String Theory and Generalized Geometries

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Introduction

Close and fruitful interplay between

String Theory ⇔ Supersymmetry ⇔ Geometry

purpose of this talk:

- ➪ review some of its aspects
- ➪ discuss string compactifications on manifolds with $SU(3)$-structure (and $SU(3) \times SU(3)$-structure)

work in collaboration with

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String Theory

basic idea: point-like objects $\longrightarrow$ extended objects (strings)

Strings move in 10-dimensional space-time background

contact with "our world": Compactification

$\Rightarrow$ space-time background:

$$\mathcal{M}_{10} = R_{1,3} \times Y_6$$

$R_{1,3}$: four-dimensional Minkowski-space
Y_6: compact manifold – determines amount of supersymmetry

Different string theories:

Type I, Type II, Heterotic

they differ in spectrum of excitations and their interactions

talk today:

focus only on Type II string theories

they come in two versions: IIA & IIB – both are supersymmetric

massless spectrum in $R_{1,9}$:

	IIA	IIB
NS:	$G_{MN},\ H_3 = dB_2,\ \Phi$	
RR:	$F_2 = dC_1,\ F_4 = dC_3$	$l,\ F_3 = dC_2,\ F_5^* = dC_4$
NSR	$\Psi^{1,2},\ \lambda^{1,2}$	$\Psi^{1,2},\ \lambda^{1,2}$

$F_p = p$-form field strength

$C_{p-1} = (p{-}1)$-form gauge potential

Compactification: determine Y_6

Lorentz group on space-time background $\mathcal{M}_{10} = R_{1,3} \times Y_6$ decomposes

$$Spin(1,9) \to Spin(1,3) \times Spin(6)$$

spinor decompose accordingly:

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$$

impose two conditions:

1. demand that two supercharges $Q^{1,2}$ exist

   $\Rightarrow$ nowwhere vanishing, invariant spinor η needs to exist

   $\Rightarrow$ structure group of Y_6 has to be reduced

   $$Spin(6) \to SU(3) \quad \text{s.t.} \quad \mathbf{4} \to \mathbf{3} + \mathbf{1}$$

   $\Rightarrow$ Y_6 has $SU(3)$-structure

2. background preserves supersymmetry

   $$\delta\Psi^{1,2} = \nabla\eta + (\gamma F) \cdot \eta = 0 \ , \qquad \gamma \in Cliff(6)$$

   $\Rightarrow$ for $F = 0$: $\quad \nabla\eta \ = \ 0 \quad \Rightarrow$ Y_6 is Calabi-Yau manifold Y

Calabi-Yau Threefold Y

- Levi-Civita connection has $SU(3)$ holonomy $\Rightarrow$ Kähler manifold
- integrability condition: $R_{ij} = 0$ $\Rightarrow$ Ricci-flat manifold
- existence of invariant spinor η implies existence of two invariant tensors: J, Ω
  - closed two-form

$$J = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \eta \,, \qquad dJ = 0$$

$\Rightarrow$ complex structure

$$I^2 = -1 \,, \qquad N(I) = 0$$

J is $(1,1)$-form with respect to I

  - $(3,0)$-form

$$\Omega = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \gamma \cdot \eta \,, \qquad d\Omega = 0$$

  - Fierz implies (for $\eta^\dagger \eta = 1$)

$$J \wedge J \wedge J \;=\; \tfrac{3i}{4}\, \Omega \wedge \bar{\Omega} \,, \qquad J \wedge \Omega = 0$$

Kaluza-Klein compactification in space-time background: $R_{1,3} \times Y$

➪ massless scalars

$$\Delta_{10}\phi = (\Delta_4 + \Delta_6)\phi = (\Delta_4 + m^2)\phi = 0$$

$\Rightarrow$ massless $d = 4$ spectrum $=$ zero modes of Δ_6 $=$ harmonic forms in $H^{(p,q)}(Y)$

➪ Hodge numbers: $h^{p,q} = \mathrm{dim} H^{p,q}(Y)$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

➪ deformations of Calabi-Yau metric and the 2-form B form the moduli space

$$\mathcal{M} = \mathcal{M}_\Omega^{h^{(1,2)}} \times \mathcal{M}_J^{h^{(1,1)}}$$

$\mathcal{M}_\Omega^{h^{(1,2)}}$: deformations of complex structure/holomorphic three-form Ω,

$\mathcal{M}_J^{h^{(1,1)}}$: deformations of complexified Kähler form $B + iJ$

appear as scalar fields in effective action: supergravity in $d = 4$

Mirror Symmetry

conjecture:

for 'every' Y there exists a mirror manifold $\tilde{Y}$ with

$$h^{1,1}(Y) = h^{1,2}(\tilde{Y}) \ , \qquad h^{1,2}(Y) = h^{1,1}(\tilde{Y})$$

manifestation in string theory:

IIA in background $R_{1,3} \times Y$ $\equiv$ IIB in background $R_{1,3} \times \tilde{Y}$

implies:

$$\mathcal{M}_{\Omega} \equiv \mathcal{M}_{\mathrm{J}}$$

Low energy effective action: $N = 2$ supergravity

$$\mathcal{S} = \int_{M_4} \tfrac{1}{2}R - \mathcal{N}_{IJ}(z)F^I_{\mu\nu}F^{\mu\nu J} - g_{ab}(z)\,\partial_\mu z^a \partial^\mu z^b - V(z) + \dots\ , \qquad \mu,\nu = 0,\dots,3$$

- scalar manifold: $N = 2$ constraint: $\mathcal{M} = \mathcal{M}_{\rm SK} \times \mathcal{M}_{\rm QK}$

$$\text{IIA}: \qquad \mathcal{M}_{\rm SK} = \mathcal{M}_J\ , \qquad \mathcal{M}_{\rm QK} \supset \mathcal{M}_\Omega$$
$$\text{IIB}: \qquad \mathcal{M}_{\rm SK} = \mathcal{M}_\Omega\ , \qquad \mathcal{M}_{\rm QK} \supset \mathcal{M}_J$$

- Kähler potentials [Strominger, Candelas,de la Ossa]

$$e^{-K_J} = \int_Y \langle \Phi^+, \overline{\Phi}^+ \rangle = \int_Y J \wedge J \wedge J\ , \qquad \Phi^+ = e^{B+iJ}\ ,$$
$$e^{-K_\Omega} = \int_Y \langle \Phi^-, \overline{\Phi}^- \rangle = \int_Y \Omega \wedge \overline{\Omega}\ , \qquad \Phi^- = \Omega$$

where $\langle \Phi^+, \overline{\Phi}^+ \rangle = \Phi_0^+ \wedge \overline{\Phi}_6^+ - \Phi_2^+ \wedge \overline{\Phi}_4^+ + \Phi_4^+ \wedge \overline{\Phi}_2^+ - \Phi_6^+ \wedge \overline{\Phi}_0^+$, etc.

Generalization: background flux and manifolds with $SU(3)$ $\big(SU(3) \times SU(3)\big)$ – structure

Recall that we imposed two conditions:

1. demand that two supercharges Q exist

   $\Rightarrow$ invariant spinor η exists $\Rightarrow$ Y_6 has $SU(3)$-structure

2. background preserves supersymmetry $\delta\Psi^{1,2} = \nabla\eta + (\gamma F) \cdot \eta = 0$

   $\Rightarrow$ for $F = 0$: $\quad \nabla\eta \; = \; 0 \quad \Rightarrow$ Y_6 is Calabi-Yau manifold

Generalizations: insist on 1. (existence of Q) but relax 2.

$$(i) \qquad F \neq 0 \quad \text{and} \quad \nabla\eta \neq 0 \qquad \text{such that} \qquad \delta\Psi = 0$$

corresponds to supersymmetric background with non-trivial flux

$$(ii) \qquad F \neq 0 \quad \text{and/or} \quad \nabla\eta \neq 0 \qquad \text{but} \qquad \delta\Psi \neq 0$$

corresponds to spontaneously broken supersymmetry possible situations:

- $F \neq 0$: Y_6 has non-trivial background flux

- $\nabla\eta \neq 0$: Y_6 is manifold of $SU(3)$ structure with torsion

  [Gray, Hervella, Salamon, Chiossi, Friedrich, Ivanov, Papadopoulos, Hitchin, ...]

  such manifolds are characterized by existence of invariant spinor η which obeys

$$\nabla^{(T)}\eta \;\equiv\; (\nabla^{(LC)} + T_0)\,\eta \;=\; 0\;, \qquad T_0 : \text{intrinsic (con)-torsion}$$

  $\Rightarrow$ existence of two invariant tensors:

almost complex structure	$J = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \eta\;,$	$I^2 = -1,$	$J \wedge \Omega = 0$
$(3,0)$-form	$\Omega = \eta^\dagger \cdot \gamma \cdot \gamma \cdot \gamma \cdot \eta\;,$		$J^3 = \frac{3i}{4}\Omega \wedge \overline{\Omega}$

  generically: $dJ \neq 0\;,$ $N(I) \neq 0$ $d\Omega \neq 0$ obstructed by T_0

- have different spinors η^1, η^2 for the two gravitini $\Psi_M^{1,2}$

  each spinor defines an $SU(3)$ structure – together an $SU(3) \times SU(3)$ structure

Background fluxes

[Rohm,Witten, Strominger,Polchinski, Becker,Becker, ...]

$$\text{allow} \qquad \int_{\gamma_p^I \in Y} F_p \neq 0 \qquad \text{keeping} \qquad dF_p = 0 = d^\dagger F_p$$

$$F_p = e_I\, \omega_p^I\ , \qquad \omega_p \in H^p(Y)\ , \qquad e_I = \text{quantized flux}$$

for large γ^I: e_I small perturbation, light spectrum does not change,

kinetic terms unchanged but potential V induced.

