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Koszul Cohomology and Applications to Moduli

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1. Introduction

One of the driving problems in the theory of algebraic curves in the past two decades has been *Green's Conjecture* on syzygies of canonical curves. Initially formulated by M. Green [**Gr84a**], it is a deceptively simple vanishing statement concerning Koszul cohomology groups of canonical bundles of curves: If $C \stackrel{|K_C|}{\longrightarrow} \mathbb{P}^{g-1}$ is a smooth canonically embedded curve of genus g and $K_{i,j}(C, K_C)$ are the Koszul cohomology groups of the canonical bundle on C, Green's Conjecture predicts the equivalence

$$K_{p,2}(C, K_C) = 0 \Longleftrightarrow p < \text{Cliff}(C), \tag{1}$$

where $\text{Cliff}(C) := \min\{\deg(L) - 2r(L) : L \in \text{Pic}(C),\ h^i(C, L) \geq 2,\ i = 0, 1\}$ denotes the *Clifford index* of C. The main attraction of Green's Conjecture is that it links the *extrinsic* geometry of C encapsulated in $\text{Cliff}(C)$ and all the linear series $\mathfrak{g}^r_d$ on C, to the *intrinsic* geometry (equations) of the canonical embedding. In particular, quite remarkably, it shows that one can read the Clifford index of any curve off the equations of its canonical embedding. Hence in some sense, a curve has no other interesting line bundle apart from the canonical bundle and its powers[1].

One implication in (1), namely that $K_{p,2}(C, K_C) \neq 0$ for $p \geq \text{Cliff}(C)$, having been immediately established in [**GL84**], see also Theorem 2.4 in this survey, the converse, that is, the vanishing statement

$$K_{p,2}(C, K_C) = 0 \ \text{ for } p < \text{Cliff}(C),$$

attracted a great deal of effort and resisted proof despite an amazing number of attempts and techniques devised to prove it, see [**GL84**], [**Sch86**], [**Sch91**], [**Ein87**], [**Ei92**], [**PR88**], [**Tei02**], [**V93**]. The major breakthrough came around 2002 when Voisin [**V02**], [**V05**], using specialization to curves on $K3$ surfaces, proved that Green's Conjecture holds for a general curve $[C] \in \mathcal{M}_g$:

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[1] "The canonical bundle is not called canonical for nothing"- Joe Harris

Theorem 1.1. *For a general curve* $[C] \in \mathcal{M}_{2p+3}$ *we have that* $K_{p,2}(C, K_C) = 0$. *For a general curve* $[C] \in \mathcal{M}_{2p+4}$ *we have that* $K_{p,2}(C, K_C) = 0$. *It follows that Green's Conjecture holds for general curves of any genus.*

Combining the results of Voisin with those of Hirschowitz and Ramanan [**HR98**], one finds that Green's Conjecture is true for *every* smooth curve $[C] \in \mathcal{M}_{2p+3}$ of maximal gonality $\mathrm{gon}(C) = p + 3$. This turns out to be a remarkably strong result. For instance, via a specialization argument, Green's Conjecture for *arbitrary* curves of maximal gonality implies Green's Conjecture for *general* curves of genus g and *arbitrary* gonality $2 \leq d \leq [g/2] + 2$. One has the following more precise result cf. [**Ap05**], see Theorem 4.5 for a slightly different proof:

Theorem 1.2. *We fix integers* $2 \leq d \leq [g/2] + 1$. *For any smooth* d*-gonal curve* $[C] \in \mathcal{M}_g$ *satisfying the condition*

$$\dim W^1_{g-d+2}(C) \leq g - 2d + 2,$$

we have that $K_{d-3,2}(C, K_C) = 0$. *In particular* C *satisfies Green's Conjecture.*

Dimension theorems from Brill-Noether theory due to Martens, Mumford, and Keem, cf. [**ACGH85**], indicate precisely when the condition appearing in the statement of Theorem 1.2 is verified. In particular, Theorem 1.2 proves Green's Conjecture for general d-gonal curves of genus g for any possible gonality $2 \leq d \leq [g/2] + 2$ and offers an alternate, unitary proof of classical results of Noether, Enriques-Babbage-Petri as well as of more recent results due to Schreyer and Voisin. It also implies the following new result which can be viewed as a proof of statement (1) for 6-gonal curves. We refer to Subsection 4.1 for details:

Theorem 1.3. *For any curve* C *with* $\mathrm{Cliff}(C) \geq 4$, *we have* $K_{3,2}(C, K_C) = 0$. *In particular, Green's Conjecture holds for arbitrary* 6*-gonal curves.*

Theorem 1.1 can also be applied to solve various related problems. For instance, using precisely Theorem 1.1, the *Green-Lazarsfeld Gonality Conjecture* [**GL86**] was verified for general d-gonal curves, for any $2 \leq d \leq (g+2)/2$, cf. [**ApV03**], [**Ap05**]. In a few words, this conjecture states that the gonality of any curve can be read off the Koszul cohomology with values in line bundles of large degree, such as the powers of the canonical bundle. We shall review all these results in Subsection 4.2.

Apart from surveying the progress made on Green's and the Gonality Conjectures, we discuss a number of new conjectures for syzygies of line bundles on curves. Some of these conjectures have already appeared in print (e.g. the *Prym-Green Conjecture* [**FaL08**], or the syzygy conjecture for special line bundles on general curves [**Fa06a**]), whereas others like the *Minimal Syzygy Conjecture* are new and have never been formulated before.

For instance we propose the following refinement of the Green-Lazarsfeld Gonality Conjecture [**GL86**]:

Conjecture 1.4. *Let* C *be a general curve of genus* $g = 2d - 2 \geq 6$ *and* $\eta \in \mathrm{Pic}^0(C)$ *a general line bundle. Then* $K_{d-4,2}(C, K_C \otimes \eta) = 0$.

Conjecture 1.4 is the sharpest vanishing statement one can make for general line bundles of degree $2g - 2$ on a curve of genus g. Since

$$\dim K_{d-4,2}(C, K_C \otimes \eta) = \dim K_{d-3,1}(C, K_C \otimes \eta),$$

it follows that the failure locus of Conjecture 1.4 is a virtual divisor in the universal degree 0 Picard variety $\mathfrak{Pic}^0_g \to \mathcal{M}_g$. Thus it predicts that the non-vanishing locus

$$\{[C,\eta] \in \mathfrak{Pic}^0_g : K_{d-4,2}(C, K_C \otimes \eta) \neq 0\}$$

is an "honest" divisor on $\mathfrak{Pic}^0_g$. Conjecture 1.4 is also sharp in the sense that from the Green-Lazarsfeld Non-Vanishing Theorem 2.8 it follows that

$$K_{d-2,1}(C, K_C \otimes \eta) \neq 0.$$

Similarly, always $K_{d-3,2}(C, K_C \otimes \eta) \neq 0$ for all $[C,\eta] \in \mathfrak{Pic}^0_g$. A yet stronger conjecture is the following vanishing statement for l-roots of trivial bundles on curves:

Conjecture 1.5. *Let C be a general curve of genus $g = 2d-2 \geq 6$. Then for every prime l and every line bundle $\eta \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$ satisfying $\eta^{\otimes l} = \mathcal{O}_C$, we have that $K_{d-4,2}(C, K_C \otimes \eta) = 0$.*

In order to prove Conjecture 1.5 it suffices to exhibit a single pair $[C,\eta]$ as above, for which $K_C \otimes \eta \in \mathrm{Pic}^{2g-2}(C)$ satisfies property (N_{d-4}). The case most studied so far is that of level $l = 2$, when one recovers the *Prym-Green Conjecture* [**FaL08**] which has been checked using Macaulay2 for $g \leq 14$. The Prym-Green Conjecture is a subtle statement which for small values of g is equivalent to the Prym-Torelli Theorem, again see [**FaL08**]. Since the condition $K_{d-4,2}(C, K_C \otimes \eta) \neq 0$ is divisorial in moduli, Conjecture 1.5 is of great help in the study of the birational geometry of the compactification $\overline{\mathcal{R}}_{g,l} := \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_l)$ of the moduli space $\mathcal{R}_{g,l}$ classifying pairs $[C,\eta]$, where $[C] \in \mathcal{M}_g$ and $\eta \in \mathrm{Pic}^0(C)$ satisfies $\eta^{\otimes l} = \mathcal{O}_C$.

Concerning both Conjectures 1.4 and 1.5, it is an open problem to find an analogue of the Clifford index of the curve, in the sense that the classical Green Conjecture is not only a Koszul cohomology vanishing statement but also allows one to read off the Clifford index from a non-vanishing statement for $K_{p,2}(C, K_C)$. It is an interesting open question to find a *Prym-Clifford index* playing the same role as the original Cliff(C) in (1) and to describe it in terms of the corresponding Prym varieties: Is there a geometric characterization of those Prym varieties $\mathcal{P}_g([C,\eta]) \in \mathcal{A}_{g-1}$ corresponding to pairs $[C,\eta] \in \mathcal{R}_g$ with $K_{p,2}(C, K_C \otimes \eta) \neq 0$?

Another recent development on syzygies of curves came from a completely different direction, with the realization that loci in the moduli space $\mathcal{M}_g$ consisting of curves having exceptional syzygies can be used effectively to answer questions about the birational geometry of the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g, cf. [**FaPo05**], [**Fa06a**], [**Fa06b**], in particular, to produce infinite series of effective divisors on $\overline{\mathcal{M}}_g$ violating the Harris-Morrison Slope Conjecture [**HaM90**]. We recall that the slope $s(D)$ of an effective divisor D on $\overline{\mathcal{M}}_g$ is defined as the smallest rational number a/b with $a, b \geq 0$, such that the class $a\lambda - b(\delta_0 + \cdots + \delta_{[g/2]}) - [D] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ is an effective $\mathbb{Q}$-combination of boundary divisors. The Slope Conjecture [**HaM90**] predicts a lower bound for the slope of effective divisors on $\overline{\mathcal{M}}_g$,

$$s(\overline{\mathcal{M}}_g) := \inf_{D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) \geq 6 + \frac{12}{g+1},$$

with equality precisely when $g+1$ is composite; the quantity $6 + 12/(g+1)$ is the slope of the Brill-Noether divisors on $\overline{\mathcal{M}}_g$, in case such divisors exist. A first counterexample to the Slope Conjecture was found in [**FaPo05**]: The locus

$$\mathcal{K}_{10} := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}$$

can be interpreted as being set-theoretically equal to the locus of curves $[C] \in \mathcal{M}_{10}$ carrying a linear series $L \in W^4_{12}(C)$ such that the multiplication map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is not an isomorphism, or equivalently $K_{0,2}(C, L) \neq 0$. The main advantage of this Koszul-theoretic description is that it provides a characterization of the $K3$ divisor $\mathcal{K}_{10}$ in a way that makes no reference to $K3$ surfaces and can be easily generalized to other genera. Using this characterization one shows that $s(\overline{\mathcal{K}}_{10}) = 7 < 78/11$, that is, $\overline{\mathcal{K}}_{10} \in \mathrm{Eff}(\overline{\mathcal{M}}_{10})$ is a counterexample to the Slope Conjecture.

