak{p})) - \dim \mathrm{Im}(\delta_1 \otimes k(\mathfrak{p}))$$
$$= r_2 \binom{c}{p} - \dim \mathrm{Im}(\delta_2 \otimes k(\mathfrak{p})) - \dim \mathrm{Im}(\delta_1 \otimes k(\mathfrak{p})).$$

So it suffices to observe that for any morphism $\psi: A \to B$ of finitely generated, free R-modules, the function $\mathfrak{p} \mapsto \mathrm{rank}\ \psi \otimes k(\mathfrak{p})$ is lower semicontinuous, which is clear.

□

We collect some results on syzygies of curves which, taken together, reduce Conjecture 0.1 to the extremal case of line bundles of degree $d = 2g - 1 + \mathrm{gon}(C)$. We first quote from [Ap], Lemma 4.1:

Lemma 1.2. *Let C be a smooth curve of genus g and L a line bundle of degree $d \geq g$ with $h^1(C, L) = 0$. Assume $K_{p,1}(C, L) = 0$. Then $K_{p+1,1}(C, L(x)) = 0$, for any point $x \in C$ such that $L(x)$ is base point free.*

It is standard, see e.g. [AN], Corollary 2.13, that if $L \ncong \mathcal{O}_C$ is a globally generated line bundle on a smooth curve C, if $K_{p,1}(C, L) = 0$, then $K_{p+1,1}(C, L) = 0$. Accordingly, there are several natural invariants which one can read directly off the Betti table of an embedded curve $C \overset{|L|}{\hookrightarrow} \mathbf{P}^{r(L)}$ and which measure the length of the linear and the quadratic strand respectively:

$$\ell_1(C, L) := \max\bigl\{p \in \mathbb{N}_{>0} : b_{p,1}(C, L) \neq 0\bigr\} \quad \text{and}$$
$$\ell_2(C, L) := \min\bigl\{p \in \mathbb{N}_{>0} : b_{p,2}(C, L) \neq 0\bigr\}.$$

Recalling that $K_{p,q}(C, L) = 0$ for $p \geq r(L)$, the invariants $\ell_1(C, L)$ are encoded in the more classical properties (N_p) and (M_q) defined in [GL1]. Precisely, $\ell_2(C, L)$ is the smallest integer such that (C, L) fails property $\bigl(N_{\ell_2(C,L)}\bigr)$, whereas $\ell_1(C, L)$ is the smallest integer such that L fails property $\bigl(M_{r(L)-\ell_1(C,L)}\bigr)$.

Equally important invariants of the Betti table of $(C; L)$ are the *extremal Betti numbers*:

$$b_1(C, L) := b_{\ell_1,1}(C, L) \quad \text{and} \quad b_2(C, L) := b_{\ell_2,2}(C, L).$$

Using these invariants, the Effective Gonality Conjecture can be formulated as saying that $\ell_1(C, L) \leq r(L) - \mathrm{gon}(C)$, for every line bundle L of degree at least $2g - 1 + \mathrm{gon}(C)$. Similarly, Schreyer's Conjecture is equivalent to $b_1(C, K_C) = g - \mathrm{gon}(C)$, for every curve of non-maximal gonality having a unique pencil of minimal degree.

2. The Effective Gonality Conjecture for generic curves

We start by proving Theorem 0.3. It turns out that our proof of the generic Green-Lazarsfeld Secant Conjecture [FK] takes us a long distance towards finding a complete solution.

Proof of Theorem 0.3. Let C be a curve of genus $2n+1$ and gonality $n+2$. Then using e.g. [HR, Remark 6.3], we observe that $\mathrm{Cliff}(C) = n$, that is, C has maximal Clifford index as well. We need to prove that for any line bundle $L \in \mathrm{Pic}(C)$ of degree at least $5n+3$, we have $K_{i,1}(C, L) = 0$ for $i \geq h^0(C, L) - n - 2$. We may assume $n \geq 2$ and as explained in the previous section, it is enough to prove that for any line bundle $L \in \mathrm{Pic}^{5n+3}(C)$, we have $K_{2n+1,1}(C, L) = 0$.

Theorem 1.4 of [FK] establishes the following equivalence for any line bundle $M \in \mathrm{Pic}^{4n+2}(C)$:

$$K_{n-1,2}(C, M) \neq 0 \Longleftrightarrow M - K_C \in C_{n+1} - C_{n-1}.$$

For any line bundle $M \in \mathrm{Pic}^{4n+2}(C)$ one has cf. [FK, formula (8)]

$$\dim K_{n,1}(C, M) = \dim K_{n-1,2}(C, M).$$

Thus, for any $M \in \mathrm{Pic}^{4n+2}(C)$, the equivalence

$$K_{n,1}(C,M) \neq 0 \Longleftrightarrow M - K_C \in C_{n+1} - C_{n-1}$$

holds. Using Lemma 1.2 again, it thus suffices to show that for any line bundle L of degree $5n+3$, there exists an effective divisor $D \in C_{n+1}$ such that

$$L - D - K_C \notin C_{n+1} - C_{n-1}.$$

Suppose this were not the case, that is,

$$L - K_C - C_{n+1} \subseteq C_{n+1} - C_{n-1}.$$

Then for every $D \in C_{n+1}$ there exists a divisor $E \in C_{n+1}$ such that $H^1(C, L(-D-E)) \neq 0$, that is, $D+E$ is an element of the (determinantal) secant variety $V^{2n+1}_{2n+2}(L)$ of effective divisors failing to impose independent conditions on $|L|$. In particular,

$$\dim V^{2n+1}_{2n+2}(L) \geq n+1,$$

which is one higher than the expected dimension n. We observe that the morphism

$$\begin{aligned}\psi :\ V^{2n+1}_{2n+2}(L) &\to C_{n-1},\\ A &\mapsto K_C - L + A\end{aligned}$$

is well-defined, since $h^0(C, K_C - L + A) = 1$, for $\mathrm{gon}(C) > n-1$. Let I be any component of $V^{2n+1}_{2n+2}(L)$ of dimension $n+1$ and set $r := n - 1 - \dim\ \psi(I)$. Then $\psi_{|_I}$ must have fibres of dimension at least $2+r$. As all divisors in the inverse image $\psi^{-1}(B)$ are clearly linearly equivalent, we have $h^0(C,A) \geq 3+r$ for all $A \in V^{2n+1}_{2n+2}(L)$ such that $\psi(A) = B \in \psi(I)$. By Riemann–Roch, this implies $h^1(C,A) \geq 1+r$, or $h^0(C, K_C - A) = h^0(2K_C - L - B) \geq 1+r$. The latter inequality holds for *any* effective divisor $B \in \psi(I)$, so we must have

$$\dim |2K_C - L| \geq r + \dim \psi(I) = n-1.$$

This implies $h^1(C, 2K_C - L) \geq 3$, or equivalently $L - K_C \in W^2_{n+3}(C)$. But then $\mathrm{Cliff}(C) \leq n-1$ (if $n=2$, then compute the Clifford index of $2K_C - L$ rather than $L - K_C$). Since we have $\mathrm{Cliff}(C) = n$, this is a contradiction. □

The proof of Theorem 0.3 gives a characterisation of those line bundles $L \in \mathrm{Pic}^{2g-2+\mathrm{gon}(C)}(C)$, such that $K_{h^0(L)-\mathrm{gon}(C),1}(C,L) \neq 0$, in the case where C has odd genus and maximal gonality.

Proposition 2.1. *Let C be a smooth curve of odd genus $2n+1$ and gonality $n+2$. Let $L \in \mathrm{Pic}^{5n+2}(C)$ be such that $K_{2n,1}(C,L) \neq 0$. Then $L - K_C \in W^1_{n+2}(C)$.*

Proof. Following the proof of Theorem 0.3, we obtain $\dim V^{2n}_{2n+1}(L) \geq n$. By studying the morphism

$$\psi : V^{2n}_{2n+1}(L) \to C_{n-1}, \quad A \mapsto K_C - L + A.$$

and arguing as in Theorem 0.3, we are again led to the statement $h^0(C, 2K_C - L) \geq n$. The Riemann–Roch theorem gives $h^0(C, L - K_C) \geq 2$, as required. □

We will prove Theorem 0.2 by an induction on the genus, fixing the gonality. To perform the induction step, let C be a smooth genus g curve of gonality k and denote by $f : C \to \mathbf{P}^1$ the induced degree k cover. We assume that C verifies the Effective Gonality Conjecture. Let $p \in C$ be a branch point of f, and consider the stable curve $X = C \cup_p E$ obtained by glueing a smooth, genus 1 curve at p. A standard argument with admissible covers or limit linear series shows that X is a limit of smooth, genus $g+1$ curves of gonality k, see [HM, §3.G].

Proposition 2.2. *Let $X = C \cup_p E$ be the genus $g+1$ stable curve as above. Let L be a line bundle on X such that $\deg(L_C) = 2g+k$ and $\deg(L_E) = 1$. Then, for a general point $q \in E \backslash \{p\}$, we have*

$$K_{g,1}(X, L(-q)) = 0.$$

Further, for such a point, $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$.