➪ IIB [Michelson; Taylor,Vafa; Mayr; Dall'Agata; Micu,JL; ...]

potential V induced which depends on

$$W = \int_Y \Omega \wedge G_3\ , \qquad G_3 \equiv F_3 - \tau H_3$$

➪ IIA [Strominger,Polchinski; Gukov, ...]

$$W = \int_Y \left(F_6 + F_4 \wedge J + F_2 \wedge J^2 + F_0 J^3\right)$$

problem: mirror symmetry

Compactifications on manifolds with $SU(3)$ $\big(SU(3)\times SU(3)\big)$ – structure

➪ impose "standard $N=2$" (no massive gravitino multiplets)

$\Rightarrow$ $SU(3)$ – structure without triplets: $dJ^2=0$ and $d\Omega^{3,1}=0$

- kinetic terms

  $\mathcal{M}$ is product of special geometries $\mathcal{M}=\mathcal{M}_J\times\mathcal{M}_\Omega$

  with [Hitchin, Graña,Waldram,JL]

$$e^{-K_J} = \int_Y \langle\Phi^+,\overline{\Phi}^+\rangle = \int_Y J\wedge J\wedge J\,, \qquad \Phi^+=e^{B+iJ}\,,$$
$$e^{-K_\Omega} = \int_Y \langle\Phi^-,\overline{\Phi}^-\rangle = \int_Y \Omega\wedge\overline{\Omega}\,, \qquad \Phi^-=\Omega$$

- potential

$$W=\int_Y \langle\Phi^+,d\Phi^-\rangle = \int_Y J\wedge d\Omega$$

➪ $SU(3)\times SU(3)$ – structure without triplets

$\Rightarrow$ same kinetic terms and same potential for

$$\Phi^+=\Phi_0^++\Phi_2^++\Phi_4^++\Phi_6^+\,, \qquad \Phi^-=\Phi_1^-+\Phi_3^-+\Phi_5^-$$

Mirror symmetry in the presence of fluxes

[Gukov,Vafa,Witten; Gurrieri,Micu,Waldram,JL; Fidanza,Graña,Minasien,Tomasiello; ...]

- RR-flux:
  IIB: $e = \int_\gamma F_3 \ , \qquad m = \int_{\gamma^*} F_3$
  IIA: $\tilde{e} = \int_{\gamma_4} F_4 \ , \qquad \tilde{m} = \int_{\gamma_2} F_2$

  mirror symmetry:

  $$H^{\text{odd}}(Y) \Longleftrightarrow H^{\text{even}}(\tilde{Y})$$

  effective actions obey:

  $$\mathcal{L}^{IIB}(Y, e, m) \ \equiv \ \mathcal{L}^{IIA}(\tilde{Y}, \tilde{e}, \tilde{m}) \ , \qquad e = \tilde{e}, \quad m = \tilde{m}$$

- NS-flux:

  no obvious mirror symmetry since flux of H_3 is along $H^3(Y)$ on both sides

  $\Rightarrow$ NS F_4 (electric) and F_2 (magnetic) are missing

  $\Rightarrow$ can only come from metric/geometry [Vafa]

Mirrors of Calabi-Yau & NS 3-form flux

➪ electric flux: mirror is "half-flat" $SU(3)$ manifold [Hitchin, Chiossi,Salamon]

which obeys $d(\mathrm{Im}\Omega) = 0 = d(J \wedge J)$

'missing' NS 4-form: $F_4 \sim d(\mathrm{Re}\Omega) = e^i_{NS}\,\omega^i_4$

"Proof":

- go to SYZ limit and perform mirror map explicitly [Gurrieri,Micu,Waldram,JL]
- match type IIB $N = 1$ domain-wall solution of [Behrndt, Cardoso, Lüst] with type IIA solution [Hitchin; Mayer,Mohaupt]
- compute low energy effective action for type IIA compactified on Y [Graña,GMLW]

➪ magnetic fluxes [Benmachiche,Grimm; Graña,Waldram,JL]

mirror is $SU(3) \times SU(3)$ manifold which obeys $d(\mathrm{Im}\Phi^-) = 0$

'missing' NS 2-form: $F_2 \sim d(\mathrm{Re}\Phi^-_1)$

➪ generalized mirror symmetry

$$\Phi^+ \Leftrightarrow \Phi^-$$

Conclusions/open problems

- compactifications on manifolds with $SU(3)$ $\big(SU(3) \times SU(3)\big)$ – structure compatible with $N = 2$ supergravity
  - Kähler potential is "unchanged"
  - scalar potential depends on the torsion
  - mirror symmetry restored

- deformation theory/moduli space of manifolds with $SU(3)$ structure
- relation with (mirror of) Calabi-Yau [Berglund,Mayr]
- mirror of ridged Calabi-Yau
- include warped space-time [Giddings,Maharana]
- relation with non-geometric background [Hull, Shelton,Taylor,Wecht]
- relation with non-commutative geometry [Bouwknegt,Mathai,Rosenberg]