SU(3) STRUCTURES AND SPINORS
by
Simon Salamon

A sequel to the papers of
Abbena-Garbiero-Salamon
Chiossi-Salamon
with material common to
Goldstein-Prokushin
Cardoso-Curio-Dell’Agata-Lust-
Fidanza-Minasian-Tomasiello
and many others

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Definitions and a general scheme
for calculation and classification:
1. Spinors and complex structures
2. Intrinsic torsion from scratch

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3. Spinors and covariant derivatives
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1. SPINORS AND COMPLEX STRUCTURES

Projective spaces
Unitary and special unitary reductions
in 6 dimensions are parametrized by
\[
\begin{align*}
\text{SO}(3)/\text{SU}(3) &= \mathbb{R}^3 \\
\downarrow \\
\text{SO}(3)/\text{U}(3) &= \mathbb{C}P^3
\end{align*}
\]
Representations of SU(4)

SU(4) acts on $V = \mathbb{C}^4$
preserving a 4-form $\delta \in \wedge^4 V$

SU(4) acts on $\wedge^2 V = \mathbb{C}^6$
preserving the antilinear map
$\wedge^4 V \to \wedge^2 V$

1-forms as 2-spinors

$SU(4) \cong Spin(6)$

$([T_\alpha M^\alpha])_{c} = \wedge^2 V$

Standard complex structure

$e^1 - ie^2 = 2v_{12}$
$e^3 - ie^4 = 2v_{13}$
$e^5 - ie^6 = 2v_{15}$

$\Lambda^{1,0} = \{v_0 \wedge v : v \in V\}$

Klein correspondence

$e^1 + ie^2 = 2v_{23}$
$e^3 + ie^4 = 2v_{51}$
$e^5 + ie^6 = 2v_{12}$

$\Lambda^{0,1} = \Lambda^2 \{v \in V : v \perp v_0\}$
### Conclusion

Choosing a spinor \( \eta \in \mathbb{V} \) and setting

\[
\mathcal{L} = \langle \eta \rangle, \quad \mathbb{V} = \mathcal{L} \oplus \mathcal{L}^\perp = 1 + \mathfrak{3}
\]

gives a complex structure with

\[
\Lambda^3 = \{ \eta \wedge v : v \in \mathbb{V} \} = \mathcal{L} \wedge \mathcal{L}^\perp
\]

\[
\Lambda^0 \Lambda^3 = \wedge^2 \mathbb{V}^{\perp} = \mathcal{L}^\perp \Lambda^3
\]

### Exterior powers

The space of \((3,0)\)-forms for \( \mathcal{L} \) is

\[
\wedge = \Lambda^3 \otimes = \Lambda^3(\mathcal{L}^\perp)
\]

\[
\mathcal{L} \otimes \mathcal{L}^\perp = \Lambda^2
\]

\[
\mathbb{V} = \mathcal{L} \oplus \mathcal{L}^\perp \rightarrow \Lambda^3(\mathcal{L}^\perp) = \mathcal{L}^\perp
\]

### Example

\[\eta \wedge \eta\]

\[4\eta \wedge \eta = 4 \eta_2 \cdot \eta_3 \cdot \eta_{03456} = \eta_2 \cdot \eta_3 \cdot (\eta_{03} \eta_{56} - \eta_{05} \eta_{36} + \eta_{06} \eta_{35})
\]

\[= -4 \eta_2 \eta_3 \eta_{03} \eta_{56} + 4 \eta_2 \eta_3 \eta_{05} \eta_{36} + 4 \eta_2 \eta_3 \eta_{06} \eta_{35}
\]

\[= 8 \eta_2 \eta_3 \eta_{05} \eta_{36} + 8 \eta_2 \eta_3 \eta_{06} \eta_{35}
\]

### Conclusion

Reductions to \( U(3) \) are parametrized by \( \text{SO}(6)/U(3) = \mathbb{C}P^3 \) by choosing

\[
\mathcal{L} = \langle \eta \rangle \subset \mathbb{V}
\]

\[
\Lambda^3 \otimes \Lambda^0 \Lambda^3 = \mathcal{L} \Lambda^3 \otimes \mathcal{L}^\perp \Lambda^3
\]

