String Theory and Generalized Geometries

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Introduction

Close and fruitful interplay between

\[ \text{String Theory} \Leftrightarrow \text{Supersymmetry} \Leftrightarrow \text{Geometry} \]

purpose of this talk:

- review some of its aspects
- discuss string compactifications on manifolds with \( SU(3) \)-structure
  
  (and \( SU(3) \times SU(3) \)-structure)

work in collaboration with

I. Benmachiche, M. Graña, S. Gurrieri, A. Micu, D. Waldram
String Theory

basic idea: point-like objects $\to$ extended objects (strings)

Strings move in 10-dimensional space-time background

contact with “our world”: Compactification

$\Rightarrow$ space-time background:

$\mathcal{M}_{10} = R_{1,3} \times Y_6$

$R_{1,3}$: four-dimensional Minkowski-space

$Y_6$: compact manifold – determines amount of supersymmetry
Different string theories:

Type I, Type II, Heterotic

they differ in spectrum of excitations and their interactions

talk today:

focus only on Type II string theories

they come in two versions: IIA & IIB – both are supersymmetric

massless spectrum in $R_{1,9}$:

<table>
<thead>
<tr>
<th></th>
<th>IIA</th>
<th>IIB</th>
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<tbody>
<tr>
<td>NS:</td>
<td>$G_{MN}, H_3 = dB_2, \Phi$</td>
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<tr>
<td>RR:</td>
<td>$F_2 = dC_1, F_4 = dC_3$</td>
<td>$l, F_3 = dC_2, F_5^* = dC_4$</td>
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<tr>
<td>NSR</td>
<td>$\Psi^{1,2}, \chi^{1,2}$</td>
<td>$\Psi^{1,2}, \chi^{1,2}$</td>
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$F_p = p$-form field strength

$C_{p-1} = (p-1)$-form gauge potential
Compactification: determine $Y_6$

Lorentz group on space-time background $\mathcal{M}_{10} = R_{1,3} \times Y_6$ decomposes

$$Spin(1, 9) \rightarrow Spin(1, 3) \times Spin(6)$$

spinor decompose accordingly:

$$16 \rightarrow (2, 4) \oplus (\bar{2}, \bar{4})$$

impose two conditions:

1. demand that two supercharges $Q^{1,2}$ exist
   $\Rightarrow$ nowhere vanishing, invariant spinor $\eta$ needs to exist
   $\Rightarrow$ structure group of $Y_6$ has to be reduced
   $$Spin(6) \rightarrow SU(3) \text{ s.t. } 4 \rightarrow 3 + 1$$
   $\Rightarrow Y_6$ has $SU(3)$-structure

2. background preserves supersymmetry
   $$\delta \Psi_{1,2} = \nabla \eta + (\gamma F') \cdot \eta = 0 , \quad \gamma \in Cliff(6)$$
   $\Rightarrow$ for $F = 0$: $\nabla \eta = 0$ $\Rightarrow Y_6$ is Calabi-Yau manifold $Y$
Calabi-Yau Threefold $Y$

- Levi-Civita connection has $SU(3)$ holonomy $\Rightarrow$ Kähler manifold
- integrability condition: $R_{ij} = 0$ $\Rightarrow$ Ricci-flat manifold
- existence of invariant spinor $\eta$ implies existence of two invariant tensors: $J, \Omega$
  - closed two-form
    \[ J = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \eta , \quad dJ = 0 \]
    $\Rightarrow$ complex structure
    \[ I^2 = -1 , \quad N(I) = 0 \]
    $J$ is $(1,1)$-form with respect to $I$
  - $(3,0)$-form
    \[ \Omega = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \gamma \cdot \eta , \quad d\Omega = 0 \]
  - Fierz implies (for $\eta^\dagger \eta = 1$)
    \[ J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega} , \quad J \wedge \Omega = 0 \]
Kaluza-Klein compactification in space-time background: $R_{1,3} \times Y$

- **massless scalars**

$$\Delta_{10}\phi = (\Delta_4 + \Delta_6)\phi = (\Delta_4 + m^2)\phi = 0$$

$$\Rightarrow$$ massless $d = 4$ spectrum = zero modes of $\Delta_6$ = harmonic forms in $H^{(p,q)}(Y)$

- **Hodge numbers:**

$$h^{p,q} = \dim H^{p,q}(Y)$$

$$
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & h^{1,1} & 0 & 0 \\
0 & h^{1,2} & h^{1,2} & 1 \\
0 & h^{1,1} & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
$$

- **deformations of Calabi-Yau metric and the 2-form $B$ form the moduli space**

$$\mathcal{M} = \mathcal{M}_\Omega^{h^{(1,2)}} \times \mathcal{M}_J^{h^{(1,1)}}$$

- $\mathcal{M}_\Omega^{h^{(1,2)}}$: deformations of complex structure/holomorphic three-form $\Omega$,
- $\mathcal{M}_J^{h^{(1,1)}}$: deformations of complexified Kähler form $B + iJ$

appear as scalar fields in effective action: supergravity in $d = 4$
Mirror Symmetry

conjecture:

for ‘every’ $Y$ there exists a mirror manifold $\tilde{Y}$ with

$$h^{1,1}(Y) = h^{1,2}(\tilde{Y}), \quad h^{1,2}(Y) = h^{1,1}(\tilde{Y})$$

manifestation in string theory:

