Distinguished dimensions for special Riemannian geometries

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- *special* Riemannian structure $(X, g, \nabla^T, T, \Psi)$ should satisfy a number of field equations including:

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Question: How to construct solutions to the above equations in n dimensions?

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- Let Υ be an object (e.g. a tensor), whose isotropy under the action of $\mathbf{SO}(n)$ is $H \subset \mathbf{SO}(n)$. Infinitesimaly, such an object determines the inclusion of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{so}(n)$.

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- If (X, g) is endowed with such a Υ we can decompose the Levi-Civita connection 1-form $\overset{LC}{\Gamma} \in \mathfrak{so}(n) \otimes \mathbb{R}^n$ onto $\Gamma \in \mathfrak{h} \otimes \mathbb{R}^n$ and the rest:

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- Then the first Cartan structure equation $\mathrm{d}\theta + (\Gamma + \frac{1}{2}T) \wedge \theta = 0$ for the Levi-Civita connection $\overset{LC}{\Gamma}$ may be rewriten to the form

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- Curvature of this connection $K \in \mathfrak{h} \otimes \bigwedge^2 \mathbb{R}^n$ - via the second structure equation:

$$K = \mathrm{d}\Gamma + \Gamma \wedge \Gamma.$$

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- If so, for which n and $H \subset \mathbf{SO}(n)$?
- What is Υ which reduces $\mathbf{SO}(n)$ to H?

Special geometries $(X, g, \nabla^T, T \equiv 0, \Psi)$

- If $T \in \bigwedge^3 \mathbb{R}^n$ was *identically zero*, then since $\mathfrak{h} \otimes \mathbb{R}^n \ni \Gamma = \overset{LC}{\Gamma}$, the *holonomy group* of (X, g) would be *reduced* to $H \in \mathbf{SO}(n)$.

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- These are:

  ⋆ either *symmetric spaces* G/H, with the holonomy group $H \subset \mathbf{SO}(n)$
  ⋆ or they are contained in the *Berger's list*:

Berger's list

Berger's list

Holonomy group for g	Dimension of X	Type of X	Remarks
$\mathbf{SO}(n)$	n	generic	
$\mathbf{U}(n)$	$2n$, $n \geq 2$	Kähler manifold	Kähler
$\mathbf{SU}(n)$	$2n$, $n \geq 2$	Calabi-Yau manifold	Ricci-flat,Kähler
$\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$	$4n$, $n \geq 2$	quaternionic Kähler	Einstein
$\mathbf{Sp}(n)$	$4n$, $n \geq 2$	hyperkähler manifold	Ricci-flat,Kähler
$\mathbf{G}_2$	7	$\mathbf{G}_2$ manifold	Ricci-flat
$\mathbf{Spin}(7)$	8	$\mathbf{Spin}(7)$ manifold	Ricci-flat

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- relax $T = 0$ condition to $T \in \bigwedge^3 \mathbb{R}^n$ for H corresponding to the irreducible symmetric spaces G/H from *Cartan's list.*

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- $X = \mathbf{SU}(3)/\mathbf{SO}(3)$ is the *integrable* $(T = 0)$ model for the irreducible $\mathbf{SO}(3)$ geometries in dimension 5.

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- Th. Friedrich: Is it possible to have 5-dimensional Riemannian geometries for which the torsionless model would be $X = \mathbf{SU}(3)/\mathbf{SO}(3)$?

- In other words, following Friedrich, we propose to study *irreducible* $\mathbf{SO}(3)$ *geometries in dimension* 5.

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- Tensor Υ whose isotropy group under the action of $\mathbf{SO}(5)$ is the irreducible $\mathbf{SO}(3)$ is determined by the following conditions (Bobieński+PN):

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (totally *symmetric*)

ii) $\Upsilon_{ijj} = 0$, (trace-free)

iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}.$

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- A 5-dimensional Riemannian manifold (X, g) equipped with a tensor field Υ satisfying conditions i)-iii) and admitting a unique decomposition $\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T$, with $T \in \bigwedge^3 \mathbb{R}^5$ and $\Gamma \in \mathfrak{so}(3) \otimes \mathbb{R}^5$ is called *nearly integrable* irreducible $\mathbf{SO}(3)$ structure.