Koszul cohomology provides an effective way of constructing cycles on $\mathcal{M}_g$. Under suitable numerical conditions, loci of the type

$$\mathcal{Z}_{g,2} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \text{ such that } K_{p,2}(C, L) \neq 0\}$$

are virtual divisors on $\mathcal{M}_g$, that is, degeneracy loci of morphisms between vector bundles of the same rank over $\mathcal{M}_g$. The problem of extending these vector bundes over $\overline{\mathcal{M}}_g$ and computing the virtual classes of the resulting degeneracy loci is in general daunting, but has been solved successfully in the case $\rho(g, r, d) = 0$, cf. [**Fa06b**]. Suitable vanishing statements of the Koszul cohomology for general curves (e.g. Conjectures 1.4, 5.4) show that, when applicable, these virtual Koszul divisors are actual divisors and they are quite useful in specific problems such as the Slope Conjecture or showing that certain moduli spaces of curves (with or without level structure) are of general type, see [**Fa08**], [**FaL08**]. A picturesque application of the Koszul technique in the study of parameter spaces is the following result about the birational type of the moduli space of Prym varieties $\overline{\mathcal{R}}_g = \overline{\mathcal{R}}_{g,2}$, see [**FaL08**]:

THEOREM 1.6. *The moduli space $\overline{\mathcal{R}}_g$ is of general type for $g \geq 13$ and $g \neq 15$.*

The proof of Theorem 1.6 depends on the parity of g. For $g = 2d - 2$, it boils down to calculating the class of the compactification in $\overline{\mathcal{R}}_g$ of the failure locus of the Prym-Green Conjecture, that is, of the locus

$$\{[C, \eta] \in \mathcal{R}_{2d-2} : K_{d-4,2}(C, K_C \otimes \eta) \neq 0\}.$$

For odd $g = 2d - 1$, one computes the class of a "mixed" Koszul cohomology locus in $\overline{\mathcal{R}}_g$ defined in terms of Koszul cohomology groups of K_C with values in $K_C \otimes \eta$.

The outline of the paper is as follows. In Section 2 we review the definition of Koszul cohomology as introduced by M. Green [**Gr84a**] and discuss basic facts. In Section 3 we recall the construction of (virtual) Koszul cycles on $\overline{\mathcal{M}}_g$ following [**Fa06a**] and [**Fa06b**] and explain how their cohomology classes can be calculated. In Section 4 we discuss a number of conjectures on syzygies of curves, starting with Green's Conjecture and the Gonality Conjecture and continuing with the Prym-Green Conjecture. We end by proposing in Section 5 a strong version of the Maximal Rank Conjecture.

Some results stated in [**Ap04**], [**Ap05**] are discussed here in greater detail. Other results are new (see Theorems 4.9 and 4.16).

2. Koszul cohomology

2.1. Syzygies. Let V be an n-dimensional complex vector space, $S := S(V)$ the symmetric algebra of V, and $X \subset \mathbb{P}V^\vee := \mathrm{Proj}(S)$ a non-degenerate subvariety, and denote by $S(X)$ the homogeneous coordinate ring of X. To the embedding of X in $\mathbb{P}V^\vee$, one associates the *Hilbert function*, defined by

$$h_X(d) := \dim_{\mathbb{C}} \left(S(X)_d\right)$$

for any positive integer d. A remarkable property of h_X is its polynomial behavior for large values of d. It is a consequence of the existence of a *graded minimal resolution* of the S-module $S(X)$, which is an exact sequence

$$0 \to E_s \to \ldots \to E_2 \to E_1 \to S \to S(X) \to 0$$

with

$$E_p = \bigoplus_{j>p} S(-j)^{\beta_{pj}(X)}.$$

The Hilbert function of X is then given by

$$h_X(d) = \sum_{p,j} (-1)^p \beta_{pj}(X) \begin{pmatrix} n+d-j \\ n \end{pmatrix}, \tag{2}$$

where

$$\begin{pmatrix} t \\ n \end{pmatrix} = \frac{t(t-1)\cdots(t-n+1)}{n\,!}, \text{ for any } t \in \mathbf{R},$$

and note that the expression on the right-hand-side is polynomial for large d. This reasoning leads naturally to the definition of *syzygies of* X, which are the graded components of the graded S-modules E_p. The integers

$$\beta_{pj}(X) = \dim_{\mathbb{C}} \mathrm{Tor}^j(S(X), \mathbb{C})_p$$

are called the *graded Betti numbers of* X and determine completely the Hilbert function, according to formula (2). Sometimes we also write $b_{i,j}(X) := \beta_{i,i+j}(X)$.

One main difficulty in developing syzygy theory was to find effective geometric methods for computing these invariants. In the eighties, M. Green and R. Lazarsfeld published a series of papers, [**Gr84a**], [**Gr84b**], [**GL84**], [**GL86**] that shed a new light on syzygies. Contrary to the classical point of view, they look at integral closures of the homogeneous coordinates rings, rather than at the rings themselves. This approach, using intensively the language of Koszul cohomology, led to a number of beautiful geometrical results with numerous applications in classical algebraic geometry as well as moduli theory.

2.2. Definition of Koszul cohomology. Throughout this paper, we follow M. Green's approach to Koszul cohomology [**Gr84a**]. The general setup is the following. Suppose X is a complex projective variety, $L \in \mathrm{Pic}(X)$ a line bundle, $\mathcal{F}$ is a coherent sheaf on X, and $p, q \in \mathbb{Z}$. The canonical contraction map

$$\bigwedge^{p+1} H^0(X,L) \otimes H^0(X,L)^\vee \to \bigwedge^{p} H^0(X,L),$$

acting on tensors as

$$(s_0 \wedge \cdots \wedge s_p) \otimes \sigma \mapsto \sum_{i=0}^{p} (-1)^i \sigma(s_i)(s_0 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_p),$$

and the multiplication map

$$H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^{q-1})\to H^0(X,\mathcal{F}\otimes L^q),$$

define together a map

$$\bigwedge^{p+1} H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^{q-1})\to \bigwedge^{p} H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^q).$$

In this way, we obtain a complex (called the *Koszul complex*)

$$\wedge^{p+1}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q-1})\to \wedge^{p}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q})\to \wedge^{p-1}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q+1}),$$

whose cohomology at the middle-term is denoted by $K_{p,q}(X,\mathcal{F},L)$. In the particular case $\mathcal{F}\cong\mathcal{O}_X$, to ease the notation, one drops $\mathcal{O}_X$ and writes directly $K_{p,q}(X,L)$ for Koszul cohomology.

Here are some samples of direct applications of Koszul cohomology:

EXAMPLE 2.1. If L is ample, then L is normally generated if and only if $K_{0,q}(X,L)=0$ for all $q\geq 2$. This fact follows directly from the definition.

EXAMPLE 2.2. If L is globally generated with $H^1(X,L)=0$, then

$$H^1(X,\mathcal{O}_X)\cong K_{h^0(L)-2,2}(X,L)$$

see [**ApN08**]. In particular, the genus of a curve can be read off (vanishing of) Koszul cohomology with values in non-special bundles. More generally, under suitable vanishing assumptions on L, all the groups $H^i(X,\mathcal{O}_X)$ can be computed in a similar way, and likewise $H^i(X,\mathcal{F})$ for an arbitrary coherent sheaf $\mathcal{F}$, cf. [**ApN08**].

EXAMPLE 2.3. If L is very ample the *Castelnuovo-Mumford regularity* can be recovered from Koszul cohomology. Specifically, if $\mathcal{F}$ is a coherent sheaf on X, then

$$\mathrm{reg}_L(\mathcal{F})=\min\{m : K_{p,m+1}(X,\mathcal{F},L)=0,\ \text{for all } p\}.$$

As a general principle, any invariant that involves multiplication maps is presumably related to Koszul cohomology.

Very surprisingly, Koszul cohomology interacts much closer with the geometry of the variety than might have been expected. This phenomenon was discovered by Green and Lazarsfeld [**Gr84a**, Appendix]:

THEOREM 2.4 (Green-Lazarsfeld). *Suppose X is a smooth variety and consider a decomposition $L=L_1\otimes L_2$ with $h^0(X,L_i)=r_i+1\geq 2$ for $i\in\{1,2\}$. Then $K_{r_1+r_2-1,1}(X,L)\neq 0$.*

In other words, non-trivial geometry implies non-trivial Koszul cohomology. We shall discuss the case of curves, which is the most relevant, and then some consequences in Section 4.

Many problems in the theory of syzygies involve vanishing/nonvanishing of Koszul cohomology. One useful definition is the following.

DEFINITION 2.5. An ample line bundle L on a projective variety is said to satisfy the property (N_p) if and only if $K_{i,q}(X,L)=0$ for all $i\leq p$ and $q\geq 2$.

From the geometric viewpoint, the property (N_p) means that L is normally generated, the ideal of X in the corresponding embedding is generated by quadrics, and all the syzygies up to order p are linear. In many cases, for example canonical curves, or 2-regular varieties, the property (N_p) reduces to the single condition $K_{p,2}(X,L)=0$, see e.g. [**Ein87**]. This phenomenon justifies the study of various loci given by the nonvanishing of $K_{p,2}$, see Section 4.

2.3. Kernel bundles. The proofs of the facts discussed in examples 2.2 and 2.3 use the kernel bundles description which is due to Lazarsfeld [**La89**]: Consider L a globally generated line bundle on the projective variety X, and set

$$M_L := \mathrm{Ker}\left(H^0(X,L)\otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} L\right).$$

Note that M_L is a vector bundle on X of rank $h^0(L)-1$. For any coherent sheaf $\mathcal{F}$ on X, and integer numbers $p\geq 0$, and $q\in\mathbb{Z}$, we have a short exact sequence on X

$$0\to \wedge^{p+1}M_L\otimes\mathcal{F}\otimes L^{q-1}\to \wedge^{p+1}H^0(L)\otimes\mathcal{F}\otimes L^{q-1}\to \wedge^p M_L\otimes\mathcal{F}\otimes L^q\to 0.$$

Taking global sections, we remark that the Koszul differential factors through the map

$$\wedge^{p+1}H^0(X,L)\otimes H^0(X,\mathcal{F}\otimes L^{q-1})\to H^0(X,\wedge^p M_L\otimes\mathcal{F}\otimes L^q),$$

hence we have the following characterization of Koszul cohomology, [**La89**]:

THEOREM 2.6 (Lazarsfeld). *Notation as above. We have*

$$K_{p,q}(X,\mathcal{F},L)\cong \mathrm{coker}\big(\wedge^{p+1}H^0(L)\otimes H^0(\mathcal{F}\otimes L^{q-1})\to H^0(\wedge^p M_L\otimes\mathcal{F}\otimes L^q)\big)$$
$$\cong \ker\big(H^1(X,\wedge^{p+1}M_L\otimes\mathcal{F}\otimes L^{q-1})\to \wedge^{p+1}H^0(L)\otimes H^1(\mathcal{F}\otimes L^{q-1})\big).$$

Theorem 2.6 has some nice direct consequences. The first one is a duality theorem which was proved in [**Gr84a**].

THEOREM 2.7. *Let L be a globally generated line bundle on a smooth projective variety X of dimension n. Set $r:=\dim|L|$. If*

$$H^i(X,L^{q-i})=H^i(X,L^{q-i+1})=0,\quad i=1,\ldots,n-1$$

then

$$K_{p,q}(X,L)^\vee\cong K_{r-n-p,n+1-q}(X,K_X,L).$$

Another consequence of Theorem 2.6, stated implicitly in [**Fa06b**] without proof, is the following:

THEOREM 2.8. *Let L be a non-special globally generated line bundle on a smooth curve C of genus $g\geq 2$. Set $d=\deg(L)$, $r=h^0(C,L)-1$, and consider $1\leq p\leq r$. Then*

$$\dim\,K_{p,1}(C,L)-\dim\,K_{p-1,2}(C,L)=p\cdot\binom{d-g}{p}\left(\frac{d+1-g}{p+1}-\frac{d}{d-g}\right).$$

In particular, if

$$p<\frac{(d+1-g)(d-g)}{d}-1,$$

then $K_{p,1}(C,L)\neq 0$, and if

$$\frac{(d+1-g)(d-g)}{d}\leq p\leq d-g,$$

then $K_{p-1,2}(C,L)\neq 0$.

