Deformations and embedding of locally conformal Kähler manifolds

(joint work with Misha Verbitsky)

Definition (M, J) is lcK if admits a Kähler covering

$$\Gamma \to (\tilde{M}, J, \Omega) \to (M, J)$$

s.t. Γ acts by holomorphic homotheties.

Remark: There exists a non-trivial homomorphism

$$\xi : \Gamma \subset Aut(\tilde{M}) \to \mathbb{R}_+, \quad \tau \mapsto \frac{\tau(\Omega)}{\Omega}.$$

Hence: Restrictions on $\pi_1(M)$.

Equiv. Definition: $\exists$ a Hermitian metric ω on M s.t.

$$d\omega = \theta \wedge \omega, \quad d\theta = 0$$

Important subclass: Vaisman manifolds

$$\text{lcK} \quad + \quad \nabla^g \theta = 0.$$

Properties:

- $\theta^\sharp$ Killing and holomorphic ($\mathcal{L}_{\theta^\sharp} J = 0$).
- If $\mathcal{F} := \{\theta^\sharp, J\theta^\sharp\}$ has compact leaves, then $M/\mathcal{F}$ is Kähler orbifold.
- If $\theta^\sharp$ has compact orbits, then $M/\theta^\sharp$ is Sasakian orbifold
- $\|\theta^\sharp\|^2$ is a potential for the Kähler form of the universal cover.

Structure theorem (-, Verbitsky):

Compact Vaisman manifolds are suspensions over S^1 with Sasakian fibre:

$\tilde{M}$ is a metric cone $N \times \mathbb{R}$, with N Sasaki, Γ is $\mathbb{Z}$ generated by $(x, t) \mapsto (\lambda(x), t + q)$ for some $\lambda \in Aut(N)$, $q \in \mathbb{R}_+$.

Examples

1) Hopf surfaces: $(\mathbb{C}^2 \setminus 0)/\mathbb{Z}$ with $\mathbb{Z}$ represented by

$$(z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2),$$

$m \in \mathbb{N}, \quad (\alpha - \beta^m)\lambda = 0, \quad |\alpha| \geq |\beta| > 1.$

- $\lambda = 0$ are Vaisman (Gauduchon, -). Other lcK metrics: Parton.
- $\lambda \neq 0$ lcK but do not admit any Vaisman structure (Belgun). Will be now generalized.

Generalization of class 1 ($\lambda = 0$) to arbitrary dimension:

$H_A := \mathbb{C}^n/\langle A\rangle$ with $A = \mathrm{diag}(a_i)$.

Let $C > 1$ const.

$$\varphi(z_1, \ldots, z_n) = \sum |z_i|^{\beta_i}, \quad \beta_i = \log_{|\alpha_i|^{-1}} C$$

is acted on by A as follows: $A^*\varphi = C^{-1}\varphi$.

Hence:

$\Omega = \sqrt{-1}\partial\bar{\partial}\varphi$ is Kähler and $\Gamma \cong \mathbb{Z}$ acts by homotheties.

Lee field: $\theta^\sharp = -\sum z_i \log|\alpha_i| \partial z_i$.

It can be shown it is Vaisman.

An lcK metric on the Hopf surface with $\lambda \neq 0$ has been found by a very particular deformation of the lcK structure of the Hopf surface with $\lambda = 0$ (suggested by Le Brun).

Still: no explicit metric.

2) Some of the Inoue surfaces (Tricerri, Belgun). All non-Vaisman.

The simplest example is S_A:

$\tilde{M} = H \times \mathbb{C}$.

Γ generated by the transformations:

$$(w, z) \mapsto (\alpha w, \beta z),$$

$$(w, z) \mapsto (w + a_j, z + b_j)$$

where $\alpha > 1$, resp. β is the real, resp. purely complex eigenvalue of a $A \in SL(3, \mathbb{Z})$ and (a_1, a_2, a_3) resp. (b_1, b_2, b_3) are the corresponding eigenvectors.

The Kähler metric is a warped product:

$$\frac{dw \otimes d\bar{w}}{w_2^2} + w_2 dz \otimes d\bar{z}$$

and $\theta = d \log w_2$.