Proof. We have the Mayer–Vietoris sequence on X

$$0 \longrightarrow L_C(-p) \longrightarrow L(-q) \longrightarrow L_E(-q) \longrightarrow 0.$$

For a general point $q \in E \setminus \{p\}$, we have $h^0(E, L_E^{\otimes j}(-jq)) = h^1(E, L_E^{\otimes j}(-jq)) = 0$ for $j = 1, 2$, which implies $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$. Further, we have a natural isomorphism $H^0(C, L(-p)) \cong H^0(X, L(-q))$, and we know, by the assumptions on C, that

$$K_{g,1}(C, L(-p)) = 0.$$

We will use this to deduce $K_{g,1}(X, L(-q)) = 0$.

We have a natural commutative diagram

$$\begin{tikzcd}
\bigwedge^{g+1} H^0(C, L(-p)) \arrow[r, "d"] \arrow[d, "\alpha"] & \bigwedge^{g} H^0(C, L(-p)) \otimes H^0(C, L(-p)) \arrow[r, "d"] \arrow[d, "\beta"] & \bigwedge^{g-1} H^0(C, L(-p)) \otimes H^0(C, L^{\otimes 2}(-2p)) \arrow[d, "\gamma"] \\
\bigwedge^{g+1} H^0(X, L(-q)) \arrow[r, "d"] & \bigwedge^{g} H^0(X, L(-q)) \otimes H^0(X, L(-q)) \arrow[r, "d"] & \bigwedge^{g-1} H^0(X, L(-q)) \otimes H^0(X, L^{\otimes 2}(-2q))
\end{tikzcd},$$

where α, β are isomorphisms, and γ is induced from the natural composition

$$H^0(C, L^{\otimes 2}(-2p)) \hookrightarrow H^0(C, L^{\otimes 2}(-p)) \cong H^0(X, L^{\otimes 2}(-2q)).$$

As $K_{g,1}(C, L(-p)) = 0$, the top row is exact and since β is surjective and γ is injective, the bottom row must also be exact, as required. $\square$

From Proposition 2.2 we readily deduce Theorem 0.2.

Proof. Fix $k \geq 4$. Assume that the general curve k-gonal C of genus g has the property that, for any line bundle $L \in \mathrm{Pic}^{2g-1+k}(C)$, one has $K_{g,1}(C, L) = 0$. We claim there exists a smooth curve C' of genus $g+1$ and gonality k, such that, for each line bundle $L' \in \mathrm{Pic}^{2g+1+k}(C')$, one has $K_{g+1,1}(C', L') = 0$. By performing induction on g and noting that the initial step is Theorem 0.3, this suffices to prove the theorem. By Theorem 1.2, it further suffices to prove that there exists a smooth curve C' of genus $g+1$ and gonality k such that, for each line bundle $L' \in \mathrm{Pic}^{2g+1+k}(C')$, there exists a point $q \in C'$ such that $K_{g,1}(C', L'(-q)) = 0$.

Let $X = C \cup_p E$ be the genus $g+1$ stable curve introduced in Proposition 2.2. Consider a flat family $\pi : \mathcal{C} \to S$ of stable curves over a smooth, pointed, one dimensional base $(S, 0)$, such that the central fibre is X and $\pi^{-1}(s)$ is a smooth curve of gonality k for all $0 \neq s \in S$. As X is a curve of compact type, after shrinking S and performing a finite base change if necessary, we have a relative Picard scheme

$$v : \mathcal{P}ic^{2g+1+k}(\mathcal{C}/S) \to S,$$

with central fibre consisting of all line bundles of multidegree $(2g+k, 1)$ on $X = C \cup_p E$; this scheme is flat and proper over S, see [D, §4] and [EH], proof of Theorem 3.3.

Let $\mathcal{C}_0$ be the open set $\mathcal{C} \backslash p$ of all points which are smooth in the fibres over S. By Proposition 2.2 together with semicontinuity for the dimension of Koszul groups, there is an open subset $U \subseteq \mathcal{P}ic^{2g+1+k}(\mathcal{C}/S) \times_S \mathcal{C}_0$, such that, for each pair $(L', q') \in U$, one has $K_{g,1}(C', L'(-q)) = 0$, where $C' = \pi^{-1}(v(L'))$, and such that

$$0 \notin v\big(\mathcal{P}ic^{2g+1+k}(\mathcal{C}/S) \setminus \mathrm{pr}_1(U)\big),$$

where $\mathrm{pr}_1 : \mathcal{P}ic^{2g+1+k}(\mathcal{C}/S) \times_S \mathcal{C}_0 \to \mathcal{P}ic^{2g+1+k}(\mathcal{C}/S)$ is the projection. As flat morphisms are open, $\mathrm{pr}_1(U)$ is open, and since v is proper, the image

$$V := v\big(\mathcal{P}ic^{2g+1+k}(\mathcal{C}/S) \setminus \mathrm{pr}_1(U)\big)$$

is closed. Thus if $0 \neq t \in S \setminus V$ and $C_t := \pi^{-1}(t)$, then, for each $L \in \mathrm{Pic}^{2g+1+k}(C_t)$ there exists $q \in C_t$ with $K_{g,1}(C_t, L(-q)) = 0$, as required. □

3. Schreyer's Conjecture for general curves of non-maximal gonality

In this section, we begin discussing Schreyer's Conjecture for general k-gonal curves of genus $g \geq 2k-1$. We start by explaining the relevance of [HR] for Conjecture 0.2.

For $g = 2k-1$, we consider two divisors on $\mathcal{M}_g$, which already played a role in [Ap2] or [FK]:

$$\mathfrak{Syz} := \big\{[C] \in \mathcal{M}_g : K_{k-1,1}(C, \omega_C) \neq 0\big\}$$
$$\mathfrak{Hur} := \big\{[C] \in \mathcal{M}_g : W^1_k(C) \neq \emptyset\big\}.$$

Recall that $\mathfrak{Syz}$ has a structure of degeneracy locus whereas $\mathfrak{Hur}$ is the push-forward of the smooth Hurwitz space $\mathcal{H}_{2k-1,k}$ of degree k covers of $\mathbf{P}^1$. It is proved in [HR] that

$$[\mathfrak{Syz}] = (k-1)[\mathfrak{Hur}] \in CH^1(\mathcal{M}_g).$$

Theorem 3.1. *(*[HR]*) Let C be a curve of genus $2k-1$ and gonality k such that the point $[C] \in \mathfrak{Hur}$ is smooth. Then $b_{k-1,1}(C, K_C) = k-1$.*

Proof. From [HR], there exist two vector bundles V and W of the same rank over $\mathcal{M}_g$, with a morphism $\phi : V \to W$, such that, for any $[C] \in \mathcal{M}_g$, we may identify $K_{k-1,1}(C, \omega_C) = \mathrm{Ker}(\phi_{[C]})$. Then $\mathfrak{Syz}$ is defined by $\det(\phi)$. Suppose $b_{k-1,1}(C, K_C) \geq k$. Thus $\det(\phi)$ vanishes to order at least k, cf. [HR, Lemma 6.1]. By the equality of cycles $\mathfrak{Syz} = (k-1)\mathfrak{Hur}$, the function defining $\mathfrak{Hur}$ must vanish to order at least two. Thus $\mathfrak{Hur}$ is not smooth at $[C]$. □

It is well-known that $[C] \in \mathfrak{Hur}$ is a smooth point if and only if $W^1_k(C)$ consists of a single pencil A and moreover $h^0(C, A^{\otimes 2}) = 3$. For such a curve C, we thus have $b_{k-1,1}(C) = k-1$.

We now turn our attention to curves of genus g and non-maximal gonality $k \leq \frac{g+1}{2}$. Let $G^{1,\mathrm{bpf}}_d(C) \subseteq G^1_d(C)$ be the subvariety of base point free pencils of degree d on C. The following observation is a slight modification of the linear growth condition of [Ap2, Theorem 2]:

Lemma 3.2. *A general curve C of genus g and gonality $k \leq \frac{g+1}{2}$ satisfies* bpf-linear growth*:*

$$\dim G^1_{k+m}(C) \leq m, \quad \textit{for } 0 \leq m \leq g-2k+1$$
$$\textit{and, further,} \quad \dim G^{1,\mathrm{bpf}}_{k+m}(C) < m, \quad \textit{for } 0 < m \leq g-2k+1.$$

Proof. From [Ap2], we have $\dim W^1_{k+m}(C) = m$, for $0 \leq m \leq g-2k+1$. It now suffices to observe that if $Z \subseteq W^r_d(C)$ is an irreducible component, then $Z \cap W^{r+1}_d(C)$ has codimension at least two in Z, provided $g-r+d \geq 0$. This follows from the fact that no component of C^r_d is entirely contained in C^{r+1}_d, see [ACGH, §IV.1].

We claim $\dim G^1_{d+m}(C) \leq m$, for $0 \leq m \leq g-2k+1$. Take an irreducible component $J \subseteq G^1_{d+m}(C)$ and consider the restriction to J of the surjection $c : G^1_{k+m}(C) \to W^1_{k+m}(C)$. Assume $c(J) \subseteq W^{1+j}_{d+m}(C)$ and choose $j \geq 0$ maximal with this property. Then by the above, $\dim \psi(J) \leq m-2j$. Since the general fibre of $c_{|J}$ is isomorphic to the Grassmannian $G(2, 2+j)$, it follows $\dim(J) \leq 2j + \dim c(J) \leq m$. By an identical argument and using [AC, Theorem 2.6], we similarly obtain that $\dim G^{1,\mathrm{bpf}}_{k+m}(C) < m$, in the range $0 < m \leq g-2k+1$. □

For an integral nodal curve D, we define $W^1_k(D) \subseteq \overline{\mathrm{Pic}}^k(D)$ the closed subset of the compactified Jacobian of rank one, torsion free sheaves A of degree k on C with $h^0(D,A) \geq 2$.

