Spinorial Geometry
and
supersymmetry

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M-theory
J. Gillard, U. Gran and G.P. hep-th/0410155
U. Gran, G.P. and D. Roest, hep-th/0503046
IIB supergravity
J. Gutowski, U. Gran and G.P. hep-th/0501177, hep-th/0505074
J. Gutowski, U. Gran, D. Roest and G.P. hep-th/0507087
Heterotic and common sector supergravities
U. Gran, P. Lohrmann and G.P., hep-th/0510176, hep-th/0602250.

MOTIVATION

- The Killing spinor equations are the extension of self-duality and BPS conditions to gravity, and their solutions are the gravitational solitons and instantons.
- The solutions exhibit special geometry, like hyper-Kähler, Calabi-Yau and their extensions. There is close relation to spinorial geometry.
- String Theory, e.g. black holes, branes
- AdS/CFT, gravity/Yang-Mills correspondences

Topics

Spinors and instantons

- Self-duality and spinorial geometry

Spinorial geometry and supersymmetry

Heterotic supergravity

- Gravitino Killing spinor equation
- Gaugino and dilatino Killing spinor equations

$Spin(9,1)$ spinors

- Parallel spinors
- Parallel forms

Null backgrounds

Timelike backgrounds

Common sector

- Killing spinor equations
- Geometry of $N = 2$ backgrounds

Conclusions

SPINORS AND INSTANTONS

The self-duality condition

$$^*F = \pm F$$

is equivalent to the spinorial equation

$$F\epsilon = \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon = 0$$

where ϵ is in a spin representation of $Spin(4)$.

$Spin(4)$ **spinors**

Take $\mathbb{R}^4 = \mathbb{R} < e_1, \ldots, e_4 >$, and $U = \mathbb{R} < e_1, e_2 >$, where $e_1, \ldots, e_4$ orthonormal basis.

The space of Dirac spinors is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$ and the Gamma matrices act as

$$\begin{aligned}\Gamma_i\eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \\ \Gamma_{i+2}\eta &= ie_i \wedge \eta - ie_i \lrcorner \eta \ , \qquad i=1,2\end{aligned}$$

Properties

- $\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu\nu}, \ \mu,\nu = 1,\dots,4$
- Weyl spinors: $\Delta_c^+ = \Lambda^{\mathrm{ev}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\mathrm{odd}}(U \otimes \mathbb{C})$. Therefore if $\eta \in \Delta_c^+$,

  $$\eta = a1 + be_{12} \ , \qquad a,b \in \mathbb{C}$$

  and if $\eta \in \Delta_c^-$,

  $$\eta = ae_1 + be_2 \ .$$

- Introduce the Hermitian Gamma matrices ($\Gamma^{\bar\alpha} = \delta^{\bar\alpha\beta}\Gamma_\beta$)

  $$\Gamma_\beta = \frac{1}{\sqrt{2}}(\Gamma_\beta - i\Gamma_{2+\beta}) \ , \quad 1 \leq \beta \leq 2$$

- Observe that $\Gamma^{\alpha}1 = e_{\alpha}\lrcorner 1 = 0$ and

  $$1\ , \quad \Gamma^{\bar\alpha}1\ , \quad \Gamma^{\bar\alpha\bar\beta}1$$

  is a Hermitian basis in the space of spinors.

Self-duality

- The spinorial equation $F\epsilon = \epsilon$ is invariant under $Spin(4)$.
- $Spin(4) = SU(2) \times SU(2)$ has one type of orbit in Δ_c^+. This allows to choose $\epsilon = 1$.

In such case,

$$F_{\mu\nu}\Gamma^{\mu\nu}1 = F_{\bar\alpha\bar\beta}\Gamma^{\bar\alpha\bar\beta}1 + \delta^{\alpha\bar\beta}F_{\alpha\bar\beta}1 = 0$$

Using the Hermitian basis, one finds the Donaldson equations

$$F_{\bar\alpha\bar\beta} = 0\ , \qquad \delta^{\alpha\bar\beta}F_{\alpha\bar\beta} = 0$$

SPINORIAL GEOMETRY AND SUPERSYMMETRY

Gillard, Gran, GP, hep-th/0410155

The Killing spinor equations of supergravity theories are a parallel transport equation of the supercovariant connection $\mathcal{D}$ and some algebraic conditions $\mathcal{A}$, i.e.

$$\begin{aligned}&\mathcal{D}_M\epsilon = \nabla_M\epsilon + \Sigma_M(F)\epsilon = 0\\&\mathcal{A}(F)\epsilon = 0\end{aligned}$$

$\mathcal{D}$ and $\mathcal{A}$ depend of the bosonic fields F of the supergravity theory and ϵ is a spinor.