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- In particular, we have a 7-parameter family of nonequivalent examples which satisfy

$$\nabla^T \Psi = 0, \quad \delta(T) = 0, \quad T \cdot \Psi = \mu \Psi$$

 i.e. equations of type IIB string theory (but in wrong dimension!). For this family of examples $T \neq 0$ and, at every point of X, we have two 2-dimensional vector spaces of ∇^T-covariantly constant spinors Ψ. Moreover, since for this family $K = 0$, we also have $Ric^{\nabla^T} = 0$.

Question

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What are the possible dimensions n in which there exists a tensor Υ satisfying:

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (total *symmetry*)

ii) $\Upsilon_{ijj} = 0$, (no trace)

iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$?

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- Given Υ_{ijk} we consider a 3rd order polynomial $w(a) = \Upsilon_{ijk} a_i a_j a_k$, where $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$.

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- Then the tensor Υ which brakes $\mathbf{SO}(5)$ to the irreducible $\mathbf{SO}(3)$ gives:

$$w(a) = 6\sqrt{3} a_1 a_2 a_3 + 3\sqrt{3}(a_1^2 - a_2^2) a_4 - \left(3a_1^2 + 3a_2^2 - 6a_3^2 - 6a_4^2 + 2a_5^2 \right) a_5$$

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- Note that:

$$w(a) = \det \begin{pmatrix} a_5 - \sqrt{3} a_4 & \sqrt{3} a_3 & \sqrt{3} a_2 \\ \sqrt{3} a_3 & a_5 + \sqrt{3} a_4 & \sqrt{3} a_1 \\ \sqrt{3} a_2 & \sqrt{3} a_1 & -2a_5 \end{pmatrix}$$

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- if $n = 5$ the tensor Υ is given by:

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- if $n = 5$, 8, 14 and 26 we take:

$$w(a) = \det \begin{pmatrix} a_5 - \sqrt{3}a_4 & \sqrt{3}\alpha_3 & \sqrt{3}\alpha_2 \\ \sqrt{3}\overline{\alpha}_3 & a_5 + \sqrt{3}a_4 & \sqrt{3}\alpha_1 \\ \sqrt{3}\overline{\alpha}_2 & \sqrt{3}\overline{\alpha}_1 & -2a_5 \end{pmatrix}$$

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where for $n = 5$:

$$\alpha_1 = a_1$$

$$\alpha_2 = a_2$$

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where for $n = 8$:

$$\alpha_1 = a_1 + a_6 \mathrm{i}$$
$$\alpha_2 = a_2 + a_7 \mathrm{i}$$
$$\alpha_3 = a_3 + a_8 \mathrm{i}$$

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where for $n = 14$:

$$\alpha_1 = a_1 + a_6 \mathrm{i} + a_9 \mathrm{j} + a_{10} \mathrm{k}$$
$$\alpha_2 = a_2 + a_7 \mathrm{i} + a_{11} \mathrm{j} + a_{12} \mathrm{k}$$
$$\alpha_3 = a_3 + a_8 \mathrm{i} + a_{13} \mathrm{j} + a_{14} \mathrm{k}$$

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where for $n = 26$:

$$\alpha_1 = a_1 + a_6\mathrm{i} + a_9\mathrm{j} + a_{10}\mathrm{k} + a_{15}\mathrm{p} + a_{16}\mathrm{q} + a_{17}\mathrm{r} + a_{18}\mathrm{s},$$
$$\alpha_2 = a_2 + a_7\mathrm{i} + a_{11}\mathrm{j} + a_{12}\mathrm{k} + a_{19}\mathrm{p} + a_{20}\mathrm{q} + a_{21}\mathrm{r} + a_{22}\mathrm{s},$$
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- For each $n = 5, 8, 14$ i 26 tensor Υ given by

$$\Upsilon_{ijk} a_i a_j a_k = w(a)$$

satisfies i)-iii)!