Proposition 3.3. *Let C be a smooth curve of genus g and gonality $k \leq \frac{g+1}{2}$. Assume C satisfies bpf-linear growth and $W^1_k(C)$ consists of a single point A. If (x_i, y_i) are general pairs of points on C, where $1 \leq i \leq g-2k+1$, let D be the nodal curve obtained by glueing x_i to y_i for all i. Then $W^1_{g-k+1}(D) = \{\nu_*(A)\}$, where $\nu : C \to D$ is the normalisation morphism. Furthermore,* $\mathrm{gon}(D) = g-k+1$.

Proof. Very similar to the proof of Theorem 2 in [Ap2] and we skip the details. □

Consider the moduli space $\overline{\mathcal{H}}_{g,k}$ of degree k admissible covers of genus g. Precisely,

$$\overline{\mathcal{H}}_{g,k} = \overline{\mathcal{M}}_{0,2g+2k-2}(\mathcal{B}\mathfrak{S}_k)/\mathfrak{S}_{2g+2k-2}$$

is the space of twisted stable maps from genus zero curves into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$ and which are simply branched over $2g+2k-2$ points which we do not order. We refer to [ACV] for the construction of this space. It is known that $\overline{\mathcal{H}}_{g,k}$ is the normalisation of the space of admissible covers constructed by Harris and Mumford in [HM]. There is a morphism $\pi : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ given by stabilisation of the source curve of each admissible cover and then $\mathrm{Im}(\pi) = \overline{\mathfrak{Hur}}$. The following result is the translation of Proposition 3.3 to the moduli space of admissible covers.

Proposition 3.4. *Let C be a smooth curve of genus g and gonality $k \leq \frac{g+1}{2}$. Assume C satisfies bpf-linear growth and that $W^1_k(C)$ consists of a single point A, which we assume to have only simple ramification. For $1 \leq i \leq g-2k+1$, we choose general pairs of points (x_i, y_i) on C and let $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ be the nodal curve obtained by glueing x_i to y_i. If*

$$\pi : \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$

is the forgetful map, then $\pi^{-1}([D])$ consists of a unique point.

Proof. We show that the construction described in [HM, Theorem 5] is *unique* in our case. Let $[f : B \to T] \in \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ be an admissible cover, where $p_a(T) = 0$ and B is a nodal curve whose stable model is isomorphic to D. There exists a unique component C_0 of B having positive genus. The restriction $f_0 := f_{|C_0}$ gives a morphism $f_0 : C_0 \to \mathbf{P}^1_0$ onto a smooth rational component $\mathbf{P}^1_0$ of T. By admissibility, $C_0 \cong C$ and $\deg(f_0) \geq k$.

Assume that $f_0(x_i) = f_0(y_i)$ if and only if $1 \leq i \leq j$. For $i = j+1, \ldots, g-2k+1$, we denote by R_{x_i} and R_{y_i} respectively the irreducible components of B meeting C at x_i and y_i respectively. As the stabilisation of B is D and $f(R_{x_i}) \cap f(R_{y_i}) = \emptyset$, for each such i there must be a component $\widetilde{R}_i$ of the subcurve $\overline{B - C_0}$ of B such that $f(\widetilde{R}_i) = \mathbf{P}^1_0$, or else T contains a loop. As $\deg(f) = g-k+1$, this implies that $d := \deg(f_0) \leq k+j$.

Since the pairs $(x_1, y_1), \ldots, (x_j, y_j)$ are general and f_0 gives rise to an element of $G^{1,\mathrm{bpf}}_d(C)$, it follows $\dim G^{1,\mathrm{bpf}}_d(C) \geq j$. If $d > k$, this contradicts the bpf-linear condition on C, which implies that $\deg(f_0) = k$ and f_0 is the map induced by the pencil of minimal degree $A \in W^1_k(C)$. Each $\widetilde{R}_i$ maps isomorphically onto $\mathbf{P}^1_0$. Clearly $\deg(f_{|R_{x_i}}) \geq 2$ and $\deg(f_{|R_{y_i}}) \geq 2$, in particular $f_{|R_{x_i}}$ and $f_{|R_{y_i}}$ will both contain at least two ramification points of f, for each $i = 1, \ldots, g-2k+1$ (Note that being general points, x_i, y_i are not among the ramification points of f_0). Counting the total number of ramification points of the cover f, it follows that $\deg(f_{|R_{x_i}}) = \deg(f_{|R_{y_i}}) = 2$. The morphism f is now uniquely determined, for $f^{-1}(\mathbf{P}^1_0) = C \cup \widetilde{R}_1 \cup \ldots \cup \widetilde{R}_{g-2k+1}$ and all the components of $f^{-1}(f(R_{x_i}))$ and $f^{-1}(f(R_{y_i}))$ other than R_{x_i} and R_{y_i} respectively are mapped isomorphically onto their images. □

Let $\mathfrak{BN}' \subseteq \mathcal{H}_{2g-2k+1,g-k+1} \times_{\mathcal{M}_{2g-2k+1}} \mathcal{H}_{2g-2k+1,g-k+1}$ be the closure of the locus of pairs

$$\Big([f : C \to \mathbf{P}^1], [g : C \to \mathbf{P}^1]\Big),$$

where C is a smooth curve of genus $2g-2k+1$ and $f \ncong g$. Applying [AC, Proposition 2.4], we know that $\dim \mathfrak{BN}' = \dim \mathcal{H}_{2g-2k+1,g-k+1} - 1$. We introduce the Brill-Noether divisor on the Hurwitz space of curves possessing an extra pencil:

$$\mathfrak{BN} := \mathrm{pr}_1(\mathfrak{BN}') \subseteq \mathcal{H}_{2g-2k+1,g-k+1}.$$

Since $\mathfrak{BN}'$ is birational to the Severi variety of nodal curves of type $(g-k+1, g-k+1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ having geometric genus $2g-2k+1$, using [Ty], we conclude that $\mathfrak{BN}$ is an irreducible divisor. We also recall Coppens' result [C] saying that if a curve $[C] \in \mathcal{M}_{2g-2k+1}$ has a pencil $A \in W^1_{g-k+1}(C)$ such that $h^0(C, A^{\otimes 2}) \geq 4$, then $[C, A] \in \mathfrak{BN}$. The locus of such pairs $[C, A] \in \mathcal{H}_{2g-2k+1,g-k+1}$ is of pure codimension one in $\mathfrak{BN}$.

Our next goal is to show that, in the notation of Proposition 3.4, the point $[f] \in \pi^{-1}([D])$ does not lie in the closure $\overline{\mathfrak{BN}} \subseteq \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ provided the normalisation C is sufficiently general. This is achieved by degenerating to a $(k-2)$-nodal curve such that its normalisation is hyperelliptic.

Proposition 3.5. *Let C be a smooth hyperelliptic curve of genus $g-k+2$ and set $\{A\} = W^1_2(C)$. Choose general points y_1 and $\{x_i\}_{i=1}^{g-k-1}$ on C and let y_i be the hyperelliptic conjugate of x_{i-1} for $2 \leq i \leq g-k-1$. Consider the semistable curve D obtained by adjoining smooth rational curves R_i to C at x_i and y_i for $1 \leq i \leq g-1-k$. If $L \in \mathrm{Pic}(D)$ is any line bundle such that*

- $L_{|C} \cong A^{\otimes 2}\big(x_1 + y_1 + \cdots + x_{g-k-1} + y_{g-k-1}\big)$
- $L_{|R_i} \cong \mathcal{O}_{R_i}$, *for all* $1 \leq i \leq g-1-k$,

then $h^0(D, L) = 3$.

Proof. Since $h^0(C, A^{\otimes 2}) = 3$, by Riemann–Roch $h^0(C, \omega_C \otimes A^{\otimes(-2)}) = g-k$. As $y_1, x_1, \ldots, x_{g-1-k}$ are general, $h^0\Big(C, \omega_C \otimes A^{\otimes(-2)}(-y_1 - \sum_{i=1}^{g-1-k} x_i)\Big) = 0$, thus $h^0\big(C, A^{\otimes 2}(y_1 + \sum_{i=1}^{g-1-k} x_i)\big) = 3$, by Riemann-Roch. For each $1 \leq i \leq g-1-k$, define the nodal subcurve of D

$$D_i := C \cup R_1 \cup \ldots \cup R_i,$$

then set $L_i := L_{|D_i}\Big(-\sum_{j=i+1}^{g-1-k}(x_j + y_j)\Big) \in \mathrm{Pic}(D_i)$. We shall prove by induction on i that $h^0(D_i, L_i) = 3$. The inequality $h^0(D_i, L_i) \geq 3$ follows from the Mayer-Vietoris exact sequence

$$0 \longrightarrow A^{\otimes 2} \longrightarrow L_i \longrightarrow \bigoplus_{j=1}^{i} \mathcal{O}_{R_j} \longrightarrow 0,$$

so we need to show that $h^0(D_i, L_i) \leq 3$.