Proof. Since we work with a spanned line bundle on a curve, Theorem 2.7 applies, hence we have

$$K_{p,1}(C,L)\cong K_{r-p-1,1}(C,K_C,L)^\vee$$

and

$$K_{p-1,2}(C,L)\cong K_{r-p,0}(C,K_C,L)^\vee.$$

Set, as usual, $M_L = \mathrm{Ker}\{H^0(C,L)\otimes \mathcal{O}_C \to L\}$, and consider the Koszul complex

$$0\to \wedge^{r-p}H^0(C,L)\otimes H^0(C,K_C)\to H^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)\to 0. \tag{3}$$

Since L is non-special, $K_{r-p,0}(C,K_C,L)$ is isomorphic to the kernel of the differential appearing in (3), hence the difference which we wish to compute coincides with the Euler characteristic of the complex (3).

Next we determine $h^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)$. Note that $\mathrm{rk}(M_L)=r$ and $\wedge^{r-p-1}M_L\otimes L\cong \wedge^{p+1}M_L^\vee$. In particular, since $H^0(C,\wedge^{p+1}M_L)\cong K_{p+1,0}(C,L)=0$, we obtain

$$h^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)=-\chi(C,\wedge^{p+1}M_L).$$

Observe that

$$\deg(\wedge^{p+1}M_L)=\deg(M_L)\binom{r-1}{p}=-d\binom{r-1}{p}$$

and

$$\mathrm{rk}(\wedge^{p+1}M_L)=\binom{r}{p+1}$$

From the Riemann-Roch Theorem it follows that

$$-\chi(C,\wedge^{p+1}M_L)=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}$$

and hence

$$\dim\, K_{p,1}(C,L)-\dim\, K_{p-1,2}(C,L)=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}-g\binom{r+1}{p+1}.$$

The formula is obtained by replacing r by $d-g$. □

Remark 2.9. A full version of Theorem 2.8 for special line bundles can be obtained in a similar manner by adding alternating sums of other groups $K_{p-i,i+1}$. For example, if $L^{\otimes 2}$ is non-special, then from the complex

$$0\to\wedge^{r-p+1}H^0(C,L)\otimes H^0(C,K_C\otimes L^{-1})\to\wedge^{r-p}H^0(C,L)\otimes H^0(C,K_C)\to$$

$$\to H^0(C,\wedge^{r-p-1}M_L\otimes K_C\otimes L)\to 0$$

we obtain the following conclusion:

$$\dim\, K_{p,1}(C,L)-\dim\, K_{p-1,2}(C,L)+\dim\, K_{p-2,3}(C,L)=$$

$$=d\binom{r-1}{p}+(g-1)\binom{r}{p+1}-g\binom{r+1}{p+1}+\binom{r+1}{p}(r-d+g).$$

2.4. Hilbert schemes. Suppose X is a smooth variety, and consider $L \in \mathrm{Pic}(X)$. A novel description of the Koszul cohomology of X with values in L was provided in [**V02**] via the Hilbert scheme of points on X.

Denote by $X^{[n]}$ the Hilbert scheme parameterizing zero-dimensional length n subschemes of X, let $X^{[n]}_{\mathrm{curv}}$ be the open subscheme parameterizing curvilinear length n subschemes, and let

$$\Xi_n \subset X^{[n]}_{\mathrm{curv}} \times X$$

be the incidence subscheme with projection maps $p : \Xi_n \to X$ and $q : \Xi_n \to X^{[n]}_{\mathrm{curv}}$. For a line bundle L on X, the sheaf $L^{[n]} := q_* p^* L$ is locally free of rank n on $X^{[n]}_{\mathrm{curv}}$, and the fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$ is isomorphic to $H^0(\xi, L \otimes \mathcal{O}_\xi)$.

There is a natural map

$$H^0(X, L) \otimes \mathcal{O}_{X^{[n]}_{\mathrm{curv}}} \to L^{[n]},$$

acting on the fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$, by $s \mapsto s|_\xi$. In [**V02**] and [**EGL**] it is shown that, by taking wedge powers and global sections, this map induces an isomorphism:

$$\wedge^n H^0(X, L) \cong H^0(X^{[n]}_{\mathrm{curv}}, \det\, L^{[n]}).$$

Voisin proves that there is an injective map

$$H^0(\Xi_{p+1}, \det\, L^{[p+1]} \boxtimes L^{q-1}) \to \wedge^p H^0(X, L) \otimes H^0(X, L^q)$$

whose image is isomorphic to the kernel of the Koszul differential. This eventually leads to the following result:

THEOREM 2.10 (Voisin [**V02**]). *For all integers p and q, the Koszul cohomology $K_{p,q}(X, L)$ is isomorphic to the cokernel of the restriction map*

$$H^0(X^{[p+1]}_{\mathrm{curv}} \times X, \det\, L^{[p+1]} \boxtimes L^{q-1}) \to H^0(\Xi_{p+1}, \det\, L^{[p+1]} \boxtimes L^{q-1}|_{\Xi_{p+1}}).$$

In particular,

$$K_{p,1}(X, L) \cong \mathrm{coker}(H^0(X^{[p+1]}_{\mathrm{curv}}, \det\, L^{[p+1]}) \xrightarrow{q^*} H^0(\Xi_{p+1}, q^* \det\, L^{[p+1]}|_{\Xi_{p+1}})).$$

REMARK 2.11. The group $K_{p,q}(X, \mathcal{F}, L)$ is obtained by replacing L^{q-1} by $\mathcal{F} \otimes L^{q-1}$ in the statement of Theorem 2.10.

The main application of this approach is the proof of the generic Green Conjecture [**V02**]; see Subsection 4.1 for a more detailed discussion on the subject. The precise statement is the following.

THEOREM 2.12 (Voisin [**V02**] and [**V05**]). *Consider a smooth projective $K3$ surface S, such that $\mathrm{Pic}(S)$ is isomorphic to $\mathbb{Z}^2$, and is freely generated by L and $\mathcal{O}_S(\Delta)$, where Δ is a smooth rational curve such that $\deg(L_{|\Delta}) = 2$, and L is a very ample line bundle with $L^2 = 2g - 2$, $g = 2k + 1$. Then $K_{k+1,1}\big(S, L \otimes \mathcal{O}_S(\Delta)\big) = 0$ and*

$$K_{k,1}(S, L) = 0. \tag{4}$$

Voisin's result, apart from settling the Generic Green Conjecture, offers the possibility (via the cohomological calculations carried out in [**Fa06b**], see also Section 3), to give a much shorter proof of the Harris-Mumford Theorem [**HM82**] on the Kodaira dimension of $\overline{\mathcal{M}}_g$ in the case of odd genus. This proof does not use intersection theory on the stack of admissible coverings at all and is considerably shorter than the original proof. This approach has been described in full detail in [**Fa08**].

3. Geometric cycles on the moduli space

3.1. Brill-Noether cycles. We recall a few basic facts from Brill-Noether theory; see [**ACGH85**] for a general reference.

For a smooth curve C and integers $r, d \geq 0$ one considers the Brill-Noether locus

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}$$

as well as the variety of linear series of type $\mathfrak{g}^r_d$ on C, that is,

$$G^r_d(C) := \{(L, V) : L \in W^r_d(C), V \in \mathbf{G}(r + 1, H^0(L))\}.$$

The locus $W^r_d(C)$ is a deteminantal subvariety of $\mathrm{Pic}^d(C)$ of expected dimension equal to the *Brill-Noether number* $\rho(g, r, d) = g - (r + 1)(g - d + r)$. According to the Brill-Noether Theorem for a general curve $[C] \in \mathcal{M}_g$, both $W^r_d(C)$ and $G^r_d(C)$ are irreducible varieties of dimension

$$\dim\, W^r_d(C) = \dim\, G^r_d(C) = \rho(g, r, d).$$

In particular, $W^r_d(C) = \emptyset$ when $\rho(g, r, d) < 0$. By imposing the condition that a curve carry a linear series $\mathfrak{g}^r_d$ when $\rho(g, r, d) < 0$, one can define a whole range of geometric subvarieties of $\mathcal{M}_g$.

We introduce the Deligne-Mumford stack $\sigma : \mathfrak{G}^r_d \to \mathbf{M}_g$ classifying pairs $[C, l]$ where $[C] \in \mathcal{M}_g$ and $l = (L, V) \in G^r_d(C)$ is a linear series $\mathfrak{g}^r_d$, together with the projection $\sigma[C, l] := [C]$. The stack $\mathfrak{G}^r_d$ has a determinantal structure inside a Grassmann bundle over the universal Picard stack $\mathfrak{Pic}^d_g \to \mathbf{M}_g$. In particular, each irreducible component of $\mathfrak{G}^r_d$ has dimension at least $3g-3+\rho(g, r, d)$, cf. [**AC81b**]. We define the *Brill-Noether cycle*

$$\mathcal{M}^r_{g,d} := \sigma_*(\mathfrak{G}^r_d) = \{[C] \in \mathcal{M}_g : W^r_d(C) \neq \emptyset\},$$

together with the substack structure induced from the determinantal structure of $\mathfrak{G}^r_d$ via the morphism σ. A result of Steffen [**St98**] guarantees that each irreducible component of $\mathcal{M}^r_{g,d}$ has dimension at least $3g - 3 + \rho(g, r, d)$.

When $\rho(g, r, d) = -1$, Steffen's result coupled with the Brill-Noether Theorem implies that the cycle $\mathcal{M}^r_{g,d}$ is pure of codimension 1 inside $\mathcal{M}_g$. One has the following more precise statement due to Eisenbud and Harris [**EH89**]:

THEOREM 3.1. *For integers g, r and d such that $\rho(g, r, d) = -1$, the locus $\mathcal{M}^r_{g,d}$ is an irreducible divisor on $\mathcal{M}_g$. The class of its compactification $\overline{\mathcal{M}}^r_{g,d}$ inside $\overline{\mathcal{M}}_g$ is given by the following formula:*

$$\overline{\mathcal{M}}^r_{g,d} \equiv c_{g,d,r}\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\right) \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

The constant $c_{g,d,r}$ has a clear intersection-theoretic interpretation using Schubert calculus. Note that remarkably, the slope of all the Brill-Noether divisors on $\overline{\mathcal{M}}_g$ is independent of d and r and

$$s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1}$$

for all $r, d \geq 1$ satisfying $\rho(g, r, d) = -1$. For genera g such that $g+1$ is composite, one has as many Brill-Noether divisors on $\overline{\mathcal{M}}_g$ as ways of non-trivially factoring $g+1$. It is natural to raise the following:

PROBLEM 3.2. Construct an explicit linear equivalence between various Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ on $\overline{\mathcal{M}}_g$ for different integers $r, d \geq 1$ with $\rho(g, r, d) = -1$.

