Nearly hypo structures and compact Nearly Kähler 6-manifolds with conical singularities

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1. Hypo structures on 5-manifolds

N^5 - a 5-manifold. $(\eta, \omega_1, \omega_2, \omega_3)$ - an $SU(2)$-structure on N^5.

This means:

η is a 1-form on N^5.

ω_i are 2-forms on N^5 satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0, \tag{1}$$

for some 4-form v, and

$$X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0, \tag{2}$$

where $X \lrcorner$ denotes the contraction by X.

These data induce an $SU(3)$-structure (F, Ψ_+, Ψ_-) on $N^5 \times \mathbb{R}$ defined by

$$\begin{aligned} F &= \omega_1 + \eta \wedge dt, \\ \Psi &= \Psi_+ + i\Psi_- = (\omega_2 + i\omega_3) \wedge (\eta + i dt), \end{aligned} \tag{3}$$

where t is the coordinate on $\mathbb{R}$.

Conversely, suppose

- $f : N^5 \hookrightarrow M^6$ is an oriented hypersurface in a 6-manifold M^6.

- M^6 is equipped with an $SU(3)$-structure (F, Ψ_+, Ψ_-).

Let $\mathbb{N}$ denote the unit normal vector field. Then the $SU(3)$-structure induces an $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 defined by the equalities [Conti-Salamon]:

$$\begin{aligned} \eta &= -\mathbb{N} \lrcorner F, \\ \omega_1 = f^*F, \ \omega_2 &= \mathbb{N} \lrcorner \Psi_-, \ \omega_3 = -\mathbb{N} \lrcorner \Psi_+. \end{aligned} \tag{4}$$

An $SU(2)$-structure determined by (η, ω_i) is called *hypo* if it satisfies the equations [Conti-Salamon]

$$d\omega_1 = 0, \quad d(\eta \wedge \omega_2) = 0, \quad d(\eta \wedge \omega_3) = 0. \tag{5}$$

Suppose Hol$(M^6) \subset SU(3)$, that is, the $SU(3)$-structure (F, Ψ_+, Ψ_-) is integrable:

$$dF = d\Psi_+ = d\Psi_- = 0.$$

Remark: (a) Any oriented hypersurface $N^5 \subset M^6$ is naturally endowed with a hypo structure.

(b) Conversely, Conti and Salamon prove that a real analytic hypo structure on N^5 can be lifted to an integrable $SU(3)$-structure on $N^5 \times \mathbb{R}$. Namely, (η, ω_i) belongs to a one-parameter family of hypo structures $(\eta(t), \omega_i(t))$ satisfying the evolution equations

$$(6) \qquad \begin{cases} \partial_t \omega_1 = -d\eta \\ \partial_t(\eta \wedge \omega_2) = -d\omega_3 \\ \partial_t(\eta \wedge \omega_3) = d\omega_2. \end{cases}$$

Recall: A nearly Kähler 6-manifold is a manifold M^6 with an $SU(3)$-structure (F, Ψ_+, Ψ_-) which satisfies the following differential equations

$$dF = 3\Psi_+, \qquad d\Psi_- = -2F \wedge F. \tag{7}$$

Lemma 1. If $f : N^5 \longrightarrow M^6$ is a totally geodesic hypersurface of a nearly Kähler manifold M^6, then the induced $SU(2)$-structure (4) on N^5 satisfies

$$\begin{aligned} d\eta &= -2\omega_3, \\ d\omega_1 &= 3\eta \wedge \omega_2, \\ d\omega_2 &= -3\eta \wedge \omega_1. \end{aligned} \tag{8}$$

Theorem 1. Any totally geodesic hypersurface N^5 of a nearly Kähler 6-manifold M^6 admits a Sasaki-Einstein hypo structure, and therefore the Conti-Salamon evolution equations (6) can be solved on $N^5 \times \mathbb{R}$.

Indeed, (8) implies that the conical $SU(3)$-structure on $M = N^5 \times \mathbb{R}$ defined by

$$F = t^2\omega_3 + t\eta \wedge dt, \qquad (9)$$
$$\Psi = t^2(\omega_2 + i\omega_1) \wedge (t\eta + idt)$$

satisfies $dF = d\Psi = 0$.