But: There exist Inoue surfaces not of lcK type (Belgun).

3) Oeljeklaus, Toma (AIF, 2005) generalization of Inoue surfaces:

K is an algebraic number field of degree $n := (K : \mathbb{Q})$.

(for any $s, t \in \mathbb{N}$, there exist algebraic number fields with precisely s real and $2t$ complex embeddings).

Let $\mathcal{O}_K$ be the ring of algebraic integers and U some subgroup of $\mathcal{O}_K^*$ s. t.

$$X = X(K, U) := (H^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K)$$

is a compact m-dimensional complex (affine) manifold, differentiably a fiber bundle over $(S^1)^s$ with fiber $(S^1)^n$.

For $s = t = 1$ and particular choice of U, $X(K, U)$ reduces to an Inoue surface S_A.

Theorem (i) For $t = 1$ ($\tilde{X} = H^s \times \mathbb{C}$), $X(K, U)$ admits lcK metric given by Kähler potential

$$\varphi = \frac{1}{\Pi_{j=1}^s (i(z_j - \bar{z}_j))} + |z_m|^2$$

(ii) $b_1(X) = s$, hence for $s = 2p$, X not Vaisman.

(iii) If U not contained in $\mathbb{Z}$ and there is no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subset \mathcal{O}_{K'}$: $b_2 = s(s-1)/2$

(iv) TX flat; $\dim H^1(X, \mathcal{O}_X) \geq s$. In particular, X are non-Kähler.

They give an example with $s = 2$ and $t = 1$, $\dim X = 6$, satisfying (iii) with: $b_0 = b_6 = 1$, $b_1 = b_5 = 2$, $b_2 = b_4 = 1$, $b_3 = 0$

This disproves Vaisman's conjecture:

"A non-Kähler compact lcK must have an odd odd Betti number".

4) J. Renaud (CRAS, 2004). Non-compact lcK.

The Kähler metric on the universal cover has a potential.

Deck group is $\mathbb{Z}$.

Ad-hoc proof that cannot be Kähler.

In dim. 2 they are Inoue-Hirzebruch surfaces minus the rational curves.

Probably not Vaisman.

5) Any product $S^1\times$ Sasakian mfd. They are Vaisman.

What is known (connected with this talk):

1. The lcK class is not stable at small deformations (Belgun): lcK Inoue surfaces can be deformed into some non-lcK Inoue surfaces.

2. But: Hopf surfaces with $\lambda = 0$ (Vaisman) can be deformed into Hopf surfaces with $\lambda \neq 0$ (lcK but not Vaisman).

3. Vaisman structures on compact manifolds admit particular deformations into Vaisman structures (one deforms the generator of the monodromy representation in the complexified of the group generated by the Lee flow).

The deformed structure can be assumed quasi-regular (–, Verbitsky 05).

4. Compact Vaisman manifold admit holomorphic immersion (finite covering on the image) into a diagonal Hopf manifolds H_A, preserving the respective Lee flows. (–, Verbitsky 05).

Method: M may be assumed quasi-regular.

The push-forward of $L_{\mathbb{C}}$ (complexified of the weight bundle) over $M/\mathcal{F}$ is positive, hence projective.

Lift this embedding to $\tilde{M}$ and show it is equivariant.

Aim: Define a class containing strictly Vaisman mfds. which

- is stable to small deformations and
- satisfies an *embedding* theorem into a model space which should include the Hopf manifolds as a particular case.

Definition: *lcK manifold with potential* is a manifold which admits a Kähler cover $\Gamma \to (\tilde{M}, \Omega) \to M$ with global potential $\varphi : \tilde{M} \to \mathbb{R}_+$ satisfying the following conditions:

(1) φ is proper: has compact level sets.

(2) The covering group acts on φ by multiplication with a constant:
$\tau(\varphi) = const \cdot \varphi$ for any $\tau \in \Gamma$.

Note: The property in the definition is inherited by any closed complex submanifold.