The simplest case is $g = 11$ when there exist two (distinct) Brill-Noether divisors $\overline{\mathcal{M}}^1_{11,6}$ and $\overline{\mathcal{M}}^2_{11,9}$ and $s(\overline{\mathcal{M}}^1_{11,6}) = s(\overline{\mathcal{M}}^2_{11,9}) = 7$. These divisors can be understood in terms of Noether-Lefschetz divisors on the moduli space $\overline{\mathcal{F}}_{11}$ of polarized $K3$ surfaces of degree $2g-2 = 20$. We recall that there exists a rational $\mathbb{P}^{11}$-fibration

$$\phi : \overline{\mathcal{M}}_{11} \dashrightarrow \overline{\mathcal{F}}_{11}, \quad \phi[C] := [S, \mathcal{O}_S(C)],$$

where S is the unique $K3$ surface containing C, see [**M94**]. Noting that $\mathcal{M}^1_{11,6} = \mathcal{M}^5_{11,14}$ it follows that

$$\mathcal{M}^1_{11,6} = \phi^*_{|\mathcal{M}_{11}}(NL_1),$$

where NL_1 is the Noether-Lefschetz divisor on $\mathcal{F}_{11}$ of polarized $K3$ surfaces S with Picard lattice $\mathrm{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)] + \mathbb{Z} \cdot [C]$, where $C^2 = 20$ and $C \cdot c_1(\mathcal{O}_S(1)) = 14$. Similarly, by Riemann-Roch, we have an equality of divisors $\mathcal{M}^2_{11,9} = \mathcal{M}^3_{11,11}$, and then

$$\mathcal{M}^2_{11,9} = \phi^*_{|\mathcal{M}_{11}}(NL_2),$$

with NL_2 being the Noether-Lefschetz divisor whose general point corresponds to a quartic surface $S \subset \mathbb{P}^3$ with $\mathrm{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)] + \mathbb{Z} \cdot [C]$, where $C^2 = 20$ and $C \cdot c_1(\mathcal{O}_S(1)) = 11$. It is not clear whether NL_1 and NL_2 should be linearly equivalent on $\mathcal{F}_{11}$.

The next interesting case is $g = 23$, see [**Fa00**]: The three (distinct) Brill-Noether divisors $\overline{\mathcal{M}}^1_{23,12}$, $\overline{\mathcal{M}}^2_{23,17}$ and $\overline{\mathcal{M}}^1_{23,20}$ are multicanonical in the sense that there exist explicitly known integers $m, m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ and an effective boundary divisor $E \equiv \sum_{i=1}^{11} c_i \delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{23})$ such that

$$m_1 \cdot \overline{\mathcal{M}}^1_{23,12} + E \equiv m_2 \cdot \overline{\mathcal{M}}^2_{23,17} + E = m_3 \cdot \overline{\mathcal{M}}^3_{23,20} + E \in |mK_{\overline{\mathcal{M}}_{23}}|.$$

QUESTION 3.3. For a genus g such that $g+1$ is composite, is there a good geometric description of the stable base locus

$$\mathbf{B}\big(\overline{\mathcal{M}}_g, |\overline{\mathcal{M}}^r_{g,d}|\big) := \bigcap_{n \geq 0} \mathrm{Bs}(\overline{\mathcal{M}}_g, |n\overline{\mathcal{M}}^r_{g,d}|)$$

of the Brill-Noether linear system? It is clear that $\mathbf{B}(\overline{\mathcal{M}}_g, |\overline{\mathcal{M}}^r_{g,d}|)$ contains important subvarieties of $\overline{\mathcal{M}}_g$ like the hyperelliptic and trigonal locus, cf. [**HaM90**].

Of the higher codimension Brill-Noether cycles, the best understood are the d-gonal loci

$$\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \cdots \subset \mathcal{M}^1_{g,d} \subset \cdots \subset \mathcal{M}_g.$$

Each stratum $\mathcal{M}^1_{g,d}$ is an irreducible variety of dimension $2g+2d-5$. The gonality stratification of $\mathcal{M}_g$, apart from being essential in the statement of Green's Conjecture, has often been used for cohomology calculations or for bounding the cohomological dimension and the affine covering number of $\mathcal{M}_g$.

3.2. Koszul cycles. Koszul cohomology behaves like the usual cohomology in many regards. Notably, it can be computed in families, see [**BG85**], or the book [**ApN08**]:

THEOREM 3.4. *Let $f : X \to S$ be a flat family of projective varieties, parameterized by an integral scheme, $L \in \mathrm{Pic}(X/S)$ a line bundle and $p, q \in \mathbb{Z}$. Then there exists a coherent sheaf $\mathcal{K}_{p,q}(X/S, L)$ on X and a nonempty Zariski open subset $U \subset S$ such that for all $s \in U$ one has that $\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$.*

In the statement above, the open set U is precisely determined by the condition that all $h^i(X_s, L_s)$ are minimal.

By Theorem 3.4, Koszul cohomology can be used to construct effective determinantal cycles on the moduli spaces of smooth curves. This works particularly well for Koszul cohomology of canonical curves, as h^i remain constant over the whole moduli space. More generally, Koszul cycles can be defined over the relative Picard stack over the moduli space. Under stronger assumptions, the canonically defined determinantal structure can be given a better description. To this end, one uses the description provided by Lazarsfeld kernel bundles.

In many cases, for example canonical curves, or 2-regular varieties, the property (N_p) reduces to the single condition $K_{p,2}(X, L) = 0$, see for instance [**Ein87**] Proposition 3. This phenomenon justifies the study of various loci given by the non-vanishing of $K_{p,2}$. Note however that extending this determinantal description over the boundary of the moduli stack (especially over the locus of reducible stable curves) poses considerable technical difficulties, see [**Fa06a**], [**Fa06b**]. We now describe a general set-up used to compute Koszul non-vanishing loci over a partial compactification $\widetilde{\mathcal{M}}_g$ of the moduli space $\mathcal{M}_g$ inside $\overline{\mathcal{M}}_g$. As usual, if $\mathbf{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its associated coarse moduli space.

We fix integers $r, d \geq 1$, such that $\rho(g, r, d) = 0$ and denote by $\mathbf{M}_g^0 \subset \mathbf{M}_g$ the open substack classifying curves $[C] \in \mathcal{M}_g$ such that $W^r_{d-1}(C) = \emptyset$ and $W^{r+1}_d(C) = \emptyset$. Since $\rho(g, r+1, d) \leq -2$ and $\rho(g, r, d-1) = -r-1 \leq -2$, it follows from [**EH89**] that $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}_g^0, \mathcal{M}_g) \geq 2$. We further denote by $\Delta_0^0 \subset \Delta_0 \subset \overline{\mathcal{M}}_g$ the locus of nodal curves $[C_{yq} := C/y \sim q]$, where $[C] \in \mathcal{M}_{g-1}$ is a curve that satisfies the Brill-Noether Theorem and $y, q \in C$ are *arbitrary* distinct points. Finally, $\Delta_1^0 \subset \Delta_1 \subset \overline{\mathcal{M}}_g$ denotes the open substack classifying curves $[C \cup_y E]$, where $[C] \in \mathcal{M}_{g-1}$ is Brill-Noether general, $y \in C$ is an arbitrary point and $[E, y] \in \overline{\mathcal{M}}_{1,1}$ is an arbitrary elliptic tail. Note that every Brill-Noether general curve $[C] \in \mathcal{M}_{g-1}$ satisfies

$$W^r_{d-1}(C) = \emptyset, \ \ W^{r+1}_d(C) = \emptyset \ \text{ and } \ \dim W^r_d(C) = \rho(g-1, r, d) = r.$$

We set $\widetilde{\mathbf{M}}_g := \mathbf{M}_g^0 \cup \Delta_0^0 \cup \Delta_1^0 \subset \overline{\mathbf{M}}_g$ and we regard it as a partial compactification of $\mathbf{M}_g$. Then following [**EH86**] we consider the Deligne-Mumford stack

$$\sigma_0 : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathbf{M}}_g$$

classifying pairs $[C, l]$ with $[C] \in \widetilde{\mathcal{M}}_g$ and l a limit linear series of type $\mathfrak{g}^r_d$ on C. We remark that for any curve $[C] \in \mathcal{M}_g^0 \cup \Delta_0^0$ and $L \in W^r_d(C)$, we have that $h^0(C, L) = r + 1$ and that L is globally generated. Indeed, for a smooth curve $[C] \in \mathcal{M}_g^0$ it follows that $W^{r+1}_d(C) = \emptyset$, so necessarily $W^r_d(C) = G^r_d(C)$. For a point $[C_{yq}] \in \Delta_0^0$ we have the identification

$$\sigma_0^{-1}[C_{yq}] = \{L \in W_d^r(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r\},$$

where we note that since the normalization $[C] \in \mathcal{M}_{g-1}$ is assumed to be Brill-Noether general, any sheaf $L \in \sigma_0^{-1}[C_{yq}]$ satisfies

$$h^0(C, L \otimes \mathcal{O}_C(-y)) = h^0(C, L \otimes \mathcal{O}_C(-q)) = r$$

and $h^0(C, L) = r+1$. Furthermore, $\overline{W}_d^r(C_{yq}) = W_d^r(C_{yq})$, where the left-hand-side denotes the closure of $W_d^r(C_{yq})$ inside the variety $\overline{\mathrm{Pic}}^d(C_{yq})$ of torsion-free sheaves on C_{yq}. This follows because a non-locally free torsion-free sheaf in $\overline{W}_d^r(C_{yq}) - W_d^r(C_{yq})$ is of the form $\nu_*(A)$, where $A \in W_{d-1}^r(C)$ and $\nu : C \to C_{yq}$ is the normalization map. But we know that $W_{d-1}^r(C) = \emptyset$, because $[C] \in \mathcal{M}_{g-1}$ satisfies the Brill-Noether Theorem. The conclusion of this discussion is that $\sigma : \widetilde{\mathfrak{G}}_d^r \to \widetilde{\mathbf{M}}_g$ is proper. Since $\rho(g, r, d) = 0$, by general Brill-Noether theory, there exists a unique irreducible component of $\mathfrak{G}_d^r$ which maps onto $\mathbf{M}_g^0$.

In [**Fa06b**], a universal Koszul non-vanishing locus over a partial compactification of the moduli space of curves is introduced. Precisely, one constructs two locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ over $\widetilde{\mathfrak{G}}_d^r$ such that for a point $[C, l]$ corresponding to a *smooth* curve $[C] \in \mathcal{M}_g^0$ and a (necessarily complete and globally generated linear series) $l = (L, H^0(C, L)) \in G_d^r(C)$ inducing a map $C \xrightarrow{|L|} \mathbb{P}^r$, we have the following description of the fibres:

$$\mathcal{A}(C, L) = H^0\big(\mathbb{P}^r, \wedge^p M_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(2)\big) \ \text{ and } \mathcal{B}(C, L) = H^0\big(C, \wedge^p M_L \otimes L^{\otimes 2}\big).$$

There is a natural vector bundle morphism $\phi : \mathcal{A} \to \mathcal{B}$ given by restriction. From Grauert's Theorem it follows that both $\mathcal{A}$ and $\mathcal{B}$ are vector bundles over $\mathfrak{G}_d^r$ and from Bott's Theorem (in the case of $\mathcal{A}$) and Riemann-Roch (in the case of $\mathcal{B}$) respectively, we compute their ranks

$$\mathrm{rank}(\mathcal{A}) = (p+1)\binom{r+2}{p+2} \ \text{ and } \ \mathrm{rank}(\mathcal{B}) = \binom{r}{p}\Big(-\frac{pd}{r} + 2d + 1 - g\Big).$$

Note that M_L is a stable vector bundle (again, one uses that $[C] \in \mathcal{M}_g^0$), hence $H^1(C, \wedge^p M_L \otimes L^{\otimes 2}) = 0$ and then $\mathrm{rank}(\mathcal{B}) = \chi(C, \wedge^p M_L \otimes L^{\otimes 2})$ can be computed from Riemann-Roch. We have the following result, cf. [**Fa06b**] Theorem 2.1:

THEOREM 3.5. *The cycle*

$$\mathcal{U}_{g,p} := \{(C, L) \in \mathfrak{G}_d^r : K_{p,2}(C, L) \neq 0\},$$

is the degeneracy locus of the vector bundle map $\phi : \mathcal{A} \to \mathcal{B}$ *over* $\mathfrak{G}_d^r$.