$\text{IIA in background } R_{1,3} \times Y \equiv \text{IIB in background } R_{1,3} \times \tilde{Y}$

implies:

$$\mathcal{M}_\Omega \equiv \mathcal{M}_J$$
Low energy effective action: $N = 2$ supergravity

$$S = \int_{M_4} \frac{1}{2} R - \mathcal{N}_{IJ}(z) F^I_{\mu\nu} F^{\mu\nu J} - g_{ab}(z) \partial_{\mu} z^a \partial^{\mu} z^b - V(z) + \ldots , \quad \mu, \nu = 0, \ldots, 3$$

- scalar manifold: $N = 2$ constraint: $\mathcal{M} = \mathcal{M}_{SK} \times \mathcal{M}_{QK}$
  - IIA: $\mathcal{M}_{SK} = \mathcal{M}_J$, $\mathcal{M}_{QK} \supset \mathcal{M}_{\Omega}$
  - IIB: $\mathcal{M}_{SK} = \mathcal{M}_{\Omega}$, $\mathcal{M}_{QK} \supset \mathcal{M}_J$

- Kähler potentials [Strominger, Candelas, de la Ossa]

$$e^{-K_J} = \int_Y \langle \Phi^+, \bar{\Phi}^+ \rangle = \int_Y J \wedge J \wedge J, \quad \Phi^+ = e^{B+iJ} ,$$

$$e^{-K_\Omega} = \int_Y \langle \Phi^-, \bar{\Phi}^- \rangle = \int_Y \Omega \wedge \bar{\Omega} , \quad \Phi^- = \Omega$$

where $\langle \Phi^+, \bar{\Phi}^+ \rangle = \Phi^+_0 \wedge \bar{\Phi}^+_6 - \Phi^+_2 \wedge \bar{\Phi}^+_4 + \Phi^+_4 \wedge \bar{\Phi}^+_2 - \Phi^+_6 \wedge \bar{\Phi}^+_0$, etc.
Generalization: background flux and manifolds with $SU(3)$ ($SU(3) \times SU(3)$) – structure

Recall that we imposed two conditions:

1. demand that two supercharges $Q$ exist
   ⇒ invariant spinor $\eta$ exists ⇒ $Y_6$ has $SU(3)$-structure

2. background preserves supersymmetry $\delta \Psi^{1,2} = \nabla \eta + (\gamma F) \cdot \eta = 0$
   ⇒ for $F = 0$: $\nabla \eta = 0$ ⇒ $Y_6$ is Calabi-Yau manifold

Generalizations: insist on 1. (existence of $Q$) but relax 2.

\[(i)\] $F \neq 0$ and $\nabla \eta \neq 0$ such that $\delta \Psi = 0$

... corresponds to supersymmetric background with non-trivial flux

\[(ii)\] $F \neq 0$ and/or $\nabla \eta \neq 0$ but $\delta \Psi \neq 0$

... corresponds to spontaneously broken supersymmetry
possible situations:

- $F \neq 0$: $Y_6$ has non-trivial background flux

- $\nabla \eta \neq 0$: $Y_6$ is manifold of $SU(3)$ structure with torsion

  [Gray, Hervella, Salamon, Chiossi, Friedrich, Ivanov, Papadopoulos, Hitchin, ...]

  such manifolds are characterized by existence of invariant spinor $\eta$ which obeys

  $$\nabla^{(T)} \eta \equiv (\nabla^{(LC)} + T_0) \eta = 0 , \quad T_0 : \text{intrinsic (con)-torsion}$$

  ⇒ existence of two invariant tensors:

  - almost complex structure $J = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \eta$, $I^2 = -1$, $J \wedge \Omega = 0$
  - $(3, 0)$-form $\Omega = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \gamma \cdot \eta$, $J^3 = \frac{3i}{4} \Omega \wedge \overline{\Omega}$

  generically: $dJ \neq 0$, $N(I) \neq 0$, $d\Omega \neq 0$ obstructed by $T_0$

  - have different spinors $\eta^1, \eta^2$ for the two gravitini $\Psi_{M}^{1,2}$

    each spinor defines an $SU(3)$ structure – together an $SU(3) \times SU(3)$ structure
Background fluxes

[ Rohm, Witten, Strominger, Polchinski, Becker, Becker, ... ]

allow

\[
\int_{\gamma_p \in Y} F_p \neq 0 \quad \text{keeping} \quad dF_p = 0 = d^i F_p
\]

\[
F_p = e_I \omega_p^I, \quad \omega_p \in H^p(Y), \quad e_I = \text{quantized flux}
\]

for large \( \gamma^I \): \( e_I \) small perturbation, light spectrum does not change,
kinetic terms unchanged but potential \( V \) induced.