To solve these equations use

- A description of spinors in terms of forms
- The use of the gauge group of the

Killing spinor equations to bring the Killing spinors into a canonical or normal form

- A basis in the space of spinors which is used to expand the Killing spinor equations

The gauge group of the Killing spinor equations are the local transformations which preserve the form of the Killing spinor transformations.

- The holonomy of the supercovariant connection may be different but includes the gauge group of the supercovariant derivative

Heterotic Supergravity

The spacetime is a $d=10$ Lorentzian manifold.

Fields

g (metric) , H (real 3-form) ,
F (gauge curvature), Φ (real scalar)
where $F=dA+A\wedge A$.

Killing spinor equations

The gravitino, dilatino and gaugino KSE are

$$\hat\nabla_M\epsilon=0$$
$$\partial_M\Phi\Gamma^M\epsilon-\frac{1}{12}H_{N_1N_2N_3}\Gamma^{N_1N_2N_3}\epsilon=0$$
$$F_{MN}\Gamma^{MN}\epsilon=0$$

where ϵ is chiral real spinor and

$$\hat\nabla_M X^N=\nabla_M X^N+\frac{1}{2}H^N{}_{MR}X^R$$

Gravitino Killing spinor equation

- $\hat{\nabla}$ is a connection of the spinor bundle associated with a metric connection with torsion H
- The gauge group of the Killing spinor equations is $Spin(9,1)$ and $\mathrm{hol}(\hat{\nabla}) \subseteq Spin(9,1)$
- If $\hat{\nabla}\epsilon = 0$, then $\hat{R}\epsilon = 0$ and so either $\hat{R} = 0$ or the parallel spinors have a non-trivial stability subgroup $G \subset Spin(9,1)$
- The bundle of parallel spinors $\mathcal{K}$ can be trivialized with $Spin(9,1)$ gauge transformations. As a result the parallel spinors can be chosen to be constant.

Algebraic Killing Spinor equations

$F\epsilon = 0$, can be analyzed as the parallel transport, i.e. the solutions ϵ either have a non-trivial stability subgroup in $Spin(9,1)$ or $F = 0$

$(d\Phi - \frac{1}{2}H)\epsilon = 0$, is different but it can be analyzed using representation theory.

$Spin(9,1)$ SPINORS

Take $U = \mathbb{R} < e_1, \dots, e_5 >$, $e_1, \dots, e_5$ orthonormal basis. The $Spin(9,1)$ Dirac spinors are $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$ and the chiral (Weyl) spinors are $\Delta_c^+ = \Lambda^{\text{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\text{odd}}(U \otimes \mathbb{C})$.

The gamma matrices are represented on Δ_c as

$$\begin{aligned} \Gamma_0 \eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta \ , \ \ \Gamma_5 \eta = e_5 \wedge \eta + e_5 \lrcorner \eta \\ \Gamma_i \eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \qquad i = 1, \dots, 4 \\ \Gamma_{5+i} \eta &= i e_i \wedge \eta - i e_i \lrcorner \eta \ . \end{aligned}$$

The Dirac inner product on the space of spinors Δ_c is defined as

$$D(\eta, \theta) = < \Gamma_0 \eta, \theta > \quad ,$$

where

$$< z^a e_a, w^b e_b > = \sum_{a=1}^{5} (z^a)^* w^a \ ,$$

on $U \otimes \mathbb{C}$ and then extended to Δ_c, and $(z^a)^*$ is the standard complex conjugate.

A Majorana inner product is

$$B(\eta, \theta) = < B(\eta^*), \theta > \quad , \qquad B = \Gamma_{06789}$$

Observe that the inner product B is in addition Pin invariant and skew-symmetric $B(\eta, \theta) = -B(\theta, \eta)$.

The Majorana reality condition can be chosen as

$$\eta = -\Gamma_0 B(\eta^*) = \Gamma_{6789} \eta^* \ .$$

$C = \Gamma_{6789}$ is the charge conjugation matrix.

Example

Consider the complex chiral spinor $a1 + be_{1234}$, $a, b \in \mathbb{C}$. The associated real spinor of positive chirality is

$$\eta = a1 + a^* e_{1234} \ .$$

The two Majorana spinors are $1 + e_{1234}$ and $i1 - ie_{1234}$.