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Theorem 1

In dimensions $n = 5$, 8, 14 i 26 tensor Υ reduces the $\mathbf{GL}(n, \mathbb{R})$ group via $\mathbf{O}(n)$ to a subgroup H_n, where:

- for $n = 5$ group H_5 is the irreducible $\mathbf{SO}(3)$ in $\mathbf{SO}(5)$; the torsionless compact model: $\mathbf{SU}(3)/\mathbf{SO}(3)$

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- for $n = 14$ group H_{14} is the irreducible $\mathbf{Sp}(3)$ in $\mathbf{SO}(14)$; the torsionless model: $\mathbf{SU}(6)/\mathbf{Sp}(3)$

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- for $n = 26$ group H_{26} is the irreducible $\mathbf{F}_4$ in $\mathbf{SO}(26)$; the torsionless compact model: $\mathbf{E}_6/\mathbf{F}_4$

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- The only dimensions in which conditions i)-iii) have solutions for Υ_{ijk} are $n = 5, 8, 14, 26$.

- Modulo the action of $\mathbf{O}(n)$ all such tensors are given by $\det A$, where A is a 3×3 traceless hermitian matrix with entries in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, for the respective dimensions 5,8,14,26.

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Idea of the proof

- It follows from Cartan's work on *isoparametric hypersurfaces in spheres*.

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- He concluded that the problem is equivalent to the problem of existence and the possible dimensions for the normed division algebras. Thus $n = 3k + 2$, where $k = 1, 2, 4, 8$ are dimensions of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

H_k structures in dimensions $n_k = 5, 8, 14, 26$

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An H_k structure is called *nearly integrable* iff

$$\nabla^{LC}_X \Upsilon(X, X, X) = 0, \qquad \forall X \in \Gamma(TM) \quad .$$

Nearly integrable H_k structures and characteristic connection

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Question: What are the neccessary and sufficient conditions for a H_k structure to admit a unique decomposition

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Proposition1

Every H_k structure that admits a characteristic connection must be *nearly integrable.*

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- There are *real* irreducible representations of the group $\mathbf{F}_4$ in dimensions: $1, 26, 52, 273, 324, 1053, 1274, 4096, 8424\ldots$

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- In dimension 26 the Levi-Civita connection $\overset{LC}{\Gamma}$ of a nearly integrable $\mathbf{F}_4$ structure may have values in 52-dimensional irreducible representation V_{52} of $\mathbf{F}_4$, which is not present in the algebraic sum of $\mathfrak{f}_4 \otimes \mathbb{R}^k$ and $\bigwedge^3 \mathbb{R}^{n_k}$.

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Restricted nearly integrable H_k structures

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The nearly integrable H_k structures described by Proposition 2 are called *restricted* nearly integrable.

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Torsion types of the characteristic connection for H_k geometries

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- $T \in {}^{14}V_{189} \oplus {}^{14}V_{84} \oplus {}^{14}\bigwedge^2_{70} \oplus {}^{14}\bigwedge^2_{21},$ for $n_k = 14$,
- $T \in {}^{26}V_{1274} \oplus {}^{26}V_{1053} \oplus {}^{26}\bigwedge^2_{273},$ for $n_k = 26$.