\( \Leftrightarrow \) IIB

[ Michelson; Taylor, Vafa; Mayr; Dall'Agata; Micu, JL; ... ]

potential \( V \) induced which depends on

\[
W = \int_Y \Omega \wedge G_3, \quad G_3 \equiv F_3 - \tau H_3
\]

\( \Leftrightarrow \) IIA

[ Strominger, Polchinski; Gukov, ... ]

\[
W = \int_Y (F_6 + F_4 \wedge J + F_2 \wedge J^2 + F_0 J^3)
\]

problem: mirror symmetry
Compactifications on manifolds with \( SU(3) \left( SU(3) \times SU(3) \right) \) – structure

\[ \Rightarrow \text{impose “standard } N = 2\text{” (no massive gravitino multiplets)} \]

\[ \Rightarrow SU(3) \text{ – structure without triplets: } \, dJ^2 = 0 \text{ and } d\Omega^{3,1} = 0 \]

- kinetic terms
  - \( \mathcal{M} \) is product of special geometries \( \mathcal{M} = \mathcal{M}_J \times \mathcal{M}_\Omega \)
  - \( e^{-K_J} = \int_Y \langle \Phi^+, \Phi^+ \rangle = \int_Y J \wedge J \wedge J \), \( \Phi^+ = e^{B+iJ} \)
  - \( e^{-K_\Omega} = \int_Y \langle \Phi^-, \Phi^- \rangle = \int_Y \Omega \wedge \overline{\Omega} \), \( \Phi^- = \Omega \)

- potential
  - \( W = \int_Y \langle \Phi^+, d\Phi^- \rangle = \int_Y J \wedge d\Omega \)

\[ \Rightarrow SU(3) \times SU(3) \text{ – structure without triplets} \]

\[ \Rightarrow \text{same kinetic terms and same potential for} \]

\[ \Phi^+ = \Phi_0^+ + \Phi_2^+ + \Phi_4^+ + \Phi_6^+, \quad \Phi^- = \Phi_1^- + \Phi_3^- + \Phi_5^- \]
Mirror symmetry in the presence of fluxes

[Gukov, Vafa, Witten; Gurrieri, Micu, Waldram, JL; Fidanza, Graña, Minasien, Tomasiello; ...]

\[ \text{RR-flux:} \]

\[ IIB: \quad e = \int_\gamma F_3 , \quad m = \int_{\gamma^*} F_3 \]

\[ IIA: \quad \tilde{e} = \int_{\gamma_4} F_4 , \quad \tilde{m} = \int_{\gamma_2} F_2 \]

mirror symmetry:

\[ H^{\text{odd}}(Y) \leftrightarrow H^{\text{even}}(\tilde{Y}) \]

effective actions obey:

\[ L^{IIB}(Y, e, m) \equiv L^{IIA}(\tilde{Y}, \tilde{e}, \tilde{m}) , \quad e = \tilde{e}, \quad m = \tilde{m} \]

\[ \text{NS-flux:} \]

no obvious mirror symmetry since flux of $H_3$ is along $H^3(Y)$ on both sides

\[ \Rightarrow \text{NS } F_4 \text{ (electric) and } F_2 \text{ (magnetic) are missing} \]

\[ \Rightarrow \text{can only come from metric/geometry [Vafa} \]
Mirrors of Calabi-Yau & NS 3-form flux

- **electric flux**: mirror is “half-flat” $SU(3)$ manifold [Hitchin, Chiossi, Salamon]
  
  which obeys $d(\text{Im}\Omega) = 0 = d(J \wedge J)$
  
  ‘missing’ NS 4-form: $F_4 \sim d(\text{Re}\Omega) = e_{NS}^{ij} \omega_4^i$

  “Proof”:
  
  - go to SYZ limit and perform mirror map explicitly [Gurrieri, Micu, Waldram, JL]
  - match type IIB $N = 1$ domain-wall solution of [Bahrndt, Cardoso, Lüst]
    with type IIA solution [Hitchin, Mayer, Mohaupt]
  - compute low energy effective action for type IIA compactified on $Y$
    [Graña, GMLW]

- **magnetic fluxes** [Benmachiche, Grimm; Graña, Waldram, JL]

  mirror is $SU(3) \times SU(3)$ manifold which obeys $d(\text{Im}\Phi^-) = 0$
  
  ‘missing’ NS 2-form: $F_2 \sim d(\text{Re}\Phi^1_-)$

- **generalized mirror symmetry**

  $\Phi^+ \Leftrightarrow \Phi^-$
Conclusions/open problems

- compactifications on manifolds with $SU(3) \left( SU(3) \times SU(3) \right)$ – structure compatible with $N = 2$ supergravity
  - Kähler potential is “unchanged”
  - scalar potential depends on the torsion
  - mirror symmetry restored

- deformation theory/moduli space of manifolds with $SU(3)$ structure
- relation with (mirror of) Calabi-Yau [Berglund, Mayr]
- mirror of ridged Calabi-Yau
- include warped space-time [Giddings, Maharana]
- relation with non-geometric background [Hull, Shelton, Taylor, Wecht]
- relation with non-commutative geometry [Bouwknegt, Mathai, Rosenberg]