- The spinor $1 + e_{1234}$ is invariant under the $Spin(7) \ltimes \mathbb{R}^8$ subgroup of $Spin(9,1)$
- The spinors $1+e_{1234}$, and $i(1-e_{1234})$ are invariant under the $SU(4) \ltimes \mathbb{R}^8$ subgroup of $Spin(9,1)$

Pseudo Hermitian basis

A creation annihilation basis for Dirac spinors is

$$\begin{aligned}
\Gamma_{\bar{\alpha}} &= \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5}) \ , \\
\Gamma_{\pm} &= \frac{1}{\sqrt{2}}(\Gamma_5 \pm \Gamma_0) \ , \\
\Gamma_{\alpha} &= \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+5}) \ .
\end{aligned}$$

Observe that

$$\Gamma_{\bar{\alpha}}1 = \Gamma^{\alpha}1 = \Gamma_{+}1 = \Gamma^{-}1 = 0 \, , \qquad \Gamma^B = g^{BA}\Gamma_A$$

The representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^{\bar{\alpha}}, \Gamma^{+}$. A basis in Δ_c is

$$\eta = \sum_{k=0}^{5} \frac{1}{k!} \, \phi_{\bar{a}_1 \ldots \bar{a}_k} \, \Gamma^{\bar{a}_1 \ldots \bar{a}_k} 1 \ , \qquad \bar{a} = \bar{\alpha}, + \ .$$

PARALLEL SPINORS

G	N = 1	N = 2	N = 3	N = 4	N = 8	N = 16
$Spin(7)\ltimes\mathbb{R}^8$	$\surd$	-	-	-	-	-
$SU(4)\ltimes\mathbb{R}^8$	-	$\surd$	-	-	-	-
G_2	-	$\surd$	-	-	-	-
$Sp(2)\ltimes\mathbb{R}^8$	-	-	$\surd$	-	-	-
$(SU(2)\times SU(2))\ltimes\mathbb{R}^8$	-	-	-	$\surd$	-	-
$SU(3)$	-	-	-	$\surd$	-	-
$\mathbb{R}^8$	-	-	-	-	$\surd$	-
$SU(2)$	-	-	-	-	$\surd$	-
$\{1\}$	-	-	-	-	-	$\surd$

N denotes the number of parallel spinors and G their stability subgroup in $Spin(9,1)$. $\surd$ denotes the cases that the parallel spinors occur. $-$ denotes the cases that do not occur.

- There are two classes of stability subgroups those of the type $K\ltimes\mathbb{R}^8$, for $K=Spin(7), SU(4), Sp(2), SU(2)\times SU(2)$ and $\{1\}$. The other type are compact stability subgroups K, $K=G_2, SU(3), SU(2)$ and $\{1\}$.
- The null supersymmetric backgrounds

have stability subgroups $K \ltimes \mathbb{R}^8$ while the timelike ones have stability subgroups K.

- The backgrounds with Killing spinors with stability subgroup $\{1\}$ are locally isometric to Minkowski spacetime with $H = 0$ and $\Phi = \mathrm{const}$

In what follows, the $\hat{\nabla}$-parallel spinors are also Killing, i.e. they solve both the gravitino and dilatino Killing spinor equations.

PARALLEL FORMS

All the form Killing spinor bi-linears

$$\alpha = B(\epsilon_1, \Gamma_{A_1 \dots A_k} \epsilon_2) e^{A_1} \wedge \dots e^{A_k}$$

are $\hat{\nabla}$-parallel.

$Spin(7) \ltimes \mathbb{R}^8$ (N=1)

e^-, $\mathrm{e}^- \wedge \phi$

$SU(4) \ltimes \mathbb{R}^8$ (N=2)

e^- , $\mathrm{e}^- \wedge \chi$, $\mathrm{e}^- \wedge \omega$

$Sp(2) \ltimes \mathbb{R}^8$ (N=3)

e^- , $\mathrm{e}^- \wedge \omega_I$, $\mathrm{e}^- \wedge \omega_J$, $\mathrm{e}^- \wedge \omega_K$

$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$ (N=4)

e^- , $\mathrm{e}^- \wedge (e^1 \wedge e^6 + e^2 \wedge e^7)$

$\mathrm{e}^- \wedge (e^3 \wedge e^8 + e^4 \wedge e^9)$

$\mathrm{e}^- \wedge (e^1 + ie^6) \wedge (e^2 + ie^7)$

$\mathrm{e}^- \wedge (e^3 + ie^8) \wedge (e^4 + ie^9)$

$\mathbb{R}^8$ (N=8)