Under suitable numerical restrictions, when $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{B})$, the cycle constructed in Theorem 3.5 is a virtual divisor on $\mathfrak{G}_d^r$. This happens precisely when

$$r := 2s + sp + p, \ g := rs + s \text{ and } d := rs + r.$$

for some $p \geq 0$ and $s \geq 1$. The first remarkable case occurs when $s = 1$. Set $g = 2p+3$, $r = g-1 = 2p+2$, and $d = 2g-2 = 4p+4$. Note that, since the canonical bundle is the only $\mathfrak{g}_{2g-2}^{g-1}$ on a curve of genus g, the Brill-Noether stack is isomorphic to $\mathbf{M}_g$. The notable fact that the cycle in question is an actual divisor follows directly from Voisin's Theorem 2.12 and from Green's Hyperplane Section Theorem [**V05**].

Hence $\mathcal{Z}_{g,p} := \sigma_*(\mathcal{U}_{g,p})$ is a virtual divisor on $\mathcal{M}_g$ whenever

$$g = s(2s + sp + p + 1).$$

In the next section, we explain how to extend the morphism $\phi : \mathcal{A} \to \mathcal{B}$ to a morphism of locally free sheaves over the stack $\widetilde{\mathfrak{G}}^r_d$ of limit linear series and reproduce the class formula proved in [**Fa06a**] for the degeneracy locus of this morphism.

3.3. Divisors of small slope. In [**Fa06a**] it was shown that the determinantal structure of $\mathcal{Z}_{g,p}$ can be extended over $\overline{\mathcal{M}}_g$ in such a way that whenever $s \geq 2$, the resulting virtual slope violates the Harris-Morrison Slope Conjecture. One has the following general statement:

THEOREM 3.6. *If $\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\boldsymbol{M}}_g$ denotes the compactification of $\mathfrak{G}^r_d$ given by limit linear series, then there exists a natural extension of the vector bundle morphism $\phi : \mathcal{A} \to \mathcal{B}$ over $\widetilde{\mathfrak{G}}^r_d$ such that $\overline{\mathcal{Z}}_{g,p}$ is the image of the degeneracy locus of ϕ. The class of the pushforward to $\widetilde{\boldsymbol{M}}_g$ of the virtual degeneracy locus of ϕ is given by*

$$\sigma_*(c_1(\mathcal{G}_{p,2} - \mathcal{H}_{p,2})) \equiv a\lambda - b_0\delta_0 - b_1\delta_1 - \cdots - b_{[\frac{g}{2}]}\delta_{[\frac{g}{2}]},$$

where $a, b_0, \ldots, b_{[\frac{g}{2}]}$ are explicitly given coefficients such that $b_1 = 12b_0 - a$, $b_i \geq b_0$ for $1 \leq i \leq [g/2]$ and

$$s\big(\sigma_*(c_1(\mathcal{G}_{p,2} - \mathcal{H}_{p,2}))\big) = \frac{a}{b_0} = 6\frac{f(s,p)}{(p+2)\ sh(s,p)},\ \textit{with}$$

$$f(s,p) = (p^4 + 24p^2 + 8p^3 + 32p + 16)s^7 + (p^4 + 4p^3 - 16p - 16)s^6 - (p^4 + 7p^3 + 13p^2 - 12)s^5 - (p^4 + 2p^3 + p^2 + 14p + 24)s^4 + (2p^3 + 2p^2 - 6p - 4)s^3 + (p^3 + 17p^2 + 50p + 41)s^2 + (7p^2 + 18p + 9)s + 2p + 2$$

and

$$h(s,p) = (p^3 + 6p^2 + 12p + 8)s^6 + (p^3 + 2p^2 - 4p - 8)s^5 - (p^3 + 7p^2 + 11p + 2)s^4 - (p^3 - 5p)s^3 + (4p^2 + 5p + 1)s^2 + (p^2 + 7p + 11)s + 4p + 2.$$

Furthermore, we have that

$$6 < \frac{a}{b_0} < 6 + \frac{12}{g+1}$$

whenever $s \geq 2$. If the morphism ϕ is generically non-degenerate, then $\overline{\mathcal{Z}}_{g,p}$ is a divisor on $\overline{\mathcal{M}}_g$ which gives a counterexample to the Slope Conjecture for $g = s(2s + sp + p + 1)$.

A few remarks are necessary. In the case $s = 1$ and $g = 2p + 3$, the vector bundles $\mathcal{A}$ and $\mathcal{B}$ exist not only over a partial compactification of $\widetilde{\mathbf{M}}_g$ but can be extended (at least) over the entire stack $\mathbf{M}_g \cup \Delta_0$ in such a way that $\mathcal{B}(C, \omega_C) = H^0(C, \wedge^p M_{\omega_C} \otimes \omega_C^2)$ for any $[C] \in \mathcal{M}_g \cup \Delta_0$. Theorem 3.6 reads in this case, see also [**Fa08**] Theorem 5.7:

$$(5)\quad [\overline{\mathcal{Z}}_{2p+3,p}]^{virt} = c_1(\mathcal{B} - \mathcal{A}) = \frac{1}{p+2}\binom{2p}{p}\Big(6(p+3)\lambda - (p+2)\delta_0 - 6(p+1)\delta_1 - \cdots\Big),$$

in particular $s([\overline{\mathcal{Z}}_{2p+3,p}]^{virt}) = 6 + 12/(g+1)$.

Particularly interesting is the case $p = 0$ when the condition $K_{0,2}(C, L) = 0$ for $[C, L] \in \mathfrak{G}^r_d$, is equivalent to the multiplication map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

not being an isomorphism. Note that $\nu_2(L)$ is a linear map between vector spaces of the same dimension and $\mathcal{Z}_{g,0}$ is the failure locus of the Maximal Rank Conjecture:

COROLLARY 3.7. *For $g = s(2s+1), r = 2s, d = 2s(s+1)$ the slope of the virtual class of the locus of those $[C] \in \overline{\mathcal{M}}_g$ for which there exists $L \in W^r_d(C)$ such that the embedded curve $C \overset{|L|}{\hookrightarrow} \boldsymbol{P}^r$ sits on a quadric hypersurface, is equal to*

$$s(\overline{\mathcal{Z}}_{s(2s+1),0}) = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}.$$

4. Conjectures on Koszul cohomology of curves

4.1. Green's Conjecture. In what follows, we consider (C, K_C) a smooth canonical curve of genus $g \geq 2$. In this case, the Duality Theorem 2.7 applies, and the distribution of the numbers $b_{p,q} := \dim K_{p,q}(C, K_C)$ organized in a table (the *Betti table*) is the following:

$$\begin{bmatrix} b_{0,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & b_{1,1} & b_{2,1} & \dots & b_{g-3,1} & b_{g-2,1} \\ b_{0,2} & b_{1,2} & b_{2,2} & \dots & b_{g-3,2} & 0 \\ 0 & 0 & 0 & \dots & 0 & b_{g-2,3} \end{bmatrix}$$

The Betti table is symmetric with respect to its center, that is, $b_{i,j} = b_{g-2-i,3-j}$ and all the other entries not marked here are zero.

Trying to apply the Non-Vanishing Theorem 2.4 to the canonical bundle K_C, we obtain one *condition* and one *quantity*. The condition comes from the hypothesis that for a decomposition $K_C = L_1 \otimes (K_C \otimes L_1^\vee)$, Theorem 2.4 is applicable whenever

$$(6) \qquad r_1 + 1 := h^0(C, L_1) \geq 2 \text{ and } r_2 + 1 := h^1(C, L_1) \geq 2.$$

A line bundle L_1 satisfying (6) is said to *contribute to the Clifford index of C*.

The quantity that appears in Theorem 2.4 is the *Clifford index* itself. More precisely

$$r_1 + r_2 - 1 = g - \mathrm{Cliff}(L_1) - 2,$$

where

$$\mathrm{Cliff}(L_1) := \deg(L_1) - 2h^0(L_1) + 2.$$

Clifford's Theorem [**ACGH85**] says that $\mathrm{Cliff}(L_1) \geq 0$, and $\mathrm{Cliff}(L_1) > 0$ unless L_1 is a $\mathfrak{g}^1_2$. Following [**GL86**] we define the *Clifford index of C* as the quantity

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(L_1) : \ L_1 \text{ contributes to the Clifford index of } C\}.$$

In general, the Clifford index will be computed by minimal pencils. Specifically, a general d-gonal curve $[C] \in \mathcal{M}^1_{g,d}$ (recall that the gonality strata are irreducible) will have $\mathrm{Cliff}(C) = d - 2$. However, this equality is not valid for all curves, that is, there exist curves $[C] \in \mathcal{M}_g$ with $\mathrm{Cliff}(C) < \mathrm{gon}(C) - 2$, basic examples being plane curves, or exceptional curves on $K3$ surfaces. Even in these exotic cases, Coppens and Martens [**CM91**] established the precise relation $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$.

Theorem 2.4 implies the following non-vanishing

$$K_{g-\mathrm{Cliff}(C)-2,1}(C, K_C) \neq 0,$$

and Green's Conjecture predicts optimality of Theorem 2.4 for canonical curves:

CONJECTURE 4.1. *For any curve $[C] \in \mathcal{M}_g$ we have the vanishing $K_{p,1}(C, K_C) = 0$ for all $p \geq g - \mathrm{Cliff}(C) - 1$.*

In the statement of Green's Conjecture, it suffices to prove the vanishing of $K_{g-\mathrm{Cliff}(C)-1,1}(C,K_C)$ or, by duality, that $K_{\mathrm{Cliff}(C)-1,2}(C,K_C)=0$.

We shall analyze some basic cases:

EXAMPLE 4.2. Looking at the group $K_{0,2}(C,K_C)$, Green's Conjecture predicts that it is zero for all non-hyperelliptic curves. Or, the vanishing of $K_{0,2}(C,K_C)$ is equivalent to the projective normality of the canonical curve. This is precisely the content of the classical Max Noether Theorem [**ACGH85**], p. 117.

EXAMPLE 4.3. For a non-hyperelliptic curve, we know that $K_{1,2}(C,K_C)=0$ if and only if the canonical curve $C\subset\mathbb{P}^{g-1}$ is cut out by quadrics. Green's Conjecture predicts that $K_{1,2}(C,K_C)=0$ unless the curve is hyperelliptic, trigonal or a smooth plane quintic. This is precisely the Enriques-Babbage-Petri Theorem, see [**ACGH85**], p. 124.