Remark: Note that any Sasaki-Einstein 5-manifold has a hypo $SU(2)$-structure which satisfies (8). In fact, a Sasaki-Einstein 5-manifold N^5 is such that the cone $N^5 \times \mathbb{R}$ is Kähler and Ricci flat, that is, has an integrable $SU(3)$-structure. Thus it induces an $SU(2)$-structure on N^5 satisfying (8). Note that the two forms ω_2, ω_3 are not given explicitly since the $SU(3)$ structure on the cone is not explicit; we just know that such a structure does exist and is given by (9).

2. Nearly hypo structures

Let (η, ω_i) be an $SU(2)$-structure on N^5 and let (F, Ψ_+, Ψ_-) be the $SU(3)$-structure on $N^5 \times \mathbb{R}$ defined by (3).

Goal: Find sufficient conditions on the $SU(2)$-structure (η, ω_i) so that the induced $SU(3)$-structure on $N^5 \times \mathbb{R}$ is nearly Kähler, i.e. it satisfies (7).

Definition. We call an $SU(2)$-structure (η, ω_i) on a 5-manifold N^5 a *nearly hypo structure* if it satisfies the following two equations:

(10) $d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1.$

Consider $SU(2)$-structures $(\eta(t), \omega_i(t))$ on N^5 depending on a real parameter $t \in \mathbb{R}$, and the corresponding $SU(3)$-structures

$$(F(t), \Psi_+(t), \Psi_-(t))$$

on $N^5 \times \mathbb{R}$.

We have

Proposition 1. An $SU(2)$-structure (η, ω_i) on N^5 can be lifted to a nearly Kähler structure $(F(t), \Psi_+(t), \Psi_-(t))$ on $N^5 \times \mathbb{R}$ defined by (3) if and only if it is a nearly hypo structure and the following *evolution nearly hypo equations* hold

$$\begin{cases} \partial_t \omega_1 = -d\eta - 3\omega_3, \\ \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1. \end{cases} \tag{11}$$

Corollary. Any Sasaki-Einstein $SU(2)$-structure is a nearly hypo structure.

More generally, we have

Proposition 2. Let $f : N^5 \longrightarrow M^6$ be an immersion of an oriented 5-manifold into a 6-manifold with a nearly Kähler structure. Then the $SU(2)$-structure induced on N^5 is a nearly hypo structure.

Question 1. *Does the converse of Proposition 2 hold? That is, can any (real analytic) nearly hypo structure on N^5 be lifted to a nearly Kähler structure on $N^5 \times \mathbb{R}$?*

An affirmative answer to this question is equivalent to showing the existence of a solution of the evolution nearly hypo equations (11).

Theorem 2. Let (N^5, η, ω_i) be a Sasaki-Einstein $SU(2)$-structure satisfying (8). Then the $SU(3)$-structure (F, Ψ_+, Ψ_-) on $N^5 \times \mathbb{R}$ defined for $0 \le t \le \pi$ by

(12)

$$\begin{aligned} F &= \sin^2 t\,(\sin t\,\omega_1 + \cos t\,\omega_3) \\ &\quad + \sin t\,\eta \wedge dt, \\ \Psi_+ &= \sin^3 t\,\eta \wedge \omega_2 \\ &\quad - \sin^2 t\,(-\cos t\,\omega_1 + \sin t\,\omega_3) \wedge dt, \\ \Psi_- &= \sin^3 t\,(-\cos t\,\omega_1 + \sin t\,\omega_3) \wedge \eta \\ &\quad + \sin^2 t\,\omega_2 \wedge dt, \end{aligned}$$

is a nearly Kähler structure on $N^5 \times \mathbb{R}$ generating the well known Einstein metric

$$g_6 = dt^2 + \sin^2 t\, g_5,$$

where g_5 is the Einstein-Sasaki metric on N^5.

If (N^5, η, ω_i) is compact then

$$(N^5 \times S^1, F, \Psi_+, \Psi_-)$$

is a compact nearly Kähler 6-manifold with two conical singularities at $t = 0$ and $t = \pi$.