When $i = 1$, we have the exact sequence $0 \to \mathcal{O}_{R_1}(-2) \to L_1 \to A^{\otimes 2}(x_1 + y_1) \to 0$, which gives $h^0(D_1, L_1) \leq h^0\big(C, A^{\otimes 2}(x_1 + y_1)\big) \leq h^0\big(C, A^{\otimes 2}(y_1 + \sum_{j=1}^{g-1-k} x_j)\big) = 3$, so the claim holds.

We now prove the induction step. Assume the claim holds for $i = j$. We claim that each section of $L_j(x_{j+1})$ vanishes at x_{j+1}, from which is follows that $h^0\big(D_j, L_j(x_{j+1})\big) = h^0(D_j, L_j) = 3$, due to the induction hypothesis. We have the following short exact sequence on D_j

$$0 \longrightarrow \bigoplus_{\ell=1}^{j} \mathcal{O}_{R_\ell}(-2) \longrightarrow L_j(x_{j+1}) \longrightarrow A^{\otimes 2}\Big(x_{j+1} + \sum_{\ell=1}^{j}(x_\ell + y_\ell)\Big) \longrightarrow 0.$$

Restriction to C gives an injection $H^0\big(D_j, L_j(x_{j+1})\big) \hookrightarrow H^0\Big(C, A^{\otimes 2}\big(x_{j+1} + \sum_{\ell=1}^{j}(x_\ell + y_\ell)\big)\Big)$ and it suffices to show the sections of the latter cohomology group vanish on x_{j+1}. We have

$$H^0\Big(C, A^{\otimes 2}\big(x_{j+1} + \sum_{\ell=1}^{j}(x_\ell + y_\ell)\big)\Big) = H^0\Big(C, A^{\otimes 2}\big(y_1 + (x_1+y_2) + \cdots + (x_{j-1}+y_j) + x_j + x_{j+1}\big)\Big)$$
$$= H^0\Big(C, A^{\otimes (j+1)} + y_1 + x_j + x_{j+1}\Big) \cong H^0(C, A^{\otimes(j+1)}),$$

for y_1, x_j and x_{j+1} are general points on C. Therefore $h^0\big(D_j, L_j(x_{j+1})\big) = 3$.

We now have two cases. If $h^0\big(D_j, L_j(x_{j+1}+y_{j+1})\big) = 3$, then from th exact sequence

$$0 \longrightarrow \mathcal{O}_{R_{j+1}}(-2) \longrightarrow L_{j+1} \longrightarrow L_j(x_{j+1}+y_{j+1}) \longrightarrow 0,$$

we see $h^0(D_{j+1}, L_{j+1}) \leq 3$ as required. In the second case, $h^0\big(D_j, L_j(x_{j+1}+y_{j+1})\big) = 4$. In this case, there exists a section

$$t \in H^0\big(D_j, L_j(x_{j+1}+y_{j+1})\big)$$

which does not vanish at y_{j+1}.

On the other hand, we claim that each section of $L_j(x_{j+1}+y_{j+1})$ vanishes at x_{j+1}. As above, it suffices to show that each global section of $A^{\otimes 2}\big(\sum_{\ell=1}^{j+1}(x_\ell+y_\ell)\big)$ vanishes at x_{j+1}. We have $A^{\otimes 2}\big(\sum_{\ell=1}^{j+1}(x_\ell+y_\ell)\big) = A^{\otimes(j+2)}(y_1+x_{j+1})$, so that

$$H^0\Big(C, A^{\otimes 2}\big(\sum_{\ell=1}^{j+1}(x_\ell+y_\ell)\big)\Big) \cong H^0(C, A^{\otimes(j+2)}(y_1+x_{j+1}) \cong H^0(C, A^{\otimes(j+2)})$$

(note that $j+2 \leq g(C)$). Consequently, each section of $A^{\otimes 2}\big(\sum_{\ell=1}^{j+1}(x_\ell+y_\ell)\big)$ vanishes at x_{j+1}.

We now conclude the proof from the Mayer-Vietoris sequence

$$0 \longrightarrow L_{j+1} \longrightarrow L_j(x_{j+1}+y_{j+1}) \oplus \mathcal{O}_{R_{j+1}} \xrightarrow{\alpha} \mathcal{O}_{x_{j+1}} \oplus \mathcal{O}_{y_{j+1}} \longrightarrow 0.$$

Indeed, it suffices to show $H^0(\alpha)$ is surjective. By considering the image of t, we see that $\mathrm{Im}(H^0(\alpha))$ contains the element $(0,1)$. Next, by considering the image of constants elements in $H^0(O_{R_{j+1}})$, we note the image contains elements (a,b) with $a \neq 0$. Thus $H^0(\alpha)$ is surjective. $\square$

Remark 3.6. Suppose we replace each curve R_i with a chain of smooth rational curves

$$\Gamma_i := \Gamma_{i,1} \cup \ldots \cup \Gamma_{i,\ell_i},$$

ordered in such way that $\Gamma_{i,1}$ meets C at x_i (and nowhere else), Γ_{i,ℓ_i} meets C at y_i (and nowhere else), no other components of Γ_i meets C, whereas $\Gamma_{i,j}$ intersects $\Gamma_{i,j-1}$ in one point for $j = 2, \ldots, \ell_i$, and there are no other intersections. The conclusion of Proposition 3.5 then holds with Γ_i replacing R_i.

Definition 3.7. *The Eagon-Northcott divisor $\mathcal{EN} \subseteq \mathcal{H}_{2g-2k+1,g-k+1}$ is defined as the locus of covers $[f : C \to \boldsymbol{P}^1]$ such that* $\dim K_{g-k,1}(C, \omega_C) > g-k$.

In the next section, we will extend $\mathcal{EN}$ as a determinantal locus over a partial compactification of $\mathcal{H}_{2g-2k+1,g-k+1}$. From Theorem 3.1 and [SSW, Proposition 4.10], observe that we have the equality of subsets of $\mathcal{H}_{2g-2k+1,g-k+1}$;

$$\mathfrak{BN} = \mathcal{EN}.$$

We now come to the main result of this section, showing that the admissible cover described in Proposition 3.4, does not lie in the closure of $\mathfrak{Kosz}$.

Theorem 3.8. *Let C be a general curve of genus g and gonality $k \leq \frac{g+1}{2}$. Choose general pairs of points (x_i, y_i) on C for $1 \leq i \leq g-2k+1$ and let D be the nodal curve obtained by identifying x_i and y_i for all i. Then $\pi^{-1}([D]) \cap \overline{\mathfrak{BN}} = \emptyset$.*

Proof. In the course of proving Proposition 3.4 we have showed that the underlying set $\pi^{-1}([D])$ consists of a single admissible cover $f : B \to T$, so it suffices to establish that

$$[f,f] \notin \overline{\mathfrak{BN}'} \subseteq \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \times_{\overline{\mathcal{M}}_{2g-2k+1}} \overline{\mathcal{H}}_{2g-2k+1,g-k+1}.$$

We specialise C to a nodal curve C_0 with $k-2$ nodes, such that the normalization $\widetilde{C}_0$ of C_0 is a hyperelliptic curve. Set $\{A\} = W^1_2(C_0)$. Let $\{(x_i, y_i)\}_{i=1}^{g-2k+1}$ be general pairs of points on $\widetilde{C}_0$ and let $\{(x_i, y_i)\}_{i=g-2k}^{g-k-1}$ be the inverse images in $\widetilde{C}_0$ of the nodes of C_0. If D_0 is the curve obtained by identifying x_i to y_i for all $1 \leq i \leq g-1-k$ and $\nu : \widetilde{C}_0 \to D_0$ is the normalisation, then $\nu_*(A) \in W^1_{g-k+1}(D_0)$ and D_0 is a specialization of D. Proposition 3.4 guarantees there is a unique cover $[f_0 : B_0 \to T_0] \in \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ such that the stable model of B_0 is D_0.

Further specialize by bringing the points $\{(x_i, y_i)\}_{i=1}^{g-1-k}$ into the configuration from the hypothesis of Proposition 3.5. This specializes f_0 to an admissible cover $g_0 : B' \to T'$, where the stabilisation of B' is a nodal curve D'_0 with normalisation $\widetilde{C}_0$. It suffices to show $[g_0, g_0] \notin \overline{\mathfrak{BN}'}$.

We record two properties of the admissible cover $g_0 : B' \to T'$ which follow from the considerations in Proposition 3.4. Firstly, B' has a unique non-rational component, which is isomorphic to $\widetilde{C}_0$. The restriction $g_{0|\widetilde{C}_0}$ is the degree 2 cover of $\mathbf{P}^1_0$ determined by $A \in W^1_2(\widetilde{C}_0)$. Secondly, for each pair (x_i, y_i), where $1 \leq i \leq g-1-k$, there is a chain of smooth rational curves in B' which meets $\widetilde{C}_0$ at x_i, y_i. This chain contains precisely one component $\widetilde{R}_i$ which is mapped isomorphically to $\mathbf{P}^1_0$ by g_0. The components $\widetilde{C}_0$ and $\widetilde{R}_i$, where $1 \leq i \leq g-1-k$ are the only components mapped to $\mathbf{P}^1_0$.