Thus Conjecture 4.1 appears as a sweeping generalization of two famous classical theorems. Apart from these classical results, strong evidence has been found for Green's Conjecture (and one should immediately add, that not a shred of evidence has been found suggesting that the conjecture might fail even for a single curve $[C]\in\mathcal{M}_g$). For instance, the conjecture is true for general curves in *any* gonality stratum $\mathcal{M}^1_{g,d}$, see [**Ap05**], [**Tei02**] and [**V02**]. The proof of this fact relies on semicontinuity. Since $\mathcal{M}^1_{g,d}$ is irreducible, it suffices to find one example of a d-gonal curve that satisfies the conjecture, for any $2\le d\le (g+2)/2$; here we also need the fact mentioned above, that the Clifford index of a general d-gonal curve is $d-2$. The most important and challenging case, solved by Voisin [**V05**], was the case of curves of odd genus $g=2d-1$ and maximal gonality $d+1$. Following Hirschowitz and Ramanan [**HR98**] one can compare the Brill-Noether divisor $\mathcal{M}^1_{g,d}$ of curves with a $\mathfrak{g}^1_d$ and the virtual divisor of curves $[C]\in\mathcal{M}_g$ with $K_{d-1,1}(C,K_C)\neq 0$. The non-vanishing Theorem 2.4 gives a set-theoretic inclusion $\mathcal{M}^1_{g,d}\subset\mathcal{Z}_{g,d-2}$. Now, we compare the class $[\mathcal{Z}_{g,d-2}]^{virt}\in\mathrm{Pic}(\mathcal{M}_g)$ of the virtual divisor $\mathcal{Z}_{g,d-2}$ to the class $[\mathcal{M}^1_{g,d}]$ computed in [**HM82**]. One finds the following relation

$$[\mathcal{Z}_{g,d-2}]^{virt}=(d-1)[\mathcal{M}^1_{g,d}]\in\mathrm{Pic}(\mathcal{M}_g)$$

cf. [**HR98**]; Theorem 3.6 in the particular case $s=1$ provides an extension of this equality to a partial compactification of $\mathcal{M}_g$. Green's Conjecture for general curves of odd genus [**V05**] implies that $\mathcal{Z}_{g,d-2}$ is a genuine divisor on $\mathcal{M}_g$. Since a general curve $[C]\in\mathcal{M}^1_{g,d}$ satisfies

$$\dim K_{d-1,1}(C,K_C)\ge d-1,$$

cf. [**HR98**], one finds the set-theoretic equality $\mathcal{M}^1_{g,d}=\mathcal{Z}_{g,d-2}$. In particular we obtain the following strong characterization of curves of odd genus and maximal gonality:

THEOREM 4.4 (Hirschowitz-Ramanan, Voisin). *If C is a smooth curve of genus $g=2d-1\ge 7$, then $K_{d-1,1}(C,K_C)\neq 0$ if and only if C carries a $\mathfrak{g}^1_d$.*

Voisin proved Theorem 2.12, using Hilbert scheme techniques, then she applied Green's Hyperplane Section Theorem [**Gr84a**] to obtain the desired example of a curve $[C]\in\mathcal{M}_g$ satisfying Green's Conjecture.

Starting from Theorem 4.4, all the other generic d-gonal cases are obtained in the following refined form, see [**Ap05**]:

THEOREM 4.5. *We fix integers g and $d \geq 2$ such that $2 \leq d \leq [g/2]+1$. For any smooth curve $[C] \in \mathcal{M}_g$ satisfying the condition*

$$\dim W^1_{g-d+2}(C) \leq g-2d+2 = \rho(g,1,g-d+2), \tag{7}$$

we have that $K_{g-d+1,1}(C,K_C)=0$. In particular, C satisfies Green's Conjecture.

Note that the condition $d \leq [g/2]+1$ excludes the case already covered by Theorem 4.4. The proof of Theorem 4.5 relies on constructing a singular stable curve $[C'] \in \overline{\mathcal{M}}_{2g+3-2d}$ of maximal gonality $g+3-d$ (that is, $[C'] \notin \overline{\mathcal{M}}^1_{2g+3-d,g+2-d}$), starting from any smooth curve $[C] \in \mathcal{M}_g$ satisfying (7). The curve C' is obtained from C by gluing together $g+3-2d$ pairs of general points of C, and then applying an analogue of Theorem 4.4 for singular stable curves, [**Ap05**], see Section 4.2. The version in question is the following, cf. [**Ap05**] Proposition 7. The proof we give here is however slightly different:

THEOREM 4.6. *For any nodal curve $[C'] \in \mathcal{M}_{g'} \cup \Delta_0$, with $g' = 2d'-1 \geq 7$ such that $K_{d'-1,1}(C',\omega_{C'}) \neq 0$, it follows that $[C'] \in \overline{\mathcal{M}}^1_{g',d'}$.*

PROOF. By duality, we obtain the following equality of cycles on $\widetilde{\mathcal{M}}_{g'}$:

$$\{[C'] : \ K_{d'-1,1}(C',\omega_{C'}) \neq 0\} = \{[C'] : \ K_{d'-2,2}(C',\omega_{C'}) \neq 0\} =: \overline{\mathcal{Z}}_{g',d'-2}.$$

Theorem 3.6 shows that this locus is a virtual divisor on $\widetilde{\mathcal{M}}_{g'}$ whose class is given by formula (5) and Theorem 2.12 implies that $\overline{\mathcal{Z}}_{g',d'-2}$ is actually a divisor. Comparing its class against the class of the Hurwitz divisor $\overline{\mathcal{M}}^1_{g',d'}$ [**HM82**], we find that

$$\overline{\mathcal{Z}}_{g',d'-2} \equiv (d'-1)\overline{\mathcal{M}}^1_{g',d'} \in \mathrm{Pic}(\widetilde{\mathbf{M}}_{g'}).$$

Note that this is a stronger statement than [**HR98**] Proposition 3.1, being an equality of codimension 1 cycles on the compactified moduli space $\widetilde{\mathcal{M}}_{g'}$, rather than on $\mathcal{M}_{g'}$. The desired statement follows immediately since for any curve $[C'] \in \mathcal{M}^1_{g',d'}$ one has $\dim K_{d'-1,1}(C',\omega_{C'}) \geq d'-1$, hence the degeneracy locus $\overline{\mathcal{Z}}_{g',d'-2}$ contains $\overline{\mathcal{M}}^1_{g',d'}$ with multiplicity at least $d'-1$. □

We return to the discussion on Theorem 4.5 (the proof will be resumed in the next subsection). By duality, the vanishing in the statement above can be rephrased as

$$K_{d-3,2}(C,K_C)=0.$$

The condition (7) is equivalent to a string of inequalities

$$\dim W^1_{d+n}(C) \leq n$$

for all $0 \leq n \leq g-2d+2$, in particular $\mathrm{gon}(C) \geq d$. This condition is satisfied for a general d-gonal curve, cf. [**Ap05**]. More generally, if $[C] \in \mathcal{M}^1_{g,d}$ is a general d-gonal curve then any irreducible component

$$Z \neq W^1_d(C) + W_{n-d}(C)$$

of $W^1_n(C)$ has dimension $\rho(g,1,n)$. In particular, for $\rho(g,1,n)<0$ it follows that $W^1_n(C) = W^1_d(C) + W_{n-d}(C)$ which of course implies (7). For $g = 2d-2$, the inequality (7) becomes necessarily an equality and it reads: the curve C carries finitely many $\mathfrak{g}^1_d$'s of minimal degree.

We make some comments regarding condition (7). Let us suppose that C is non-hyperelliptic and $d \geq 3$. From Martens' Theorem [**ACGH85**] p.191, it follows that dim $W^1_{g-d+2}(C) \leq g-d-1$. Condition (7) requires the better bound $g-2d+2 \leq g-d-1$. However, for $d=3$, the two bounds are the same, and Theorem 4.5 shows that $K_{0,0}(C,K_C)=0$, for any non-hyperelliptic curve, which is Max Noether's Theorem, see also Example 4.2. Applying Mumford's Theorem [**ACGH85**] p.193, we obtain the better bound dim $W^1_{g-d+2}(C) \leq g-d-2$ for $d \geq 4$, unless the curve is trigonal, a smooth plane quintic or a double covering of an elliptic curve. Therefore, if C is not one of the three types listed above, then $K_{1,2}(C,K_C)=0$, and we recover the of Enriques-Babbage-Petri Theorem, see also Example 4.3 (note, however, the exception made to bielliptic curves).

Keem has improved the dimension bounds for $W^1_{g-d+2}(C)$. For $d \geq 5$ and C a curve that has neither a $\mathfrak{g}^1_4$ nor is a smooth plane sextic, one has the inequality dim $W^1_{g-d+2}(C) \leq g-d-3$, cf. [**Ke90**] Theorems 2.1 and 2.3. Consequently, Theorem 4.5 implies the following result which is a complete solution to Green's Conjecture for 5-gonal curves:

Theorem 4.7 (Voisin [**V88**], Schreyer [**Sch91**]). *If $K_{2,2}(C,K_C) \neq 0$, then C is hyperelliptic, trigonal, tetragonal or a smooth plane sextic, that is,* $\text{Cliff}(C) \leq 2$.

Geometrically, the vanishing of $K_{2,2}(C,K_C)$ is equivalent to the ideal of the canonical curve being generated by quadrics, and the minimal relations among the generators being linear.

Theorem 3.1 from [**Ke90**] gives the next bound dim $W^1_{g-d+2}(C) \leq g-d-4$, for $d \geq 6$ and C with $\text{gon}(C) \geq 6$ which does not admit a covering of degree two or three on another curve, and which is not a plane curve. The following improvement of Theorem 4.7 is then obtained directly from Theorem 4.5 and [**Ap05**] Theorem 3.1:

Theorem 4.8. *If $g \geq 12$ and $K_{3,2}(C,K_C) \neq 0$, then C is one of the following: hyperelliptic, trigonal, tetragonal, pentagonal, double cover of an genus* 3 *curve, triple cover of an elliptic curve, smooth plane septic. In other words, if* $\text{Cliff}(C) \geq 4$ *then $K_{3,2}(C,K_C)=0$.*

Theorem 4.8 represents the solution to Green's Conjecture for hexagonal curves. Likewise, Theorem 4.5 can be used together with the Brill-Noether theory to prove Green's Conjecture for any gonality d and large genus. The idea is to apply Coppens' results [**Co83**].

Theorem 4.9. *If $g \geq 10$ and $d \geq 5$ are two integers such that $g > (d-2)(2d-7)$, and C is any d-gonal curve of genus g which does not admit any morphism of degree less than d onto another smooth curve, then* $\text{Cliff}(C) = d-2$ *and Green's Conjecture is verified for C, i.e. $K_{g-d+1,1}(C,K_C)=0$.*

The statement of Conjecture 4.1 (meant as a vanishing result), is empty for hyperelliptic curves is, hence the interesting cases begin with $d \geq 3$.

It remains to verify Green's Conjecture for curves which do not verify (7). One result in this direction was proved in [**ApP06**].

Theorem 4.10. *Let S be a $K3$ surface with* $\text{Pic}(S) = \mathbb{Z} \cdot H \oplus Z \cdot \ell$, *with H very ample, $H^2 = 2r-2 \geq 4$, and $H \cdot \ell = 1$. Then any smooth curve in the linear system $|2H+\ell|$ verifies Green's conjecture.*

Smooth curves in the linear system $|2H+\ell|$ count among the few known examples of curves whose Clifford index is not computed by pencils, i.e. $\mathrm{Cliff}(C) = \mathrm{gon}(C)-3$, [**ELMS89**] (other obvious examples are plane curves, for which Green's Conjecture was verified before, cf. [**Lo89**]). Such curves are the most special ones in the moduli space of curves from the point of view of the Clifford dimension. Hence, this case may be considered as opposite to that of a general curve of fixed gonality. Note that these curves carry a one-parameter family of pencils of minimal degree, hence the condition (7) is not satisfied.

