Symplectic connections and Yang-Mills theory

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Let M be a smooth manifold and let $\mathcal{C}(M)$ denote the space of linear connections on M.
$\mathcal{C}(M)$ is an affine space over the vector space $\Omega^1(M, \mathrm{End}(TM))$.
The torsion $T^\nabla \in \Omega^2(M, TM)$ of $\nabla \in \mathcal{C}(M)$ is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \,.$$

The mapping $\nabla \in \mathcal{C}(M) \mapsto T^\nabla \in \Omega^2(M, TM)$ is affine and the associated linear map $\Phi : \Omega^1(M, \mathrm{End}(TM)) \to \Omega^2(M, TM)$ is given by

$$\Phi(\theta)(X, Y) = \theta(X)Y - \theta(Y)X \,.$$

Let g be a Riemannian metric on M and let $\mathcal{C}(M,g) \subset \mathcal{C}(M)$ denote the affine subspace of metric connections on M.
The vector space associated to $\mathcal{C}(M,g)$ is

$$\mathcal{E}(M,g) = \{\theta \in \Omega^1(M, \mathrm{End}(TM)) : \\ g(\theta(X)Y, Z) = -g(\theta(X)Z, Y)\} .$$

Proposition

The restriction of Φ to $\mathcal{E}(M,g)$ is an isomorphism onto $\Omega^2(M, TM)$. Thus $\nabla \in \mathcal{C}(M,g) \mapsto T^\nabla \in \Omega^2(M, TM)$ is $1:1$. In particular, there is a unique $\nabla \in \mathcal{C}(M,g)$ with $T^\nabla = 0$, called Levi-Civita connection.

Let ω be an almost symplectic structure (i.e. a non-degenerate 2-form) on M.
$\nabla \in \mathcal{C}(M)$ is called **symplectic** if $\nabla\omega = 0$, i.e. if

$$X(\omega(Y,Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) \ .$$

We do not require that ∇ is torsion-free.
The space $\mathcal{C}(M,\omega)$ of symplectic connections on M is an affine subspace of $\mathcal{C}(M)$ and the associated vector space is

$$\mathcal{E}(M,\omega) = \{\theta \in \Omega^1(M, \mathrm{End}(TM)) : \\ \omega(\theta(X)Y, Z) = \omega(\theta(X)Z, Y)\} \ .$$

Proposition

- $\theta \in \mathcal{E}(M,\omega)$ satisfies $\Phi(\theta) = 0$ iff $\omega(\theta(X)Y, Z)$ is totally symmetric in X, Y, Z.
- $\Phi(\mathcal{E}(M,\omega))$ is the space of all $\zeta \in \Omega^2(M, TM)$ such that

$$\omega(\zeta(X,Y),Z) + \omega(\zeta(Y,Z),X) + \omega(\zeta(Z,X),Y) = 0 \,.$$

Corollary

$\nabla \in \mathcal{C}(M,\omega) \mapsto T^{\nabla} \in \Omega^2(M, TM)$ is neither injective nor onto. Moreover, $\{\nabla \in \mathcal{C}(M,\omega) : T^{\nabla} = \zeta\}$ for $\zeta \in \Omega^2(M, TM)$ is empty or infinite dimensional.

Proposition

There exists a torsion-free symplectic connection iff ω is a symplectic structure, i.e. iff $\mathrm{d}\omega = 0$.

Consequence: In symplectic geometry, there is no analog of the Levi-Civita connection.

Question: Can one find preferred symplectic connections in other ways?

In 1999, Bourgeois and Cahen suggested to use a variational principle for a Yang-Mills type functional. However, their considerations are restricted to torsion-free connections.

Reasons for considering symplectic connections with torsion:

- In **String theory**, one is interested in metric connections with “good” but non-trivial torsion.
- **Hermitian connections** with respect to a compatible almost complex structure can be used. There, one has distinguished connections.
- For the construction of **symplectic Dirac operators**, some symplectic connections with torsion seem to be more suitable than torsion-free connections.
- One can take into considerations also **almost symplectic** manifolds.
- The suggested approach can be generalized to **connections on vector bundles**.

The last point will give a Yang-Mills theory in a purely symplectic context.

In

> *H. Urakawa: Yang-Mills theory over compact symplectic manifolds. Ann. Global Anal. Geom. 25, 365-402 (2004).*

one deals with usual Yang-Mills theory but in a symplectic framework. Namely, there one considers

$$\frac{1}{2\,n!}\int_M g(R^\nabla, R^\nabla)\,\omega^n\ ,$$

where g is a compatible Riemannian metric.