Assuming for a contradiction that $[g_0, g_0]$ lies in the closure of $\mathfrak{BN}'$, then there exists a smooth curve Δ with chosen point $0 \in \Delta$, and a family of pairs of admissible covers

$$\big(g_{1,t} : B_t \to T_t, g_{2,t} : B_t \to T_t\big)_{t \in \Delta}$$

with $g_{1,0} = g_{2,0} = g_0$, and such that for $t \neq 0$, we have $T_t \cong \mathbf{P}^1$, the source curve B_t is smooth and $g_{1,t} \neq g_{2,t}$. Each base curve T_t comes with the data of unlabelled branch points. Choose 3 marked points on the component $\mathbf{P}^1_0 \subseteq T_0$, ignore the other marked points and perform stabilisation on this family of 3-marked genus 0 curves. After possible replacing Δ by an étale cover and contracting unstable components, we produce a family of pairs of *stable maps*

$$\big(h_{1,t} : B'_t \to \mathbf{P}^1, h_{2,t} : B'_t \to \mathbf{P}^1\big)_{t \in \Delta},$$

such that the general fibre is unchanged, that is, $h_{i,t} = g_{i,t}$, for $i = 1, 2$ and $t \in \Delta \setminus \{0\}$, but in addition, the special fibre is a morphism with smooth target.

The limiting stable map $h := h_{1,0} = h_{2,0}$ is easy to describe. The curve B''_0 consists of the smooth, hyperelliptic curve $\widetilde{C}_0$ together with rational components R_i meeting $\widetilde{C}_0$ precisely at x_i, y_i for $i = 1, \ldots, g-1-k$. Let $\mathcal{B}'' \to \Delta$ be the total family with fibre over t given by B''_t. Then $\mathcal{B}''$ may have isolated singularities over the nodes of the central fibre B''_0. We have two line bundles, $\mathcal{L}_1$ and $\mathcal{L}_2$, on $\mathcal{B}''$ with $\mathcal{L}_{i,t} \cong h^*_{i,t}(\mathcal{O}_{\mathbf{P}^1}(1))$ for $i = 1, 2$. Consider $\mathcal{N} := \mathcal{L}_1 \otimes \mathcal{L}_2$. The morphisms $h_{1,t} : B_t = B''_t \to \mathbf{P}^1$ and $h_{2,t} : B_t = B''_t \to \mathbf{P}^1$ are *distinct*, so $h^0(B''_t, \mathcal{N}_t) \geq 4$.

Assume firstly $\mathcal{B}''$ is smooth. The rational components R_i of the central fibre define Cartier divisors on $\mathcal{B}''$ for $1 \leq i \leq g-1-k$. Consider the line bundle on $\mathcal{B}''$

$$\mathcal{N}' := \mathcal{N}\Big(\sum_{i=1}^{g-1-k} R_i\Big).$$

The line bundle $\mathcal{N}'_0$ on the central fibre B''_0 satisfies the hypothesis of Proposition 3.5, so that $h^0(B''_0, \mathcal{N}'_0) = 3$. On the other hand, $\mathcal{N}'_t \cong \mathcal{N}_t$ for $t \neq 0$, which contradicts semicontinuity.

In the general case, blow $\mathcal{B}''$ up over the nodes on the central fibre to obtain a smooth surface $\widetilde{\mathcal{B}} \to \Delta$. This introduces chains

$$\Gamma_{x_i} = \Gamma_{x_i,1} \cup \ldots \cup \Gamma_{x_i,\ell_i}, \quad \Gamma_{y_i} = \Gamma_{y_i,1} \cup \ldots \cup \Gamma_{y_i,m_i}$$

of rational curves into the central fibre for all $1 \leq i \leq g-1-k$. Here $\Gamma_{x_i,1}$ respectively $\Gamma_{y_i,1}$ are the components of Γ_{x_i} respectively Γ_{y_i}, meeting $\widetilde{C}_0$ precisely at x_i and y_i respectively. Furthermore, Γ_{x_i,ℓ_i} and Γ_{y_i,m_i} are the components meeting R_i. Finally, the components are ordered in such a way that $\Gamma_{x_i,j} \cap \Gamma_{x_i,j-1} \neq \emptyset$ for $j = 2, \ldots, \ell_i$ and $\Gamma_{y_i,j} \cap \Gamma_{y_i,j-1} \neq \emptyset$ for $j = 2, \ldots, m_i$.

If the two chains have different lengths, say $\ell_i < m_i$, then we increase the length of Γ_{x_i} as follows. First, blow up $\mathcal{B}''$ at x_i. The central fibre is no longer reduced, so follow stable reduction by performing a degree two base change and then normalizing. This has the effect of increasing the length of the chain Γ_{x_i} by one (the total family remains smooth). By repeating this procedure, we may assume $\ell_i = m_i$.

Let $\mathcal{N}'$ denote the pull-back of $\mathcal{N}$ to $\mathcal{B}''$. For $i = 1, \ldots, g-k-1$, set

$$Z_i := (\Gamma_{x_i,1} + \Gamma_{y_i,1}) + 2(\Gamma_{x_i,2} + \Gamma_{y_i,2}) + \cdots + \ell_i(\Gamma_{x_i,\ell_i} + \Gamma_{y_i,\ell_i}) + (\ell_i + 1)R_i,$$

then consider the line bundle $\mathcal{N}'' := \mathcal{N}'\Big(\sum_{i=1}^{g-1-k} Z_i\Big)$. Then one checks that

$$\mathcal{N}''_{|\Gamma_{x_i}} \cong \mathcal{O}_{\Gamma_{x_i}}, \quad \mathcal{N}''_{|\Gamma_{y_i}} \cong \mathcal{O}_{\Gamma_{y_i}}, \quad \mathcal{N}''_{|R_i} \cong \mathcal{O}_{\Gamma_{R_i}},$$

whereas $\mathcal{N}''_{|\widetilde{C}_0} \cong A^{\otimes 2}\Big(\sum_{i=1}^{g-1-k}(x_i + y_i)\Big)$. We now reach a contradiction from Remark 3.6. $\square$

4. Extending the Eagon-Northcott Divisor

In this section we construct an extension of the Eagon-Northcott divisor $\mathcal{EN}$ on a partial compactification of the Hurwitz space $\mathcal{H}_{2g-2k+1,g-k+1}$. We keep the notation of the previous section, set $a := g-k+1$ and further assume $a \geq 3$. In order to describe the partial compactification of the Hurwitz space $\mathcal{H}_{2a-1,a-1}$ it is convenient to work with stable maps. Let $\widetilde{\mathcal{G}}^{\dagger,\mathrm{ns}}_{2a-1,a}$ denote the moduli space of *finite* stable maps $f : C \to \mathbf{P}^1$ of degree a such that C has genus $2a-1$ and only *non-separating* nodes and further with $h^0(C, f^*\mathcal{O}_{\mathbf{P}^1}(1)) = 2$. Set

$$\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} := \widetilde{\mathcal{G}}^{\dagger,\mathrm{ns}}_{2a-1,a}/PGL(2),$$

where PGL(2) acts on the target. Then $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ is an open subset of the space $\overline{\mathcal{M}}_{2a-1,a}(\mathbf{P}^1) := \overline{\mathcal{M}}_{2a-1}(\mathbf{P}^1, a)/PGL(2)$ of stable maps $f : C \to \mathbf{P}^1$ with $f_*[C] = a[\mathbf{P}^1]$, modulo isomorphisms of the target. Note that the Hurwitz space $\mathcal{H}_{2a-1,a}$ can be regarded as an open set of $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$.

We shall construct the extended Eagon-Northcott divisor

$$\widetilde{\mathcal{EN}} \subseteq \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$$

by studying the minimal free resolutions of the scrolls attached to a cover $[f : C \to \mathbf{P}^1] \in \widetilde{\mathcal{H}}^{\mathrm{ns}}_{2a-1,a}$. Set $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \in W^1_a(C)$. Since f is finite and flat, $f_*\mathcal{O}_C$ is locally free and we write $f_*\mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{E}_f^\vee$, where $\mathcal{E}_f$ is the *Tschirnhausen bundle* of f, admitting a splitting

$$\mathcal{E}_f = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{a-1}),$$

where $e_1 \leq \ldots \leq e_{a-1}$ are the *scrollar invariants* of f and satisfy $e_1 + \cdots + e_{a-1} = 3a-2$. Dualising the morphism $\mathcal{O}_{\mathbf{P}^1} \to f_*\mathcal{O}_C$ leads to an exact sequence

$$0 \longrightarrow \mathcal{E}_f \longrightarrow f_*\omega_f \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow 0.$$

We tensor the morphism $f^*(\mathcal{E}_f) \to \omega_f$ by $f^*\omega_{\mathbf{P}^1}$ and produce a morphism $f^*(\mathcal{E}_f(-2)) \twoheadrightarrow \omega_C$, inducing a closed immersion, see [Sch1], or [CE]

$$j : C \to \mathbf{P}\big(\mathcal{E}_f(-2)\big).$$

Note that $\mathcal{E}_f(-2)$ is a globally generated vector bundle on $\mathbf{P}^1$ with $\deg(\mathcal{E}_f(-2)) = a$. Denoting by $\varphi : X := \mathbf{P}\big(\mathcal{E}_f(-2)\big) \to \mathbf{P}^1$ the associated $(a-1)$-dimensional scroll, we have a morphism

$$\iota : X \to \mathbf{P}\Big(H^0(\mathbf{P}^1, \mathcal{E}_f(-2))\Big) \cong \mathbf{P}^{2a-2},$$

such that $\iota \circ j : C \to \mathbf{P}^{2a-2}$ is the canonical morphism of C, cf. [Sch1]. Observe that since C has no disconnected nodes, ω_C is globally generated. Also observe that if $h^0(C, A^{\otimes 2}) = 3$, then $e_1 \geq 3$ and ι is a closed immersion.