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Magic square

$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathfrak{su}(3)$	$2\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
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Distinguished dimensions

$\mathbf{SU}(3)/\mathbf{SO}(3)$	$\mathbf{Sp}(3)/\mathbf{U}(3)$	$\mathbf{F}_4/(\mathbf{Sp}(3)\times\mathbf{SU}(2))$
$\mathbf{SU}(3)$	$\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))$	$\mathbf{E}_6/(\mathbf{SU}(6)\times\mathbf{SU}(2)$
$\mathbf{SU}(6)/\mathbf{Sp}(3)$	$\mathbf{SO}(12)/\mathbf{U}(6)$	$\mathbf{E}_7/(\mathbf{SO}(12)\times\mathbf{SU}(2))$
$\mathbf{E}_6/\mathbf{F}_4$	$\mathbf{E}_7/(\mathbf{E}_6\times\mathbf{SO}(2))$	$\mathbf{E}_8/(\mathbf{E}_7\times\mathbf{SU}(2))$

These 12 symmetric spaces can be considered torsionless models for special geometries on Riemannian manifolds M with the following dimensions and structure groups:

Distinguished dimensions (continued)

n_k	Structure group H_k	$2(n_k+1)$	Structure group	$4(n_k+2)$	Structure group
5	$\mathbf{SO}(3)$	12	$\mathbf{U}(3)$	28	$\mathbf{Sp}(3)\times\mathbf{SU}(2)$
8	$\mathbf{SU}(3)$	18	$\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))$	40	$\mathbf{SU}(6)\times\mathbf{SU}(2)$
14	$\mathbf{Sp}(3)$	30	$\mathbf{U}(6)$	64	$\mathbf{SO}(12)\times\mathbf{SU}(2)$
26	$\mathbf{F}_4$	54	$\mathbf{E}_6\times\mathbf{SO}(2)$	112	$\mathbf{E}_7\times\mathbf{SU}(2)$

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1) $\dim M = 8$, with the structure group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ and with the torsionless model of compact type $M = \mathbf{G}_2/(\mathbf{SU}(2) \times \mathbf{SU}(2))$.

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- This space, in
an orthonormal coframe adapted to the $\mathbf{SU}(3)$ structure is spanned by a 3-form
$$\psi = \tau_1 \wedge \theta^6 + \tau_2 \wedge \theta^7 + \tau_3 \wedge \theta^8 + \theta^6 \wedge \theta^7 \wedge \theta^8,$$
where (τ_1, τ_2, τ_3) are 2-forms
$$\tau_1 = \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3 + \sqrt{3}\theta^1 \wedge \theta^5$$
$$\tau_2 = \theta^1 \wedge \theta^3 + \theta^4 \wedge \theta^2 + \sqrt{3}\theta^2 \wedge \theta^5$$
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spanning the 3-dimensional irreducible representation ${}^5\bigwedge^2_3 \simeq \mathfrak{so}(3)$ associated with $\mathbf{SO}(3)$ structure in dimension 5.

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- It is remarkable that this 3-form *alone* reduces the $\mathbf{GL}(8,\mathbb{R})$ to the irreducible $\mathbf{SU}(3)$ in the same way as Υ does.
- Thus, in dimension 8, the H_k structure can be defined either in terms of the *totally symmetric* Υ or in terms of the *totally skew symmetric* ψ.

In this sense the 3-form ψ and the 2-forms (τ_1, τ_2, τ_3) play the same role in the relations between $\mathbf{SU}(3)$ structures in dimension *eight* and $\mathbf{SO}(3)$ structures in dimension *five* as the 3-form

$$\phi = \sigma_1 \wedge \theta^5 + \sigma_2 \wedge \theta^6 + \sigma_3 \wedge \theta^7 + \theta^5 \wedge \theta^6 \wedge \theta^7$$

and the self-dual 2-forms

$$\sigma_1 = \theta^1 \wedge \theta^3 + \theta^4 \wedge \theta^2$$
$$\sigma_2 = \theta^4 \wedge \theta^1 + \theta^3 \wedge \theta^2$$
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play in the relations between $\mathbf{G}_2$ structures in dimension *seven* and $\mathbf{SU}(2)$ structures in dimension *four*.