3. Nearly half flat structures on 6-manifolds

We generalize the construction of nearly parallel G_2-structures on $M^6 \times \mathbb{R}$ induced from a nearly Kähler structure on M^6 described by Bilal-Metzger.

Let (F, Ψ_+, Ψ_-) be an $SU(3)$-structure on a 6-manifold M^6. Consider the G_2-structure ϕ on $M^6 \times \mathbb{R}$ defined by the 3-form ϕ:

(13) $$\phi = F \wedge dq - \Psi_-,$$

where dq is the standard 1-form on $\mathbb{R}$. We also have a 4-form

(14) $$*_7 \phi = \frac{1}{2} F \wedge F + \Psi_+ \wedge dq,$$

where $*_7$ denotes the Hodge star operator on $M^6 \times \mathbb{R}$.

Vice versa, let $f : M^6 \longrightarrow P^7$ be a hypersurface of a G_2-manifold (P^7, ϕ) and denote by $\mathbb{N}$ the unit normal. Then the G_2-structure ϕ induces an $SU(3)$-structure (F, Ψ_+, Ψ_-) on M^6 defined by the equalities

(15)

$$F = \mathbb{N}\lrcorner\phi, \quad \Psi_+ = -\mathbb{N}\lrcorner * \phi, \quad \Psi_- = -f^*\phi.$$

Recall that a G_2-structure is called nearly parallel if

(16) $$d\phi = 4 * \phi.$$

It is well known that nearly parallel G_2-structures are Einstein with positive scalar curvature.

Hitchin shows that an $SU(3)$-structure on M^6 can be lifted to a parallel G_2-structure (13) on $M^6 \times \mathbb{R}$, (a G_2-structure satisfying $d\phi = d * \phi = 0$), exactly when the underlying $SU(3)$-structure is half flat:

$$dF \wedge F = d\Psi_+ = 0.$$

Goal: Find sufficient conditions on the $SU(3)$-structure (F, Ψ_+, Ψ_-) so that the induced G_2-structure on $M^6 \times \mathbb{R}$ determined by (13) is nearly parallel, i.e. it satisfies (16).

Definition. We call an $SU(3)$-structure (F, Ψ_+, Ψ_-) on a 6-manifold M^6 *nearly half flat* if it satisfies the equation

$$d\Psi_- = -2F \wedge F. \tag{17}$$

In particular, any nearly Kähler 6-manifold carries a nearly half flat structure.

Consider $SU(3)$-structures $(F(q), \Psi_+(q), \Psi_-(q))$ on M^6 depending on a real parameter $q \in \mathbb{R}$ and the corresponding G_2-structure $\phi(q)$ on $M^6 \times \mathbb{R}$.

Proposition 3. An $SU(3)$-structure (F, Ψ_+, Ψ_-) on M^6 can be lifted to a nearly parallel G_2-structure $\phi(q)$ on $M^6 \times \mathbb{R}$ defined by (13) if and only if it is a nearly half flat structure and the following *evolution nearly half flat equation* holds

$$\partial_q \Psi_- = 4\Psi_+ - dF. \tag{18}$$

As a consequence of the above considerations, we can recover one of the Bilal-Metzger main results

Theorem 3.[B-M] Let (M^6, F, Ψ_+, Ψ_-) be a nearly Kähler structure.
Then the G_2-structure ϕ on $M^6 \times \mathbb{R}$ defined for $0 \leq q \leq \pi$ by

$$\phi = \sin^2 q\, F \wedge dq - \sin^3 q \left(-\cos q\, \Psi_+ + \sin q\, \Psi_-\right) \tag{19}$$

is a nearly parallel G_2-structure on $M^6 \times \mathbb{R}$ generating the well known Einstein metric

$$g_7 = dq^2 + \sin^2 q\, g_6,$$

where g_6 is the nearly Kähler metric on M^6.

If (M^6, F, Ψ_+, Ψ_-) is compact then $(M^6 \times S^1, \phi)$ is a compact nearly parallel G_2-manifold with two conical singularities at $q = 0$ and $q = \pi$.