$\mathrm{e}^- \wedge \psi\,, \qquad \psi \in \Lambda^{\mathrm{ev}+}(\mathbb{R}^8)$

G_2 (N=2)

$\mathrm{e}^-\,, \quad \mathrm{e}^+\,, \quad e^1\,, \quad \varphi$

$SU(3)$ (N=4)

$\mathrm{e}^-\,, \mathrm{e}^+\,, \; e^1\,, \; e^6\,, \; \hat{\omega}\,, \; \hat{\chi}$

$SU(2)$ (N=8)

$\mathrm{e}^-\,, \; \mathrm{e}^+\,, \; e^1\,, \; e^6\,, \; e^2\,, \; e^7\,,$

$-e^3{\wedge}e^8{-}e^4{\wedge}e^9\,, \; (e^3{+}ie^8){\wedge}(e^4{+}ie^9)$

$\{1\}$ (N=16)

$e^A\,, \quad A = 0, \ldots, 9$

where

$$\begin{aligned}
&\chi=(e^1+ie^6)\wedge(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ ,\\
&\omega=-e^1\wedge e^6-e^2\wedge e^7-e^3\wedge e^8-e^4\wedge e^9\ ,\\
&\phi=\mathrm{Re}\,\chi-\frac{1}{2}\omega\wedge\omega\ ,\\
&\hat\omega=-e^2\wedge e^7-e^3\wedge e^8-e^4\wedge e^9\ ,\\
&\hat\chi=(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ ,\\
&\varphi=\mathrm{Re}\hat\chi+e^6\wedge\hat\omega\ ,\quad \omega_I=\omega\ ,\\
&\omega_J=\mathrm{Re}[(e^1+ie^6)\wedge(e^2+ie^7)]+(e^3+ie^8)\wedge(e^4+ie^9)]\ ,\\
&\omega_K=-\mathrm{Im}[(e^1+ie^6)\wedge(e^2+ie^7)+(e^3+ie^8)\wedge(e^4+ie^9)]\ .
\end{aligned}$$

- The generators of the $\hat\nabla$-parallel forms of the null backgrounds are null, i.e. $e^-\wedge\psi$
- Null backgrounds admit a single $\hat\nabla$-parallel null one-form e$^-$

- Time-like backgrounds admit one time-like and two (G_2), three ($SU(3)$) and five ($SU(2)$) spacelike parallel one-forms.
- The commutator $[X, Y] = i_X i_Y H$ of any two X, Y, $\hat{\nabla}$-parallel vector fields is also $\hat{\nabla}$-parallel.

In what follows, assume that the parallel vector field spinor bilinears span a Lie algebra $\mathfrak{h}$ of a Lie group $\mathcal{H}$.

Null Backgrounds

Theorem: Null backgrounds with $K \ltimes \mathbb{R}^8$-invariant Killing spinors are two-parameter Lorentzian deformation families of an 8-dim manifold B with a K-structure. The metric and three-form of the background can be written as

$$ds^2 = 2\mathrm{e}^+\mathrm{e}^- + \delta_{ij}e^ie^j$$

$$H = \mathrm{e}^+ \wedge d\mathrm{e}^- + \mathrm{e}^- \wedge (H^{\mathfrak{k}} + H^{\mathfrak{k}^\perp}) + H^{\mathrm{rest}}$$

where two-form $H^{\mathfrak{k}}$ takes values in the Lie algebra of K and $H^{\mathfrak{k}^\perp}$ takes values in the complement, $\Lambda^2(\mathbb{R}^8) = \mathfrak{k} \oplus \mathfrak{k}^\perp$, and

$$\mathrm{e}^+ = du + V dv + n \ , \quad \mathrm{e}^- = dv + m$$

The only components of H which are not determined from the metric and the parallel forms are those of $H^{\mathfrak{k}}$.

If the rotation of the parallel vector field vanishes, $de^{-}=0$, then B has a compatible metric connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion, $\tilde{H}$, i.e. $\mathrm{hol}(\hat{\tilde{\nabla}})\subseteq K$, and the K structure is integrable and conformally balanced.

- The integrability of the K structure is defined in a suitable way, i.e. if $K=SU(4)$, then B is complex.
- The K-structure is conformally balanced if a suitably defined Lee form $\tilde{\theta}=\star(\star d\psi\wedge\psi)$ satisfies $\tilde{\theta}=2df$ for some function f on B, $f=\Phi$, where ψ is an invariant form of the K-structure.