The Picard group of the scroll X is generated by the class of a ruling $R := \varphi^*(\mathcal{O}_{\mathbf{P}^1}(1))$ together with $H := \mathcal{O}_X(1)$. Note that $H^0(X,H) \cong H^0(C, \omega_C)$, whereas $H^0(X,R) \cong H^0(C,A)$ and $H^0(X, \mathcal{O}_X(H-R)) \cong H^0(C, \omega_C \otimes A^\vee)$. As already mentioned in the Introduction, the Eagon-Northcott complex, explicitly describes the minimal free resolution of

$$\Gamma_X(H) := \bigoplus_{q \in \mathbb{Z}} H^0(X, H^{\otimes q}),$$

as a Sym $H^0(X,H)$-module, see [Sch1]. This gives that

$$K_{p,0}(X,H) = 0 \text{ for } p > 0, \text{ whereas } K_{p,q}(X,H) = 0, \text{ for } q \geq 2 \text{ and any } p,$$

as well as the canonical identifications

$$K_{p,1}(X,H) \cong \bigwedge^{p+1} H^0(X, H-R) \otimes \mathrm{Sym}^{p-1} H^0(X,R)^\vee \otimes \bigwedge^2 H^0(X,R)$$

$$\cong \bigwedge^{p+1} H^0(C, \omega_C \otimes A^\vee) \otimes \mathrm{Sym}^{p-1} H^0(C,A)^\vee \otimes \bigwedge^2 H^0(C,A).$$

In particular, $b_{p,1}(X,H) = p\binom{a}{p+1}$.

We record the following lemma, while skipping the proof:

Lemma 4.1. *We have the vanishing $H^i(X, H^{\otimes q}) = 0$, for $i \geq 1$ and $q \geq 0$. Furthermore, $H^i(X, \mathcal{O}_X(-H)) = 0$, for $i \geq 2$.*

Define the kernel bundles M_H and M_{ω_C} on X and C respectively by the exact sequences

$$0 \longrightarrow M_H \longrightarrow H^0(X,H) \otimes \mathcal{O}_X \longrightarrow H \longrightarrow 0$$
$$0 \longrightarrow M_{\omega_C} \longrightarrow H^0(C,\omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0.$$

As $C \subseteq X$ is linearly normal, $j^* M_H \cong M_{\omega_C}$. Note that $H^0\big(X, \bigwedge^p M_H\big) = H^0\big(C, \bigwedge^p M_{\omega_C}\big) = 0$, for $p \geq 1$. Further, we record the following short exact sequences:

(4)

$$0 \longrightarrow \bigwedge^{p+1} M_H \otimes \mathcal{O}_X\big((q-1)H\big) \longrightarrow \bigwedge^{p+1} H^0(X,H) \otimes \mathcal{O}_X\big((q-1)\big) \longrightarrow \bigwedge^{p} M_H \otimes \mathcal{O}_X\big(qH\big) \longrightarrow 0,$$

(5)

$$0 \longrightarrow \bigwedge^{p+1} M_{\omega_C} \otimes \omega_C^{\otimes(q-1)} \longrightarrow \bigwedge^{p+1} H^0(C,\omega_C) \otimes \omega_C^{\otimes(q-1)} \longrightarrow \bigwedge^{p} M_{\omega_C} \otimes \omega_C^{\otimes q} \longrightarrow 0.$$

We shall make use of the following vanishing statement.

Lemma 4.2. *We have $H^i\big(X, \bigwedge^p M_H \otimes H^{\otimes q})\big) = 0$ for $i \geq 2$ and arbitrary $p, q \geq 0$.*

Proof. By the sequence (4) and Lemma 4.1, it suffices to show $H^{i-1}\big(\bigwedge^{p-1} M_H \otimes H^{\otimes(q+1)})\big) = 0$. Continuing in this fashion, it suffices to show $H^1\big(X, \bigwedge^{p-i+1} M_H \otimes H^{\otimes(q+i-1)})\big) = 0$. Since $H^1\big(X, H^{\otimes(q+i-1)}\big) = 0$, this amounts to $K_{p,q+i}(X,H) = 0$, which holds as $q+i \geq 2$. $\square$

Lemma 4.3. *There is an injective restriction map of linear syzygies*

$$\alpha_f : K_{a-1,1}(X,H) \to K_{a-1,1}(C,\omega_C).$$

The map α_f is surjective if and only if the restriction map

$$\beta_f:\ H^0\Big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\Big) \to H^0\Big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\Big)$$

is injective.

Proof. The map α_f fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^a H^0(X,H) \arrow[r] \arrow[d, "\cong"] & H^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) \arrow[r] \arrow[d, "\mathrm{res}_C"] & K_{a-1,1}(X,H) \arrow[r] \arrow[d, "\alpha_f"] & 0\ . \\
0 \arrow[r] & \bigwedge^a H^0(C,\omega_C) \arrow[r] & H^0\big(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C\big) \arrow[r] & K_{a-1,1}(C,\omega_C) \arrow[r] & 0
\end{tikzcd}$$

Since $C \subseteq X$ is linearly normal, it follows that res_C is injective, therefore α_f is injective as well. On the other hand, by the snake lemma the surjectivity of α_f is equivalent to the surjectivity of res_C. From the kernel bundle description of Koszul cohomology, we write

$$K_{a-2,2}(X,H) = \mathrm{Ker}\Big\{H^1\big(X, \bigwedge^{a-1} M_H \otimes H\big) \to \bigwedge^{a-1} H^0(X,H) \otimes H^1(X,H)\Big\}.$$

Since $H^1(X,H) = 0$ and $K_{a-2,2}(X,H) = 0$, it follows $H^1\big(X, \bigwedge^{a-1} M_H \otimes H\big) = 0$. We write the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) \arrow[r] \arrow[d, "\mathrm{res}_C"] & \bigwedge^{a-1} H^0(X,H) \otimes H^0(X,H) \arrow[r] \arrow[d, "\cong"] & H^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big) \arrow[r] \arrow[d, "\beta_f"] & 0\ , \\
0 \arrow[r] & H^0\big(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C\big) \arrow[r] & \bigwedge^{a-1} H^0(C,\omega_C) \otimes H^0(C,\omega_C) \arrow[r] & H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big)
\end{tikzcd}$$

By the snake lemma, the surjectivity of res_C is equivalent to the injectivity of β_f. $\square$

Koszul duality gives an isomorphism $K_{a-2,2}(C,\omega_C) \cong K_{a-1,1}(C,\omega_C)^\vee$, therefore we have a surjection

$$H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) \longrightarrow H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) / \bigwedge^{a-1} H^0(C,\omega_C) \otimes H^0(C,\omega_C) \cong K_{a-1,1}(C,\omega_C)^\vee.$$

The composition of this map with $\alpha_f^\vee$ gives rise to a surjection

$$\psi_f : H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) \to K_{a-1,1}(X,H)^\vee.$$

Because $K_{a-2,2}(X,H) = 0$, from the second diagram in the proof of Lemma 4.3, it follows that $\mathrm{Im}(\beta_f) \subseteq \mathrm{Ker}(\psi_f)$.

Lemma 4.4. *We have a natural isomorphism* $\mathrm{Ker}(\psi_f) \cong H^2\big(X, \bigwedge^a M_H \otimes I_{C/X}\big)^\vee$.

Proof. Since $H^1(X, \mathcal{O}_X) = 0$, the description of Koszul cohomology via kernel bundles yields the identification $K_{a-1,1}(X,H)^\vee \cong H^1\big(X, \bigwedge^a M_H\big)^\vee$. Using that $\bigwedge^{a-2} M_{\omega_C} \otimes \omega_C \cong \bigwedge^a M_{\omega_C}^\vee$, Serre-Duality gives the isomorphism

$$H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big)^\vee \cong H^1\big(C, \bigwedge^{a} M_{\omega_C}\big),$$

which enables us to identify the dual map $\psi_f^\vee$ with the restriction

$$H^1\big(X, \bigwedge^{a} M_H\big) \to H^1\big(C, \bigwedge^{a} M_{\omega_C}\big).$$

Then $\mathrm{Ker}(\psi_f) \cong \mathrm{Coker}\big(\psi_f^\vee)\big)^\vee \cong H^2\big(X, \bigwedge^a M_H \otimes I_{C/X}\big)^\vee$, using also Lemma 4.2. $\square$

Putting the above pieces together, we have constructed a natural map

$$\beta_f :\ H^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big) \to H^2\big(X, \bigwedge^{a} M_H \otimes I_{C/X}\big)^\vee$$

such that $b_{a-1,1}(C,\omega_C) > a-1$ if and only if β_f fails to be injective. We shall see that both sides of this map have the same dimension. This allows us to construct $\widetilde{\mathcal{EN}}$ as the degeneracy locus of a morphism between vector bundles of the same rank on the Hurwitz space.