Meaning of Condition (1) for M compact:

Let $\chi : \Gamma \to \mathbb{R}_+$ be the homomorphism $\Gamma \ni \gamma \mapsto \dfrac{\gamma(\varphi)}{\varphi}$ (may be supposed injective). Then:
φ proper $\Leftrightarrow$ $\mathrm{Im}(\chi)$ is discrete in $\mathbb{R}_+$.

In particular: $\Gamma \cong \mathbb{Z}$.

Examples: 1) Vaisman manifolds: $\|\theta^\sharp\|^2$ is the potential, Γ is $\mathbb{Z}$.
2) Renaud's non-compact examples ??
3) Oeljeklaus-Toma's examples are not: the deck group is not $\mathbb{Z}$.
4) Hopf surfaces with $\lambda \neq 0$ and generalization (we shall prove).

Stability at small deformations

(M, J, g) lcK with potential, (M, J') small deformation of (M, J).
Then: φ is:

- proper on $(\tilde{M}, J')$ and
- $\gamma(\varphi) = \chi(\gamma)\varphi$
- strictly psh: small deformations preserve spsh

Therefore: $(\tilde{M}, J')$ is Kähler, and φ is an lcK-potential on $(\tilde{M}, J')$. Hence:

Theorem. The class of compact lcK manifolds with potential is stable under small deformations.

In particular:

(1) A small deformation of a compact Vaisman manifold is still lcK (with potential), not necessarily Vaisman. This explains why the Vaisman structure of a Hopf surface with $\lambda = 0$ could be deformed to a non-Vaisman lcK structure on the Hopf surface with $\lambda \neq 0$.

(2) LcK. structure of the Inoue surfaces do not admit potential: they can be deformed to the non-lcK type Inoue surface (Belgun).

FILLING THE KÄHLER COVER AND THE EMBEDDING THEOREM

To obtain an embedding of (M, J, g):

Method:

- Embed the Kähler cover in a Stein, then in a $\mathbb{C}^N$.
- Show the monodromy acts equivariantly on this $\mathbb{C}^N$ with eigenvalues < 1.
- Hence $(\mathbb{C}^N \backslash 0)/\Gamma$ is lcK and M embeds in it.

Remark: In general, the action of Γ is not diagonal, hence:

This $(\mathbb{C}^N \backslash 0)/\Gamma$ is the natural generalization of the Hopf surface with $\lambda \neq 0$.

Step 1. Suppose $\dim M \geq 3$.
Let $\Gamma \to \tilde{M} \to M$ be the corresponding covering.

Then $\tilde{M}$ is an open subset of a Stein variety $\tilde{M}_c$, with at most one singular point. Moreover, $\tilde{M}_c \setminus \tilde{M}$ is just one point.

Idea of proof:

- Apply Rossi-Andreotti-Siu: $\tilde{M}(a) = \{x \in \tilde{M} \mid \varphi(x) > a\}$, which is holomorphically concave, can be filled, hence is an open set in a Stein variety $\tilde{M}_c$ with at most isolated singularities (restriction on dimension is essential).
- Extend this embedding to $\tilde{M} \hookrightarrow \tilde{M}_c$ (techniques of complex variables).
- $\tilde{M}_c$ is $\tilde{M}$ plus just one point:

- The generator γ of the monodromy $\Gamma \cong \mathbb{Z}$ is conformal: say a contraction.
- For any f holomorphic on $\tilde{M}_c$, show that $\gamma^n f$ converges to a constant.
- Let $Z := \tilde{M}_c \setminus \tilde{M}$. It is compact and is fixed by Γ.
- Hence $\sup_Z |\gamma^n f| = \sup_Z |f|$ and $\inf_Z |\gamma^n f| = \inf_Z |f|$.
- Thus: $\inf_Z |f| = \sup_Z |f|$ for any holomorphic function f.
- All holomorphic functions have same value in all the points of Z.
- If Z has 2 points, there exists a holomorphic function to separate them (as $\tilde{M}_c$ Stein).
- Hence $Z = \{z\}$.