Timelike Backgrounds

Theorem: A timelike background with K-invariant Killing spinors is a principal bundle, $P = (\mathcal{H}, B, \pi)$, with fibre a Lorentzian Lie group $\mathcal{H}$, and it is equipped with an instanton connection λ.

The metric and three-form of the background can be written as

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + \pi^* d\tilde{s}^2 \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^*\tilde{H} \end{aligned}$$

The base space B admits an integrable, conformally balanced K-structure, compatible with a connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion associated with the pair $(d\tilde{s}^2, \tilde{H})$.

- The integrability and conformal balanced properties of the K-structure are defined as in the null backgrounds
- $\tilde{H}$ is determined in terms of the metric on B and the forms of the K-structure
- In addition one can show that

  $$dH = \eta_{ab}\mathcal{F}^a \wedge \mathcal{F}^b + \pi^* d\tilde{H}$$

  i.e. part of dH is specified by the first Pontrjagin form of λ

In particular one has the following:

G_2

Parallel Spinors: $1+e_{1234}, e_{15}+e_{2345}$

$\mathfrak{h}= \mathfrak{sl}(2,\mathbb{R})$ or $\mathbb{R}^{2,1}$

$\tilde{H} = -\frac{r}{6}(d\varphi, \star\varphi)\varphi + \star d\varphi + \star(\theta_\varphi \wedge \varphi)$

[Friedrich,Ivanov]

$r = 0$ if $\mathfrak{h}$ abelian, and $r = 1$ if $\mathfrak{h}$ non-abelian

$\tilde{\theta}_\varphi = 2d\Phi$, $\quad d\star\varphi = -\tilde{\theta}_\varphi \wedge \star\varphi$

λ , $\quad \mathfrak{g}_2$ instanton

$\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$

$SU(3)$

$\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$ or $\mathfrak{su}(2)\oplus\mathbb{R}$ or $\mathfrak{cw}_{(-1,-1)}$ or $\mathbb{R}^{3,1}$

$\tilde{H} = -i_I d\omega = \star d\omega - \star(\tilde{\theta}_\omega \wedge \omega)$

[Gauduchon, Strominger, Howe, GP, Grantcharov, Poon, Lust et al]

$\tilde{\theta}_\omega = 2d\Phi$, B complex

If $\mathfrak{h}$ abelian: $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(3)$ and λ a $\mathfrak{s}u(3)$ instanton

If $\mathfrak{h}$ non-abelian: $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq U(3)$ and λ a $\mathfrak{u}(3)$ instanton

$SU(2)$

$\mathfrak{h}{=}\mathfrak{sl}(2,\mathbb{R}){\oplus}\mathfrak{su}(2)$ or $\mathfrak{cw}_{(-1,-1,-1,-1)}$ or $\mathbb{R}^{5,1}$

$\tilde{H} = -i_I d\omega = \star d\omega - \star(\theta_\omega \wedge \omega)$

$\theta_\omega = 2d\Phi$, B (hyper-)complex

$\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(2)$ and λ a $\mathfrak{s}u(2)$ instanton

Type II common sector

Killing spinor equations

$$\hat{\nabla}\hat{\epsilon} = 0 \ , \qquad \check{\nabla}\check{\epsilon} = 0$$

$$(d\Phi - \frac{1}{2}H)\hat{\epsilon} = 0 \ , \quad (d\Phi + \frac{1}{2}H)\check{\epsilon} = 0$$

Comments

- $\hat{\nabla} = \nabla + \frac{1}{2}H$, $\check{\nabla} = \nabla - \frac{1}{2}H$
- In type IIB, $\hat{\epsilon}$ and $\check{\epsilon}$ are Majorana-Weyl of the same chirality. In type IIA $\hat{\epsilon}$ and $\check{\epsilon}$ are Majorana-Weyl of opposite chirality
- If either $\hat{\epsilon} = 0$ or $\check{\epsilon} = 0$, then the Killing spinor equations can be analyzed as in the heterotic case.

- The gauge symmetry of the Killing spinor equations is $Spin(9,1)$ while the holonomy of $\hat{\nabla}\oplus\check{\nabla}$ is $Spin(9,1)\times Spin(9,1)$.
- The geometry of the spacetime depends on the stability subgroup of all Killing spinors $G=\hat{G}\cap\check{G}$.