Lemma 4.5. *We have:*

$$h^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big) = h^2\big(X, \bigwedge^{a} M_H \otimes I_{C/X}\big) = (2a-2)\binom{2a-1}{a} - a + 1.$$

Proof. As already pointed out $H^1\big(X, \bigwedge^{a-1} M_H \otimes H\big) = 0$. Therefore

$$h^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big) = (2a-1)\binom{2a-1}{a} - h^0\big(X, \bigwedge^{a-1} M_H \otimes H\big),$$

by the short exact sequence (4). We further have a short exact sequence

$$0 \longrightarrow \bigwedge^{a} H^0(X,H) \longrightarrow H^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) \longrightarrow K_{a-1,1}(X,H) \longrightarrow 0,$$

thus using that $b_{a-1,1}(X,H) = a-1$, we find $h^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) = a - 1 + \binom{2a-1}{a}$, which leads to the claimed formula for $h^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big)$.

Using Lemma 4.4, we compute:

$$h^2\big(X, \bigwedge^{a} M_H \otimes I_{C/X}\big) = \dim(\mathrm{Ker}\ \psi_f = h^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) - b_{a-1,1}(X,H).$$

Recall that $b_{a-1,1}(X,H)=a-1$. The Riemann-Roch theorem (still valid for a nodal curve C with no disconnecting nodes) gives

$$h^0\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)=\chi\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)=(4a-2)\binom{2a-2}{a},$$

which finishes the proof. $\square$

We now explain how the above considerations can be carried out in a relative setting. Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[dr, "\nu"'] & \mathcal{P} \arrow[d, "\mu"] \\ & \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} \end{tikzcd}$$

be the universal degree a cover, where $\mathcal{P}\to\widetilde{\mathcal{G}}^{ns}_{2a-1,a}$ is a $\mathbf{P}^1$-bundle. The universal Tschirnhausen bundle $\mathcal{E}_f$ on $\mathcal{P}$ fits into an exact sequence:

$$0\longrightarrow\mathcal{E}_f\longrightarrow f_*\omega_f\longrightarrow\mathcal{O}_{\mathcal{P}}\longrightarrow 0.$$

We further have the projective bundle $\varphi:\ \mathcal{X}:=\mathbf{P}(\mathcal{E}_f\otimes\omega_\mu)\to\mathcal{P}$ and a closed immersion $j:\mathcal{C}\hookrightarrow\mathcal{X}$. Set $h:=\mu\circ\varphi:\ \mathbf{P}(\mathcal{E}_f\otimes\omega_\mu)\to\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$. By Grauert's Theorem, $h_*\big(\mathcal{O}_{\mathcal{X}}(1)\big)$ is a vector bundle of rank $2a-1$. Define the determinant $\xi:=\det h_*\big(\mathcal{O}_{\mathcal{X}}(1)\big)$. The evaluation map $h^*h_*\mathcal{O}_{\mathcal{X}}(1)\to\mathcal{O}_{\mathcal{X}}(1)$ is furthermore surjective, thus we can define the kernel bundle $\mathcal{M}$ by

$$0\longrightarrow\mathcal{M}\longrightarrow h^*h_*(\mathcal{O}_{\mathcal{X}}(1)\to\mathcal{O}_{\mathcal{X}}(1)\longrightarrow 0.$$

Then $\mathcal{M}$ restricts to the kernel bundle M_H for each scroll induced by an element $[C\to\mathbf{P}^1]$. Note that j is defined by the surjection $f^*(\mathcal{E}_f\otimes\omega_\mu)\twoheadrightarrow\omega_f\otimes f^*\omega_\mu\cong\omega_\nu$, hence $\mathcal{O}_{\mathcal{C}}(1)\cong\omega_\nu$. Set

$$\mathcal{F}_1:=h_*\Big(\bigwedge^{a-2}\mathcal{M}\otimes\mathcal{O}_{\mathcal{X}}(2)\Big)\otimes\xi^\vee,$$

which is a vector bundle of rank $(2a-2)\binom{2a-1}{a}-a+1$, by Lemma 4.5. Set

$$\mathcal{F}_2:=h_*\Big(\bigwedge^{a-2}\mathcal{M}\otimes\mathcal{O}_{\mathcal{C}}(2)\Big)\otimes\xi^\vee,$$

which is a vector bundle of rank $(2a-2)\binom{2a-1}{a}$. Restriction to $\mathcal{C}$ induces a morphism

$$\beta:\ \mathcal{F}_1\to\mathcal{F}_2.$$

Relative duality gives the isomorphism

$$R^1\nu_*\big(\bigwedge^a\mathcal{M}_{|\mathcal{C}}\big)\cong\Big(\nu_*\big(\bigwedge^a\mathcal{M}^\vee_{|\mathcal{C}}\otimes\omega_\nu\big)\Big)^\vee\cong\mathcal{F}_2^\vee,$$

using $\det(\mathcal{M})\cong h^*\xi\otimes\mathcal{O}_{\mathcal{X}}(-1)$. Define the rank $a-1$ vector bundle by $\mathcal{F}_3:=R^1h_*\Big(\bigwedge^a\mathcal{M}\Big)^\vee$, The dual of the restriction morphism $\psi^\vee:\ R^1h_*\big(\bigwedge^a\mathcal{M}\big)\to R^1\nu_*\big(\bigwedge^a\mathcal{M}_{|\mathcal{C}}\big)$ gives a morphism

$$\psi:\ \mathcal{F}_2\to\mathcal{F}_3$$

with fibre over a moduli point $[f:C\to\mathbf{P}^1]$ equal to ψ_f. As already explained, $\psi\circ\beta=0$.

We get a short exact sequence of vector bundles over $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$:

$$0\longrightarrow R^1h_*\big(\bigwedge^a\mathcal{M}\big)\longrightarrow R^1\nu_*\big(\bigwedge^a\mathcal{M}\otimes\mathcal{O}_{\mathcal{C}}\big)\longrightarrow R^2h_*\big(\bigwedge^a\mathcal{M}\otimes I_{\mathcal{C}/\mathcal{X}}\big)\longrightarrow 0,$$

where $\mathcal{F}_4 := R^2 h_*\big(\bigwedge^a \mathcal{M} \otimes I_{\mathcal{C}/\mathcal{X}}\big)$ is a vector bundle of rank $(2a-2)\binom{2a-1}{a} - a + 1$ by Lemma 4.5. Thus we may canonically identify

$$\mathrm{Ker}(\psi) \cong \mathcal{F}_4^\vee$$

and we have an induced morphism between vector bundles $\beta : \mathcal{F}_1 \to \mathcal{F}_4^\vee$ globalizing the morphisms β_f as the moduli point $[f] \in \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ varies. Since $\mathrm{rk}(\mathcal{F}_1) = \mathrm{rk}(\mathcal{F}_4)$, we define the extended Eagon-Northcott divisor

$$\widetilde{\mathcal{EN}} \subseteq \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$$

as the degeneracy locus of β. By the results of the previous chapter, this is a genuine divisor.

Define $\mathcal{EN}^{\mathrm{sm}}$ as the union of all components of $\widetilde{\mathcal{EN}}$ containing an element $[f : C \to \mathbf{P}^1]$, with C being a smooth curve. The following lemma is a direct consequence of Theorem 3.8.

Lemma 4.6. *Let C be a general smooth curve of genus g and gonality $k \leq \frac{g+1}{2}$. For $1 \leq i \leq g-2k+1$, choose (x_i, y_i) be general pairs of points on C and let B be the semistable curve given as the union of C with $g-2k+1$ smooth rational curves R_i such that each R_i meets the rest of B precisely at x_i, y_i for $1 \leq i \leq g-2k+1$. Let*

$$[f : B \to \boldsymbol{P}^1] \in \widetilde{\mathcal{H}}^{\mathrm{ns}}_{2g-2k+1,g-k+1}$$

be a morphism with $\deg(f_{|C}) = k$ and $f_{|R_i}$ an isomorphism. Then $[f] \notin \mathcal{EN}^{\mathrm{sm}}$.

Proof. Consider the closure $\overline{\mathcal{EN}}^{\mathrm{sm}} \subseteq \overline{\mathcal{M}}_{2g-2k+1,g-k+1}(\mathbf{P}^1)$ in the moduli space of stable maps. We have the projections $\pi' : \overline{\mathcal{M}}_{2g-2k+1,g-k+1}(\mathbf{P}^1) \to \overline{\mathcal{M}}_{2g-2k+1}$, as well as the projection π from the space of admissible covers. There is an equality of closed sets $\pi'(\overline{\mathcal{EN}}^{\mathrm{sm}}) = \pi(\mathcal{EN})$, since $\mathcal{M}_{2g-2k+1,g-k+1}(\mathbf{P}^1)$ and $\overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ are isomorphic over the open set of morphisms with smooth source. By Theorem 3.8, the point $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ defined by the stabilization of B does not lie in $\pi'(\overline{\mathcal{EN}}^{\mathrm{sm}})$, therefore, $[f] \notin \overline{\mathcal{EN}}^{\mathrm{sm}}$. □

To complete the proof of Theorem 0.5 we need to show that, in the situation of Lemma 4.6, the point $[f]$ does not lie in the extended Koszul divisor $\widetilde{\mathcal{EN}}$. Note that $[f]$ lies in precisely one of the three boundary divisors, namely the divisor Δ parametrising covers with singular base. Hence we need to show that the divisor $\widetilde{\mathcal{EN}}$ does not contain the boundary divisor Δ. We carry this out in the next section, using K3 surfaces.