### Conclusion

Reductions to \( SU(3) \) are parametrized by \( \text{SO}(6)/\text{SU}(3) = \mathbb{RP}^3 \) by choosing a unit spinor \( \pm \eta \in \mathcal{L} \) or a real line

\[
\langle \eta \rangle \subset \mathbb{V}
\]

\[
\eta \otimes \eta \in \Lambda^3 = \mathcal{L}^2
\]

### 2. INTRINSIC TORSION FROM SCRATCH
Second fundamental form

Regard L as a line bundle

\[ \Gamma(L) \xrightarrow{\pi} \Gamma(T^* \otimes V) \to \Gamma(T^* \otimes L^1) \]

\[ \eta \in \Gamma(L) \]

\[ \varepsilon \in \Gamma(L) \]

\[ (\eta \wedge \varepsilon) \in T^* \otimes L^1 \]

\[ (\eta \wedge \varepsilon) \in T^* \otimes T^* \]

Torsion as a 2-tensor

The intrinsic torsion of the U(3) or almost Hermitian structure lies in

\[ \mathbb{A}^2 T^* \otimes S^2 T^* = 15 + 21 \]

\[ \eta \in \Gamma(L) \]

Gray-Hervella classes

\[ \mathbb{A}^2 T^* \]

\[ \mathbb{N} \]

\[ S^2 T^* \]

Halflat or hypo structures

\[ \mathbb{A}^2 T^* \]

\[ \mathbb{N} \]

\[ S^2 T^* \]
The Dirac operator

Clifford multiplication is skewing:

\[ \gamma(\nu) \gamma(\nu^*) = \frac{1}{2} \gamma(\nu) + \frac{1}{2} \gamma(\nu^*) \]

\[ D\eta = \gamma \eta \in \Gamma(\gamma) \]

The Dirac operator

Take a spinor \( \eta \in \gamma \) of unit norm

\[ \gamma = \frac{1}{\sqrt{2}} L_0 \equiv L_0 \left( \gamma^{a} \right) \]

\[ D\eta = \eta \phi [\gamma \gamma_{\phi} + (\gamma \gamma_{\phi})^{a} + \cdots] \]

The Dirac operator

Lemma

The Dirac operator maps a defining spinor to components \( W_\alpha \), \( b W_\alpha + c W_\alpha \) of the intrinsic torsion for SU(3)

\[ D\eta = \eta \phi [\gamma \gamma_{\phi} + (\gamma \gamma_{\phi})^{a} + \cdots] \]

The Heisenberg Lie algebra

\[ [\gamma_{\alpha_{1}} \gamma_{\alpha_{2}}] = -\delta_{\alpha_{3}}^{\gamma_{a_{4}}} \]

is determined dually by differentials

\[ d\alpha = \left\{ \begin{array}{ll}
0 & , \quad r = 1,2 \\
\alpha_{11} & , \quad r = 3
\end{array} \right. \]
The Iwasawa Lie algebra

is the underlying real Lie algebra obtained by substituting

\[
\begin{align*}
\alpha_1 &= e_1 - e_2 \\
\alpha_2 &= e_1 - e_4 \\
\alpha_3 &= e_3 - e_4
\end{align*}
\]

Parallelizations

Associated to a global orthonormal basis of 1-forms

\[\alpha^r, 1 \leq r \leq 6,\]

on the group H or nilmanifold \(\Gamma \backslash H\) is the global basis of spinors

\[\psi_k, 1 \leq k \leq 4\]

The Levi-Civita Connection

is completely characterized by its rather simple action on these spinors:

\[
\nabla \psi = \v_1 \psi_2 \v_3 \\
\nabla \psi_1 = -\v_2 \psi_4 \\
\nabla \psi_2 = -\v_3 \psi_4 \\
\nabla \psi_3 = -\v_1 \psi_2 + \v_3 \psi_4
\]

Corollary

The Dirac operator has an especially simple action on invariant spinors:

\[
D\psi_k = \begin{cases} 
0, & k = 0,1,2 \\
-\v_0 \v_3, & k = 3
\end{cases}
\]

How can we use this result?