Geometry of N=2 backgrounds

IIB, $N=2$	$\hat{\mathrm{G}}$	$\check{\mathrm{G}}$	G
	$SU(4)\ltimes\mathbb{R}^8$	-	$SU(4)\ltimes\mathbb{R}^8$
	G_2	-	G_2
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$SU(4)\ltimes\mathbb{R}^8$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	G_2

IIA, $N=2$	$\hat{\mathrm{G}}$	$\check{\mathrm{G}}$	G
	$SU(4)\ltimes\mathbb{R}^8$	-	$SU(4)\ltimes\mathbb{R}^8$
	G_2	-	G_2
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$SU(4)$
	$Spin(7)\ltimes\mathbb{R}^8$	$Spin(7)\ltimes\mathbb{R}^8$	$G_2\ltimes\mathbb{R}^8$

Comment: The stability subgroups of the spinors in IIA and IIB backgrounds interchange as $K\leftrightarrow K\ltimes\mathbb{R}^8$ for $K=Spin(7), SU(4)$ and G_2.

Theorem The backgrounds for which the Killing spinors have stability subgroup $K \times \mathbb{R}^8$ admit a null, ∇-parallel vector field. The spacetime is a two-parameter deformation family of a eight-dimensional manifold B with a K-structure. The fields can be written as

$$ds^2 = 2dv(du + V dv + n) + \delta_{ij}e^i e^j$$
$$H = e^- \wedge (H^{\mathfrak{k}} + H^{\mathfrak{k}^\perp}) + H^{\text{rest}}$$

where $H^{\mathfrak{k}^\perp}$ and H^{rest} are determined in terms of the geometry.

Spin(7)

$H^{\text{rest}} = H^{\mathfrak{k}^\perp} = 0$ and $\mathrm{hol}(\tilde{\nabla}) \subseteq Spin(7)$

$SU(4)$

B almost hermitian manifold with $\tilde{W}_1$ and $\tilde{W}_2$ related to the trivialization of the canonical bundle. The (4,0)-form is $\tilde{\nabla}$-parallel.

[Chiossi, Salamon, Cabrera]

G_2

B admits a G_2 structure. This means that the is a non-vanishing vector field Z on B. H^{rest} is determined in terms of Z. There are 2^{10} G_2-structures in eight-dimensions.

Theorem The backgrounds for which the Killing spinors have stability subgroup K admit two commuting $\hat{\nabla}$- and $\check{\nabla}$-parallel vector fields, $\hat{X}$ and $\check{Y}$. The spacetime is a fibre bundle with fibres the orbits of the two parallel vector fields and base space an eight-dimensional manifold B. B has the same geometry as the deformed manifold in the previous null case. The fields can be written as

$$\begin{aligned} ds^2 &= 2e^- e^+ + ds^2(B) \\ H &= d(e^- \wedge e^+) + H_B \end{aligned}$$

where $e^- = f^2(dv + m)$ and $e^+ = f^2(du + n)$.

Comment: T-duality along the spatial isometry direction relates the backgrounds with K- and $K \ltimes \mathbb{R}^8$-invariant spinors.

SUMMARY

- The Killing spinor equations of heterotic supergravity have been solved. The are two types of supersymmetric backgrounds, the null and timelike. The former admit a parallel null 1-form while the latter admit a timelike 1-form and at least two spacelike ones.

- The null backgrounds are Lorentzian deformation families of eight-dimensional manifolds with a suitable K-structure; $K = Spin(7), SU(4), SU(2)\times SU(2), \{1\}$. If the rotation of the null parallel vector vanishes, the 8-manifold admits a compatible connection with skew-symmetric torsion.

- The time-like backgrounds are principal bundles with fibre a Lorentzian Lie group and base space a manifold with a suitable K-structure compatible with a connection with skew-symmetric torsion; $K = G_2, SU(3), SU(2)$. They are also equipped with a suitable K-instanton connection.

- The Killing spinor equations of type II common sector have been solved for all backgrounds with two Killing spinors. The stability subgroups of the spinors are K and $K \ltimes \mathbb{R}^8$ for $K = Spin(7), SU(4)$ and G_2.

- The spacetime of $K\ltimes\mathbb{R}^8$ backgrounds admits a ∇-parallel null vector field, i.e. it is a pp-wave or a deformation family, and the deformed space admits a K-structure.
- The spacetime of K backgrounds is a fibre bundle with fibre the orbits of a $\hat\nabla$- and a $\check\nabla$-parallel null commuting vector fields and base space an eight-dimensional manifold B with a K-structure.
- The deformed manifold in the $K\ltimes\mathbb{R}^8$ cases and the base space in the K case have the same geometry.