4.2. The Gonality Conjecture. The Green-Lazarsfeld Gonality Conjecture [**GL86**] predicts that the gonality of a curve can be read off the Koszul cohomology with values in any sufficiently positive line bundle.

Conjecture 4.11 (Green-Lazarsfeld). *Let C be a smooth curve of gonality d, and L a sufficiently positive line bundle on C. Then*

$$K_{h^0(L)-d,1}(C,L)=0.$$

Theorem 2.8 applied to L written as a sum of a minimal pencil and the residual bundle yields

$$K_{h^0(L)-d-1,1}(C,L)\neq 0.$$

Note that if L is sufficiently positive, then the Green-Lazarsfeld Nonvanishing Theorem is optimal when applied for a decomposition where one factor is a pencil. Indeed, consider any decomposition $L = L_1 \otimes L_2$ with $r_1 = h^0(C,L_1)-1 \geq 2$, and $r_2 = h^0(C,L_2)-1 \geq 2$. Since L is sufficiently positive, the linear system $|K_C^{\otimes 2} \otimes L^\vee|$ is empty, and finiteness of the addition map of divisors shows that at least one of the two linear systems $|K_C \otimes L_i^\vee|$ is empty. Suppose $|K_C \otimes L_2^\vee| = \emptyset$, choose a point $x \in C - \mathrm{Bs}(|L_1|)$ and consider a new decomposition $L = L_1' \otimes L_2'$, with $L_1' = L_1 \otimes \mathcal{O}_C(-x)$, and $L_2' = L_2 \otimes \mathcal{O}_C(x)$. Denoting as usual $r_i' = h^0(C,L_i')-1$, we find that $r_1'+r_2'-1 = r_1+r_2-1$, whereas $r_1' = r_1-1$, and L_2' is again non-special. We can apply an inductive argument until r_1 becomes 1. Hence the Gonality Conjecture predicts the optimality of the Green-Lazarsfeld Nonvanishing Theorem. However, one major disadvantage of this statement is that "sufficiently positive" is not a precise condition. It was proved in [**Ap02**] that by adding effective divisors to bundles that verify the Gonality Conjecture we obtain again bundles that verify the conjecture. Hence, in order to find a precise statement for Conjecture 4.11 one has to study the edge cases.

In most generic cases (general curves in gonality strata, to be more precise), the Gonality Conjecture can be verified for line bundles of degree $2g$, see [**ApV03**] and [**Ap05**]. The test bundles are obtained by adding two generic points to the canonical bundle.

Theorem 4.12 ([**Ap05**]). *For any d-gonal curve $[C] \in \mathcal{M}_g$ with $d \leq [g/2]+2$ which satisfies the condition (7), and for general points $x, y \in C$, we have that $K_{g-d+1,1}\big(C, K_C \otimes \mathcal{O}_C(x+y)\big) = 0$.*

The case not covered by Theorem 4.12 is slightly different. A general curve C of odd genus carries infinitely many minimal pencils, hence a bundle of type $K_C \otimes \mathcal{O}_C(x+y)$ can never verify the vanishing predicted by the Gonality Conjecture. Indeed, for any two points x and y there exists a minimal pencil L_1 such that $H^0(C, L_1(-x-y)) \neq 0$, and we apply Theorem 2.4. However, adding *three* points to the canonical bundle solves the problem, cf. [**Ap04**], [**Ap05**].

Theorem 4.13. *For any curve $[C] \in \mathcal{M}_{2d-1}$ of maximal gonality* $\mathrm{gon}(C) = d+1$ *and for general points $x, y, z \in C$, we have that* $K_{d,1}\big(C, K_C \otimes \mathcal{O}_C(x+y+z)\big) = 0$.

The proofs of Theorems 4.5, 4.12 and 4.13 are all based on the same idea. We start with a smooth curve C and construct a stable curve of higher genus out of it, in such a way that the Koszul cohomology does not change. Then we apply a version of Theorem 4.4 for singular curves.

Proof of Theorems 4.5 and 4.12. We start with $[C] \in \mathcal{M}_g$ satisfying the condition (7). We claim that if we choose $\delta := g + 3 - 2d$ pairs of general points $x_i, y_i \in C$ for $1 \leq i \leq \delta$, then the resulting stable curve

$$\big[C' := \frac{C}{x_1 \sim y_1, \ldots, x_\delta \sim y_\delta}\big] \in \overline{\mathcal{M}}_{2g+3-2d}$$

is a curve of maximal gonality, that is, $g + 3 - d$. Indeed, otherwise $[C'] \in \overline{\mathcal{M}}^1_{2g+3-2d,g+2-d}$ and this implies that there exists a degree $g + 2 - d$ admissible covering $f : \tilde{C} \to R$ from a nodal curve $\tilde{C}$ that is semi-stably equivalent to C', onto a genus 0 curve R. The curve C is a subcurve of $\tilde{C}$ and if $\deg(f_{|C}) = n \leq g+2-d$, then it follows that $f_{|C}$ induces a pencil $\mathfrak{g}^1_n$ such that $f_{|C}(x_i) = f_{|C}(y_i)$ for $1 \leq i \leq \delta$. Since the points $x_i, y_i \in C$ are general, this implies that $\dim W^1_n(C) + \delta \geq 2\delta$, which contradicts (7).

To conclude, apply Theorem 4.6 and use the following inclusions, [**V02**], [**ApV03**]:

$$K_{g-d+1,1}(C, K_C) \subset K_{g-d+1,1}(C, K_C(x+y)) \subset K_{g-d+1,1}(C', \omega_{C'}).$$

□

Remark 4.14. The proofs of Theorems 4.5, 4.6 and 4.12 indicate an interesting phenomenon, completely independent of Voisin's proof of the generic Green Conjecture. They show that Green's Conjecture for general curves of genus $g = 2d - 1$ and maximal gonality $d + 1$ is *equivalent* to the Gonality Conjecture for bundles of type $K_C \otimes \mathcal{O}_C(x + y)$ for general pointed curves $[C, x, y] \in \mathcal{M}_{2d-2,2}$. We refer to [**Ap02**] and [**ApV03**] for further implications between the two conjectures, in both directions.

Proof of Theorem 4.13. For C as in the hypothesis, and for general points $x, y, z \in C$, we construct a stable curve $[C'] \in \mathcal{M}_{2d+1}$ by adding a smooth rational component passing through the points x, y and z. Using admissible covers one can show, as in the proofs of Theorems 4.5 and 4.12, that C' is of maximal gonality, that is $d + 2$. From Theorem 4.6, we obtain $K_{d,1}(C', \omega_{C'}) = 0$. The conclusion follows from the observation that $K_{d,1}(C, K_C \otimes \mathcal{O}_C(x + y + z)) \cong K_{d,1}(C', \omega_{C'})$. □

It is natural to ask the following:

Question 4.15. For a curve C and points $x, y \in C$, can one give explicit conditions on Koszul cohomology ensuring that $x + y$ is contained in a fiber of a minimal pencil?

We prove here the following result, which can be considered as a precise version of the Gonality Conjecture for generic curves.

Theorem 4.16. *Let $[C] \in \mathcal{M}_{2d-2}$, and $x, y \in C$ arbitrarily chosen distinct points. Then $K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y)) \neq 0$ if and only if there exists $A \in W^1_d(C)$ such that $h^0(C, A \otimes \mathcal{O}_C(-x-y)) \neq 0$.*

PROOF. Suppose there exists $A \in W^1_d(C)$ such that $h^0(C, A(-x-y)) \neq 0$. Theorem 2.8 applied to the decomposition $K_C \otimes \mathcal{O}_C(x+y) = A \otimes B$, with $B = K_C(x+y) \otimes A^\vee$ produces nontrivial classes in the group $K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y))$.

For the converse, we consider C' the stable curve obtained from C by gluing together the points x and y and denote by $\nu : C \to C'$ the normalization morphism. Clearly $[C'] \in \overline{\mathcal{M}}_{2d-1}$. We observe that

$$K_{d-1,1}(C, K_C \otimes \mathcal{O}_C(x+y)) \cong K_{d-1,1}(C', \omega_{C'}).$$

From Theorem 4.6, it follows that $[C'] \in \overline{\mathcal{M}}^1_{2d-1,d}$, hence there exists a map

$$f : \widetilde{C} \xrightarrow{d:1} R$$

from a curve $\widetilde{C}$ semistably equivalent to C' onto a rational nodal curve R. The curve C is a subcurve of $\widetilde{C}$ and $f_{|C}$ provides the desired pencil. □

As mentioned above, the lower possible bound for explicit examples of line bundles that verify the Gonality Conjecture found so far was $2g$. One can raise the question whether this bound is optimal or not and the sharpest statement one can make is Conjecture 1.4 discussed in the introduction of this paper.

5. The Strong Maximal Rank Conjecture

Based mainly on work carried out in [**Fa06a**] and [**Fa06b**] we propose a conjecture predicting the resolution of an embedded curve with general moduli. This statement unifies two apparently unrelated deep results in the theory of algebraic curves: The *Maximal Rank Conjecture* which predicts the number of hypersurfaces of each degree containing a general embedded curve $C \subset \mathbb{P}^r$ and *Green's Conjecture* on syzygies of canonical curves.

We begin by recalling the statement of the classical *Maximal Rank Conjecture*. The modern formulation of this conjecture is due to Harris [**H82**] p. 79, even though it appears that traces of a similar statement can be found in the work of Max Noether: We fix integers g, r and d such that $\rho(g, r, d) \geq 0$ and denote by $\mathfrak{I}_{d,g,r}$ the unique component of the Hilbert scheme $\mathrm{Hilb}_{d,g,r}$ of curves $C \subset \mathbb{P}^r$ with Hilbert polynomial $h_C(t) = dt + 1 - g$, containing curves with general moduli. In other words, the variety $\mathfrak{I}_{d,g,r}$ is characterized by the following properties:
(1) The general point $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ corresponds to a smooth curve $C \subset \mathbb{P}^r$ with $\deg(C) = d$ and $g(C) = g$.
(2) The moduli map $m : \mathfrak{I}_{d,g,r} \dashrightarrow \mathcal{M}_g$, $m([C \hookrightarrow \mathbb{P}^r]) := [C]$ is dominant.

CONJECTURE 5.1. (Maximal Rank Conjecture) *A general embedded smooth curve $[C \hookrightarrow \boldsymbol{P}^r] \in \mathfrak{I}_{d,g,r}$ is of maximal rank, that is, for all integers $n \geq 1$ the restriction maps*

$$\nu_n(C) : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(C, \mathcal{O}_C(n))$$

are of maximal rank, that is, either injective or surjective.

Thus if a curve $C \subset \mathbb{P}^r$ lies on a hypersurface of degree d, then either hypersurfaces of degree d cut out the complete linear series $|\mathcal{O}_C(d)|$ on the curve, or else C is special in its Hilbert scheme. Since C can be assumed to be a Petri general curve, it follows that $H^1(C, \mathcal{O}_C(n)) = 0$ for $n \geq 2$, so $h^0(C, \mathcal{O}_C(n)) = nd + 1 - g$ and Conjecture 5.1 amounts to knowing the Hilbert function of $C \subset \mathbb{P}^r$, that is, the value of $h^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n))$ for all n.