4.

CLASSIFICATION OF HALFFLAT STRUCTURES
Invariant spinors

A generic invariant spinor is

$$\eta = \frac{1}{3} \sum_{i=3}^{9} \zeta_i \psi_i = (\zeta_{8} \zeta_{1} \zeta_{2} \zeta_{3})$$

The associated point of $\mathbb{CP}^3$ is

$$t = (\eta_0, \eta_1, \eta_2, \eta_3)$$

The invariant Dirac operator

$$D^\alpha_k = \begin{cases} 0, & k = 0, 1, 2 \\ -\gamma_{45}, & k = 3 \end{cases}$$

$$D \eta = 0 \Rightarrow \zeta_3 = 0$$

The language of 3-forms

$$D : V \rightarrow V$$

$$\mathcal{D} \in \mathcal{D}^3 = \Lambda^3_0(T^*V)$$

$$\text{Re}(\mathcal{D}) = \sum_{i=1}^{9} \mathcal{D} \wedge \mathcal{D}^i$$

The language of 3-forms

$$D = \nabla_0 \nabla_3 = \partial_{0} \partial_{3} - \partial_{0} \partial_{5} + \partial_{1} \partial_{3} - \partial_{1} \partial_{5} + \partial_{2} \partial_{3} - \partial_{2} \partial_{5}$$

$$\mathcal{D} \in \mathcal{D}^3 = \Lambda^3_0(T^*V)$$

$$\text{Re}(\mathcal{D}) = \sum_{i=1}^{9} \mathcal{D} \wedge \mathcal{D}^i$$

Toric picture

$$\mu : \mathbb{CP}^3 \rightarrow$$

The subset of $\mathbb{CP}^3$ for which $W_1$ vanishes is

$$\mu^{-1}(\mathcal{F}) = \{ \zeta_3 = 0 \} \cong \mathbb{P}^2$$

The subset of $\mathbb{CP}^3$ for which $W_1, W_3, W_4$ vanish (meaning d1=0) is

$$\mu^{-1}(\mathcal{G}) \cong \mathbb{A}^3$$
Almost Kähler structures

\[ \mu : \mathbb{C} \rightarrow \mathbb{C}^3 \]

\[ S^3 = \mu^{-1}(\mathbb{C}) \]

The halfflat condition

The image of

\[ \text{Re} [\nabla g \wedge \eta] \in T^* \otimes T^* \rightarrow \mathbb{R}^3 T^* \]

(and \( W_i^+ , W_i^0 , W_i \)) must be zero

We also need \( W_5 = 0 \) or \( (2\Omega)^{3,4} = 0 \)

Lemma

\[ A(\nabla g \wedge \eta) = z_0^2 (z_1 z_3 \wedge z_2 z_4 \wedge z_5 z_6 + z_3 z_1 \wedge z_5 z_2 + z_4 z_2 \wedge z_6 z_3) + z_5^2 (z_0 z_4 - z_2 z_6 - z_3 z_5 - z_1 z_6) \]

Real part vanishes iff

\[ 0 = z_0 z_3 = z_0 z_2 = z_0 z_4 = \text{Re}[z_0^2] \]

Final theorem

The subset for which the structure is halfflat is a union

\[ \mathbb{R}P^3 \cup \mathbb{R}P^1 \cup \{v_0, v_4\} \subset \mathbb{R}P^7 \]

Projecting onto

\[ \mathbb{C}P^1 \cup \{v_0\} \cup \{v_4\} \subset \mathbb{C}P^3 \]

\[ 0 = z_0 z_3 = z_0 z_2 = z_0 z_4 = \text{Re}[z_0^2] \]

Halfflat structures

Mary Poppins’ bag

The Iwasawa manifold has a. everything in it!

Symplectic non-Kähler structures

Halfflat SU(3) structures

... for constructing metrics with holonomy \( G_2 \)
Disconnected families of complex structures
Connected \( G_2 \) manifolds
Obstructed Kodaira deformation classes
Strong \( K3 \) structures