More generally we have

Proposition 3. Let $f : M^6 \longrightarrow P^7$ be an immersion of an oriented 6-manifold into a 7-manifold with a nearly parallel G_2-structure. Then the $SU(3)$-structure induced on M^6 is a nearly half flat $SU(3)$-structure.

Question 2. *Does the converse of Proposition 3 hold? That is, it is true that any (real analytic) nearly half flat structure on M^6 can be lifted to a nearly parallel G_2-structure on $M^6 \times \mathbb{R}$?*

This is equivalent to prove the existence of a solution of the evolution nearly half flat equations (18).

4. Examples

For $N^5 = S^5 \subset S^6$,

for $N^5 = S^2 \times S^3 \subset S^3 \times S^3$

we find an explicit description of the Sasaki-Einstein hypo $SU(2)$-structure on N^5

and obtain a new nearly Kähler structure with two conical singularities on

$S^5 \times S^1$

$S^2 \times S^3 \times S^1$

as well as a nearly parallel G_2-structure on $N^5 \times S^1 \times S^1$ according to Theorem 2 and Theorem 3.

We describe explicitely the Nearly Kähler structure on S^6.

Let $U = \sum_{i=1}^{7} x_i \frac{\partial}{\partial x_i}$ be the unit normal vector field to $S^6 - \{p\}$. We identify $\mathbb{R}^7$ with the imaginary part of the space of Cayley numbers, and define a vector cross product $x \times y$, where $x,\ y \in \mathbb{R}^7$, by the imaginary part of the Cayley number xy. The standard almost complex structure on S^6 is defined by $J(X) = U \times X$ for any vector field X on S^6.

Consider $S^5 = \{(x_1, \cdots, x_6) \in \mathbb{R}^6 \mid \sum_{i=1}^{6} x_i^2 = 1\} \subset S^6$, and $\mathbb{N} = \frac{\partial}{\partial x_7}$ the unit normal vector field to S^5. Then, using (4), the $SU(2)$-structure (η, ω_i) on S^5 is given by

$$\eta = -\frac{\partial}{\partial x_7} \lrcorner F = x_6 dx_1 - x_1 dx_6 + x_2 dx_5$$
$$-x_5 dx_2 + x_3 dx_4 - x_4 dx_3,$$

$$\omega_1 = f^*(F) = x_3 dx_{12} - x_2 dx_{13} + x_1 dx_{23} + x_5 dx_{14}$$
$$-x_4 dx_{15} + x_1 dx_{45} + x_6 dx_{24} - x_4 dx_{26}$$
$$+x_2 dx_{46} + x_5 dx_{36} - x_6 dx_{35} - x_3 dx_{56},$$

$$\text{(20)} \qquad \omega_2 = \frac{\partial}{\partial x_7} \lrcorner \Psi_- = -x_4 dx_{12} + x_5 dx_{13}$$
$$+x_2 dx_{14} - x_3 dx_{15} + x_6 dx_{23} - x_1 dx_{24}$$
$$-x_3 dx_{26} + x_1 dx_{35} + x_2 dx_{36}$$
$$+x_6 dx_{45} - x_5 dx_{46} + x_4 dx_{56},$$

$$\omega_3 = -\frac{\partial}{\partial x_7} \lrcorner \Psi_+ = dx_{16} - dx_{34} - dx_{25}.$$

The $SU(2)$-structure on S^5 defined by (20) satisfies Lemma 1.

Remark. *We notice that the structure* (η, ω_3) *is the standard Einstein-Sasaki structure on* S^5 *induced from the standard Euclidean* R^6*.*

We apply Theorem 2 and Theorem 3.

Theorem 4. Let $(S^5, \eta, \omega_i, g_5)$ be the standard Sasaki-Einstein manifold endowed with the $SU(2)$-structure determined by (20). Then

i) The $SU(3)$-structure on $S^5 \times S^1$ defined by (12) is a nearly Kähler structure generating the metric $g_6 = dt^2 + \sin^2 t \ g_5$ with two conical singularities at $t = 0, t = \pi$.

ii) The G_2-structure on $(S^5 \times S^1) \times S^1$ defined by (19) is a nearly parallel G_2-structure generating the metric $g_7 = dq^2 + \sin^2 q(dt^2 + \sin^2 t \ g_5)$ with singularities at $t = 0, t = \pi, q = 0, q = \pi$.