EXAMPLE 5.2. We consider the locus of curves $C \subset \mathbb{P}^3$ with $\deg(C) = 6$ and $g(C) = 3$ that lie on a quadric surface, that is, $\nu_2(C)$ fails to be an isomorphism. Such curves must be of type $(2,4)$ on the quadric, in particular, they are hyperelliptic. This is a divisorial condition on $\mathfrak{I}_{6,3,3}$, that is, for a general $[C \hookrightarrow \mathbb{P}^3] \in \mathfrak{I}_{6,3,3}$ the map $\nu_2(C)$ is an isomorphism.

Conjecture 5.1 makes sense of course for any component of $\mathfrak{I}_{d,g,r}$ but is known to fail outside the Brill-Noether range, see [**H82**]. The Maximal Rank Conjecture is known to hold in the non-special range, that is when $d \geq g + r$, due to work of Ballico and Ellia relying on the *méthode d'Horace* of Hirschowitz, see [**BE87**]. Vosin has also proved cases of the conjecture when $h^1(C, \mathcal{O}_C(1)) = 2$, cf. [**V92**]. Finally, Conjecture 5.1 is also known in the case $\rho(g,r,d) = 0$ when it has serious implications for the birational geometry of $\overline{\mathcal{M}}_g$. This case can be reduced to the case when $\dim \mathrm{Sym}^n H^0(C, \mathcal{O}_C(1)) = \dim H^0(C, \mathcal{O}_C(n))$, that is,

$$\binom{n+r}{n} = nd + 1 - g,$$

when Conjecture 5.1 amounts to constructing one smooth curve $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ such that $H^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$. In this situation, the failure locus of Conjecture 5.1 is precisely the virtual divisor $\mathcal{Z}_{g,0}$ on $\mathcal{M}_g$ whose geometry has been discussed in Section 3, Corollary 3.7. The most interesting case (at least from the point of view of slope calculations) is that of $n = 2$. One has the following result [**Fa06b**] Theorem 1.5:

THEOREM 5.3. *For each $s \geq 1$ we fix integers*

$$g = s(2s+1), \quad r = 2s \text{ and } d = 2s(s+1),$$

hence $\rho(g,r,d) = 0$. The locus

$$\mathcal{Z}_{g,0} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \text{ such that } \nu_2(L) : \mathrm{Sym}^2 H^0(C,L) \stackrel{\not\cong}{\to} H^0(C, L^{\otimes 2})\}$$

is an effective divisor on $\mathcal{M}_g$. In particular, a general curve $[C] \in \mathcal{M}_g$ satisfies the Maximal Rank Conjecture with respect to all *linear series $L \in W^r_d(C)$.*

For $s = 1$ we have the equality $\mathcal{Z}_{3,0} = \mathcal{M}^1_{3,2}$, and we recover the hyperelliptic locus on $\mathcal{M}_3$. The next case, $s = 2$ and $g = 10$, has been treated in detail in [**FaPo05**]. One has a scheme-theoretic equality $\mathcal{Z}_{10,0} = 42 \cdot \mathcal{K}_{10}$ on $\mathcal{M}_{10}$, where $42 = \#(W^4_{12}(C))$ is the number of minimal pencils $\mathfrak{g}^1_6 = K_C(-\mathfrak{g}^4_{12})$ on a general curve $[C] \in \mathcal{M}_{10}$. Thus a curve $[C] \in \mathcal{M}_{10}$ fails the Maximal Rank Conjecture for a linear series $L \in W^4_{12}(C)$ if and only if it fails it for all the 42 linear series $\mathfrak{g}^4_{12}$! This incarnation of the $K3$ divisor $\mathcal{K}_{10}$ is instrumental in being able to compute the class of $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$, cf. [**FaPo05**].

In view of Theorem 5.3 it makes sense to propose a much stronger form of Conjecture 5.1, replacing the generality assumption of $[C \hookrightarrow \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$ by a generality assumption of $[C] \in \mathcal{M}_g$ with respect to moduli and asking for the maximal rank of the curve with respect to all linear series $\mathfrak{g}^r_d$.

We fix positive integers g, r, d such that $g - d + r \geq 0$ and satisfying

$$0 \leq \rho(g,r,d) < r - 2.$$

We also fix a general curve $[C] \in \mathcal{M}_g$. The numerical assumptions imply that all the linear series $l \in G^r_d(C)$ are complete (the inequality $\rho(g, r+1, d) < 0$ is satisfied), as well as very ample. For each (necessarily complete) linear series $l = (L, H^0(C, L)) \in G^r_d(C)$ and integer $n \geq 2$, we denote by

$$\nu_n(L) : \mathrm{Sym}^n H^0(C, L) \to H^0(C, L^{\otimes n})$$

the multiplication map of global sections. We then choose a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$ and construct two vector bundles $\mathcal{E}_n$ and $\mathcal{F}_n$ over $G^r_d(C)$ with $\mathrm{rank}(\mathcal{E}_n) = \binom{r+n}{n}$ and $\mathrm{rank}(\mathcal{F}_n) = h^0(C, L^{\otimes n}) = nd + 1 - g$, together with a bundle morphism $\phi_n : \mathcal{E}_n \to \mathcal{F}_n$, such that for $L \in G^r_d(C)$ we have that

$$\mathcal{E}_n(L) = \mathrm{Sym}^n H^0(C, L) \ \text{ and } \mathcal{F}_n(L) = H^0(C, L^{\otimes n})$$

and $\nu_n(L)$ is the map given by multiplication of global sections.

Conjecture 5.4. (Strong Maximal Rank Conjecture) *We fix integers $g, r, d \geq 1$ and $n \geq 2$ as above. For a general curve $[C] \in \mathcal{M}_g$, the determinantal variety*

$$\Sigma^r_{n,g,d}(C) := \{L \in G^r_d(C) : \nu_n(L) \text{ is not of maximal rank}\}$$

has expected dimension, that is,

$$\dim \Sigma^r_{n,g,d}(C) = \rho(g, r, d) - 1 - |\mathrm{rank}(\mathcal{E}_n) - \mathrm{rank}(\mathcal{F}_n)|,$$

where by convention, negative dimension means that $\Sigma^r_{n,g,d}(C)$ is empty.

For instance, in the case $\rho(g, r, d) < nd + 2 - g - \binom{r+n}{n}$, the conjecture predicts that for a general $[C] \in \mathcal{M}_g$ we have that $\Sigma^r_{n,g,d}(C) = \emptyset$, that is,

$$H^0(\mathbb{P}^r, \mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$$

for *every* embedding $C \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r$ given by $L \in G^r_d(C)$.

When $\rho(g, r, d) = 0$ (and in particular whenever $r \leq 3$), using a standard monodromy argument showing the uniqueness the component $\mathfrak{I}_{d,g,r}$, the Strong Maximal Rank Conjecture is equivalent to Conjecture 5.1, and it states that $\nu_n(L)$ is of maximal rank for a general $[C \stackrel{L}{\hookrightarrow} \mathbb{P}^r] \in \mathfrak{I}_{d,g,r}$.

For $\rho(g, r, d) \geq 1$ however, Conjecture 5.4 seems to be a more difficult question than Conjecture 5.1 because one requires a way of seeing *all* linear series $L \in G^r_d(C)$ at once.

Remark 5.5. The bound $\rho(g, r, d) < r - 2$ in the statement of Conjecture 5.4 implies that all linear series $L \in G^r_d(C)$ on a general curve C, are very ample.

Remark 5.6. We discuss Conjecture 5.4 when $r = 4$ and $\rho(g, r, d) = 1$. The conjecture is trivially true for $g = 1$. The first interesting case is $g = 6$ and $d = 9$. For a general curve $[C] \in \mathcal{M}_6$ we observe that there is an isomorphism $C \cong W^4_9(C)$ given by $C \ni x \mapsto K_C \otimes \mathcal{O}_C(-x)$. Since $\mathrm{rank}(\mathcal{E}_2) = 15$ and $\mathrm{rank}(\mathcal{F}_2) = 13$, Conjecture 5.4 predicts that,

$$\nu_2(K_C(-x)) : \mathrm{Sym}^2 H^0\big(C, K_C \otimes \mathcal{O}_C(-x)\big) \twoheadrightarrow H^0\big(C, K_C^{\otimes 2} \otimes \mathcal{O}_C(-2x)\big),$$

for all $x \in C$, which is true (use the Base-Point-Free Pencil Trick).

The next case is $g = 11, d = 13$, when the conjecture predicts that the map

$$\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is injective for all $L \in W^4_{13}(C)$. This follows (non-trivially) from [**M94**]. Another case that we checked is $r = 5, g = 14$ and $d = 17$, when $\rho(14, 5, 17) = 2$.

Proposition 5.7. *The Strong Maximal Rank Conjecture holds for general non-special curves, that is, when $r = d - g$.*

Proof. This is an immediate application of a theorem of Mumford's stating that for any line bundle $L \in \mathrm{Pic}^d(C)$ with $d \geq 2g+1$, the map $\nu_2(L)$ is surjective, see e.g. [**GL86**]. The condition $\rho(g,r,d) < r-2$ forces in the case $r = d-g$ the inequality $d \geq 2g+3$. Since the expected dimension of $\Sigma^{d-g}_{n,g,d}(C)$ is negative, the conjecture predicts that $\Sigma^{d-g}_{n,g,d}(C) = \emptyset$. This is confirmed by Mumford's result. □

5.1. The Minimal Syzygy Conjecture. Interpolating between Green's Conjecture for generic curves (viewed as a vanishing statement) and the Maximal Rank Conjecture, it is natural to expect that the Koszul cohomology groups of line bundles on a general curve $[C] \in \mathcal{M}_g$ should be subject to the vanishing suggested by the determinantal description provided by Theorem 3.5. For simplicity we restrict ourselves to the case $\rho(g,r,d) = 0$:

Conjecture 5.8. *We fix integers $r, s \geq 1$ and set $d := rs + r$ and $g := rs + s$, hence $\rho(g,r,d) = 0$. For a general curve $[C] \in \mathcal{M}_g$ and for every integer*

$$0 \leq p \leq \frac{r-2s}{s+1}$$

we have the vanishing $K_{p,2}(C,L) = 0$, for every linear series $L \in W^r_d(C)$.

As pointed out in Theorem 3.5, in the limiting case $p = \frac{r-2s}{s+1} \in \mathbb{Z}$, Conjecture 5.8 would imply that the failure locus

$$\mathcal{Z}_{g,p} := \{[C] \in \mathcal{M}_g : \exists L \in W^r_d(C) \ \text{ such that } K_{p,2}(C,L) \neq \emptyset\}$$

is an effective divisor on $\mathcal{M}_g$ whose closure $\overline{\mathcal{Z}}_{g,p}$ violates the Slope Conjecture.

Conjecture 5.8 generalizes Green's Conjecture for generic curves: When $s = 1$, it reads like $K_{p,2}(C, K_C) = 0$ for $g \geq 2p+3$, which is precisely the main result from [**V05**]. Next, in the case $p = 0$, Conjecture 5.8 specializes to Theorem 5.3. The conjecture is also known to hold when $s = 2$ and $g \leq 22$ (cf. [**Fa06a**] Theorems 2.7 and 2.10).

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