Step 2. γ acts with eigenvalues strictly smaller than 1 on the cotangent space $T^*_z \tilde{M}_c$.

Idea of proof: Use a version of Schwarz lemma plus $\gamma^n f = const.$ for any holomorphic f:
if $\gamma(x) = x$ and $d_x\gamma(v) = \lambda v \neq 0$,
then $d_x(\gamma^n f)(v) = \lambda^n d_x f(v) \to 0$ because $\gamma^n f$ converges to a constant. Hence $|\lambda| < 1$.

Step 3. By Step 2., the formal logarithm of γ converges. Now show (main technical point):

- γ acts with finite Jordan blocks on the formal completion $\widehat{\mathcal{O}}_z$ of $\mathcal{O}_z$ (the local ring of analytic functions in $z \in \tilde{M}_c$).

- On a Stein variety S, for a holomorphic flow with eigenvalues smaller than 1 on T_sS for some s, there exists a sequence of $V_n \subset \mathcal{O}_sS$ of finite dimensional subspaces such that the s-adic completion of $\oplus V_n$ is exactly the completion of $\mathcal{O}_sS$, each V_n being preserved by the flow which acts by linear transformations on it.

Here we use Poincaré-Dulac theorem which gives the normal form of a vector field around a singularity:
$\lambda_i x_i + P(x_{i+1}, \ldots, x_n)$,
where P are resonant polynomials corresponding to the eigenvalues λ_i.

Hence:

- Choose enough holomorphic functions in the eigenspaces V_n which, together, embed the cover $\tilde{M}_c$ in a $\mathbb{C}^N$;
- Γ acts linearly on $\mathbb{C}^N$, with eigenvalues smaller than 1. Hence:

M embeds in $(\mathbb{C}^N \setminus 0)/\Gamma$ which is called a *linear Hopf manifold.*
It is lcK (deformation of diagonal Hopf).
Generalises both classes of Hopf surfaces.

Theorem: Any compact lcK manifold with potential, of complex dimension at least 3, admits a holomorphic embedding in a linear Hopf manifold.
If M is Vaisman, it can be embedded in a Vaisman-type Hopf manifold $(\mathbb{C}^N \setminus 0)/\langle A\rangle$, where A is a diagonal linear operator with eigenvalues < 1.

For the 2nd statement: use the Lee flow: preserves the spaces V_n, acts by homotheties, hence is self-adjoint and can be diagonalized.

A related result (Kato, 1975): Let X be a complex space with a fixed point z, equipped with a holomorphic contraction, that is, an invertible morphism $\psi : X \to X$ such that for any sufficient small neighbourhood U of z, $\psi^k(U)$ lies inside U for k sufficiently big, and for all $x \in X$,

$$\lim_{k\to\infty} \psi^k(x) = z.$$

Then $(X\backslash z)/\langle\psi\rangle$ can be embedded into a Hopf manifold.

APPLICATION FOR SASAKIAN MANIFOLDS

Riemannian manifolds (N, h) with a unit Killing field ξ satisfying a certain second order condition (Equivalently: the Riemannian cone is Kähler).

- $(\xi^{\perp}, J := \nabla^h \xi)$ makes N a CR-mfd. (strictly pseudoconvex).
- $\xi^{\perp}$ is contact distribution, ξ is the Reeb field.
- $N \times S^1$ is lcK (Vaisman).
- Examples: Weighted odd spheres, $S^2 \times S^3$, $S^6 \times S^7$, non-trivial S^1-bundles over compact Hodge mfds. (see Boyer-Galicki et. al.), exotic spheres wich bound parallelizable mfds. etc.
- In particular: $S^{2n-1}_w \times S^1$ is Vaisman (hence lcK with potential).

Theorem: A compact Sasakian manifold of dimension at least 5 admits a CR-embedding into a Sasakian weighted sphere, preserving the respective Reeb flows.

A related result (Marinescu, Yeganefar 2004): A compact Sasakian manifold admits a CR-embedding into $\mathbb{C}^N$.

The proof views M as pseudo-convex boundary of $M \times (0, \infty)$ with Biquard-Herzlich Kähler metric.