We take the explicit description of the standard $SU(3)$-structure on $S^3 \times S^3$ from Acharaya at all.

Consider the sphere S^3, viewed as the Lie group $SU(2)$, with the basis of left-invariant 1-forms $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying

$$d\alpha_1 = -\alpha_2 \wedge \alpha_3,$$
$$d\alpha_2 = \alpha_1 \wedge \alpha_3,$$
$$d\alpha_3 = -\alpha_1 \wedge \alpha_2.$$

Denote by $\{\beta_1, \beta_2, \beta_3\}$ another basis on a second sphere S^3 satisfying the same relations. Then, a nearly Kähler structure on $S^3 \times S^3$:

$$F = \frac{i}{2}(\mu_1 \wedge \overline{\mu}_1 + \mu_2 \wedge \overline{\mu}_2 + \mu_3 \wedge \overline{\mu}_3),$$
$$\Psi = i(\mu_1 \wedge \mu_2 \wedge \mu_3),$$

where $\mu_j = \frac{1}{3}(\alpha_j + e^{\frac{2\pi i}{3}}\beta_j)$, for $j = 1, 2, 3$.

Consider $S^3 \times S^3$ as the submanifold of $\mathbb{R}^8$,

$$S^3 \times S^3 = \{(x_1, \dots, x_4, x_5, \dots, x_8) \in \mathbb{R}^8 \mid x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2 = 1\}.$$

Denote by $\{U_j, V_j\}_{j=1}^3$ the basis of vector fields on $S^3 \times S^3$ dual to $\{\alpha_j, \beta_j\}_{j=1}^3$.

Consider the hypersurface $S^2 \times S^3 \subset S^3 \times S^3$ given by $x_4 = 0$. The vector field

$$\mathbb{N} = -\sqrt{3}(2x_1U_1 + 2x_2U_2 + 2x_3U_3 + x_1V_1 + x_2V_2 + x_3V_3)$$

is a unit normal vector field along $S^2 \times S^3$.

However, this does not give a totally geodesic embedding and we can not apply Lemma 1.1.

The $SU(2)$-structure (η, ω_i) induced on $S^2 \times S^3$ by $\mathbb{N}$ satisfies

$$\begin{aligned} d\omega_1 &= 3\eta \wedge \omega_2, \\ d\omega_2 &= -3\eta \wedge \omega_1 \\ d\omega_3 &= 0, \\ d\eta &\neq -2\omega_3. \end{aligned}$$

We check that

$$\eta \wedge (d\eta)^2 \neq 0,$$

so η is a contact form on $S^2 \times S^3$.

We define the quadruplet $(\eta, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ of forms on $S^2 \times S^3$ by

$$\tilde{\omega}_i = \frac{\sqrt{3}}{2}\omega_i \quad (i = 1, 2), \qquad \tilde{\omega}_3 = -\frac{1}{2}d\eta. \tag{21}$$

The quadruplet $(\eta, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ is an $SU(2)$-structure (i.e. it satisfies (1) and (2)).

The $SU(2)$-structure on $S^2 \times S^3$ given by (21) satisfies equations (8) and therefore it is an Einstein-Sasaki hypo structure.

We apply Theorem 2 and Theorem 3:

Theorem 5. Let $(S^2 \times S^3, \eta, \omega_i, g)$ be the Sasaki-Einstein manifold endowed with the $SU(2)$-structure determined by (21). Then

i) The $SU(3)$-structure on $S^2 \times S^3 \times S^1$ defined by (12) is a nearly Kähler structure generating the metric

$$g_6 = dt^2 + \sin^2 t \ g$$

with two conical singularities at $t = 0, t = \pi$.

ii) The G_2-structure on $(S^2 \times S^3 \times S^1) \times S^1$ defined by (19) is a nearly parallel G_2-structure generating the metric

$$g_7 = dq^2 + \sin^2 q(dt^2 + \sin^2 t \ g)$$

with singularities at $t = 0, t = \pi, q = 0, q = \pi$.