5. K3 Surfaces and Schreyer's Conjecture

We start by considering a K3 surface $X = X_d$ with Picard group generated by two classes L and E with self intersections given by $(L)^2 = 4d-4$, $(E)^2 = 0$ and $(L \cdot E) = d$, for $d \geq 3$. By performing Picard-Lefschetz transformations and a reflection if necessary, we may assume that L is big and nef.

Lemma 5.1. *For X as above, the class L is base point free and E is the class of a smooth elliptic curve.*

Proof. We firstly show that L is base point free. As L is big and nef, it suffices to show there is no smooth elliptic curve F with $(L \cdot F) = 1$. Assume such F exists, and write $F = aL + bE$ for $a, b \in \mathbb{Z}$. As F is smooth and elliptic, $(F)^2 = 0$, implying $0 = (aL + bE) \cdot F = a + b(E \cdot F) = a(1 + db)$. If $a = 0$, then $(L \cdot F) = bd \neq 1$, since $d \geq 2$, so $db = -1$, which is again impossible. Thus L is base point free.

We next show that E is the class of a smooth elliptic curve. As $(E)^2 = 0$ and E is primitive, it suffices to show that E is nef. Since $(E \cdot L) > 0$, and L is big and nef, E is effective. Suppose E is not nef. Then there exists a smooth, rational curve R with $(R \cdot E) < 0$. Write $R = aL + bE$ for $a, b \in \mathbb{Z}$. Then $(R \cdot E) < 0$ implies $a < 0$. As $(R)^2 = -2$ and R is effective, we must have

$b > 0$. We have $-2 = (R)^2 = R \cdot (aL + bE) = a(R \cdot L) + b(R \cdot E) = a\big((R \cdot L) + bd\big)$, which is impossible for $d \geq 3$. $\square$

We now discuss the Brill-Noether theory of a smooth curve $C \in |L|$. To that end, we follow [K1, §2] which works in the situation of a higher rank Picard lattice containing the lattice $\mathrm{Pic}(X_d)$.

Lemma 5.2. *Let $D \in \mathrm{Pic}(X_d)$ be effective with $(D)^2 \geq 0$. Assume in addition $L - D$ is effective and $(L-D)^2 > 0$. Then $D = cE$, for some integer c.*

Proof. This is a slight modification of [K1, Lemma 2.5]. Write $D = aL + bE$. As $L - D$ is effective and E nef, $(L - D) \cdot E = (1 - a)(L \cdot E) \geq 0$, so $a \leq 1$. From $(D \cdot E) \geq 0$, we obtain $a \geq 0$. If $a = 1$, then $(L - D)^2 = b^2(E)^2 = 0$, so we must have $a = 0$ as required. $\square$

The next lemma describes the Brill-Noether behaviour of any smooth curve in the linear system $|L|$.

Lemma 5.3. *Let $C \in |L|$ be a smooth curve. Then* $\mathrm{Cliff}(C) = d - 2$ *and $W^1_d(C)$ is reduced and consists of the single point $\mathcal{O}_C(E)$.*

Proof. The proof that $\mathrm{Cliff}(C) = d - 2$ is essentially the same as [K1, Lemma 2.6]. Arguing as in [K1, Lemmas 2.7, 2.8], we see that $W^1_d(C)$ is set-theoretically a single point, namely $\mathcal{O}_C(E)$.

It remains to estalish that $W^1_d(C)$ is reduced, which amounts to showing that $h^0(\mathcal{O}_C(2E)) = 3$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \to \mathcal{O}_E(2E) \cong \mathcal{O}_E \longrightarrow 0,$$

we deduce $h^1(X, 2E) = 1$ and then $h^0(X, 2E) = 3$ by Riemann–Roch. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(2E - C) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_C(2E) \longrightarrow 0,$$

it suffices to show $h^0(X, 2E - C) = h^1(X, 2E - C) = 0$. As $(C - 2E)^2 = -4$, by Riemann-Roch, it suffices to show that neither $2E - C$ nor $C - 2E$ are effective. As $(E \cdot 2E - C) < 0$ and E is nef, $2E - C$ is not effective. Now suppose $C - 2E$ is effective with integral components $R_1, \ldots, R_\ell$, for $\ell \geq 1$. We write $R_i = a_i L + b_i E$, for integers a_i, b_i, with $\sum_{i=1}^{\ell} a_i = 1$ and $\sum_{i=1}^{\ell} b_i = -2$. As $(E \cdot R_i) \geq 0$, we find $a_i \geq 0$ for all i. Without loss of generality, we may assume $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq \ell$. As R_i is integral, we must then have $b_i = 1$ for $i > 1$. Thus $R_1 = L - (\ell + 1)E$, which implies $(R_1)^2 = 4d - 4 - 2d(\ell + 1) \leq -4$, contradicting that R_1 is integral. $\square$

We can now prove Theorem 0.5.

Proof of Theorem 0.5. Let $[f : B \to \mathbf{P}^1]$ be as in the statement of Lemma 4.6. By an argument along the lines of [V1, Corollary 1]), we have an injection $K_{g-k,1}(C, \omega_C) \hookrightarrow K_{g-k,1}(B, \omega_B)$. For the sake of completeness we recall the proof.

The Mayer-Vietoris sequence induces an injection $H^0(C, \omega_C) \hookrightarrow H^0(B, \omega_B)$, as well as the composition of injections $H^0(C, \omega_C^{\otimes 2}) \hookrightarrow H^0\big(C, \omega_C^{\otimes 2}(\sum_{i=1}^{g-2k+1}(x_i + y_i)\big) \hookrightarrow H^0(B, \omega_B^{\otimes 2})$. We then get a commutative diagram

$$\begin{tikzcd}
\bigwedge^{g-k+1} H^0(\omega_C) \arrow[r, "\delta_0"] \arrow[d] & \bigwedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) \arrow[r] \arrow[d] & \bigwedge^{g-k-1} H^0(\omega_C) \otimes H^0(\omega_C^{\otimes 2}) . \arrow[d] \\
\bigwedge^{g-k+1} H^0(\omega_B) \arrow[r, "\delta_0'"] & \bigwedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B) \arrow[r] & \bigwedge^{g-k-1} H^0(\omega_C) \otimes H^0(\omega_C^{\otimes 2})
\end{tikzcd}$$

The conclusion now follows from the existence of the maps

$$\wedge : \ \wedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) \to \wedge^{g-k+1} H^0(\omega_C) \ \text{ and } \wedge' : \wedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B) \to \wedge^{g-k+1} H^0(\omega_B),$$

with $\wedge \circ \delta_0 = \pm(g-k)\mathrm{Id}$ and $\wedge' \circ \delta_0' = \pm(g-k)\mathrm{Id}$, enjoying furthermore natural commutativity properties with the maps in the diagram above.

We secondly claim that $[f]$ does not lie in the extended Koszul divisor $\widetilde{\mathcal{EN}}$. In light of the injective map above, this will complete the proof. As $[f]$ lies in exactly one boundary divisor, namely Δ, all that remains is to show that the divisor $\widetilde{\mathcal{EN}}$ does not contain Δ. By Lemma 5.3, we know that any smooth curve $C \in |L|$ on the K3 surface $X = X_{g-k+1}$ satisfies $K_{g-k,1}(C, \omega_C) = g-k$. By the Lefschetz Theorem [G], the same holds for any integral nodal curve $C_0 \in |L|$. As any integral, nodal curve C_0 (with at least one node) defines a point in Δ, it suffices to show that such curves exist for the general X_{g-k+1}.

To do this, it suffices to take $2g-2k+1 \geq 8$, as the conclusion of the Theorem is well-known for $g \leq 8$ by [Sch1]. Then the class $L-E$ is very ample for a general $K3$ surface X_d general with the given Picard lattice, by degenerating to the $K3$ surface $Y_{\Omega_{g+n}}$ from [K1, Lemma 2.3]. Choose a curve $C_1 \in |L-E|$ meeting a smooth elliptic curve $E_0 \in |E|$ transversally, and consider the nodal curve $C_1 \cup E_0$. Pick any node $p_1 \in C_1 \cup E_0$. Then, by [Ta, Theorem 3.8], the moduli space $\bar{\mathcal{V}}_1(X_d)$ parametrising deformation of $C_1 \cup E_0$ preserving the assigned node p_1 is smooth near $(C_1 \cup E_0, p_1)$ of dimension $2g-2k$. As $\dim|L-E| + \dim|E| = g-k+1 < g-1$ for $k \geq 3$, there exist integral, nodal curves $C_0 \in |L|$ with exactly one node, completing the proof. $\square$

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