In the following, let $\mathrm{s} = (\mathrm{e}_1, \ldots, \mathrm{e}_{2n})$ be a local symplectic frame on M. Here "symplectic" means that

$$\omega(\mathrm{e}_i, \mathrm{e}_j) = \omega(\mathrm{e}_{n+i}, \mathrm{e}_{n+j}) = 0 \quad \text{and}$$
$$\omega(\mathrm{e}_i, \mathrm{e}_{n+j}) = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, n\,.$$

Let J^{s} be the local almost complex structure

$$J^{\mathrm{s}}\mathrm{e}_i = \mathrm{e}_{n+i} \quad \text{for} \quad i = 1, \ldots, n\,.$$

Let $\nabla \in \mathcal{C}(M)$. The **curvature** $R^\nabla \in \Omega^2(M, \mathrm{End}(TM))$ is

$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

The **Ricci tensor** $\mathrm{ric}^\nabla \in \Gamma(T^*M \otimes T^*M)$ is

$$\mathrm{ric}^\nabla(X, Y) = \mathrm{Tr}\left(Z \mapsto R^\nabla(Z, X)Y\right) = \sum_{i=1}^{2n} \omega\left(R^\nabla(\mathrm{e}_i, X)Y, J^{\mathrm{s}}\mathrm{e}_i\right)\,.$$

We define the **symplectic Ricci operator** $\mathrm{sRic}^{\nabla} \in \Gamma(\mathrm{End}(TM))$ by

$$\mathrm{sRic}^{\nabla}(X) = R^{\nabla}(\omega)X = \sum_{i=1}^{n} R^{\nabla}(e_i, J^s e_i)X$$

and the **symplectic Ricci tensor** $\mathrm{sric}^{\nabla} \in \Gamma(T^*M \otimes T^*M)$ by

$$\mathrm{sric}^{\nabla}(X, Y) = \omega\left(\mathrm{sRic}^{\nabla}(X), Y\right) \ .$$

Remark: If ω is the Kähler form of an almost Hermitian structure (g, J) and ∇ is the Levi-Civita connection of g, then sric is the so-called $*$-Ricci tensor.

Proposition

If ∇ is symplectic and torsion-free, then $\mathrm{sric}^\nabla = \mathrm{ric}^\nabla$.

Proposition

For any symplectic connection ∇, $\mathrm{sric}^\nabla(X, Y) = \mathrm{sric}^\nabla(Y, X)$.

Proof: This follows from

$$\omega\left(R^\nabla(X, Y)Z_1, Z_2\right) = \omega(R^\nabla(X, Y)Z_2, Z_1)$$

for any $\nabla \in \mathcal{C}(M, \omega)$. □

However: For a general $\nabla \in \mathcal{C}(M, \omega)$, $\mathrm{sric} \neq \mathrm{ric}$ and ric is not symmetric.

The symplectic Ricci tensor seems to be more adapted to symplectic geometry than the usual one, since

- sric^{∇} is symmetric.
- sric^{∇} arises in a natural way in the study of symplectic Dirac operators.
- sRic^{∇} can be generalized to connections in a vector bundle.

The **symplectic Ricci operator** $\mathrm{sRic} \in \Gamma(\mathrm{End}(E))$ of a connection ∇ in a vector bundle E on M is defined by

$$\mathrm{sRic}(\xi) = R^{\nabla}(\omega)\xi = \sum_{i=1}^{n} R^{\nabla}(e_i, J^s e_i)\xi \ .$$

We consider a closed symplectic manifold (M, ω) and a real vector bundle $E \to M$ with a symplectic (or Riemannian) structure h.
Let $\mathfrak{s} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_{2m})$ be a local symplectic frame in E and let $J^{\mathfrak{s}}$ be the local almost complex structure in E given by

$$J^{\mathfrak{s}}\mathfrak{e}_i = \mathfrak{e}_{m+i} \quad \text{for} \quad i = 1, \ldots, m\,.$$

For $K, L \in \Gamma(\mathrm{End}(E))$,

$$h(K, L) = \sum_{i=1}^{2m} h\left(K\mathfrak{e}_i, LJ^{\mathfrak{s}}\mathfrak{e}_i\right)\,.$$

Furthermore, for $\alpha, \beta \in \Omega^k(M, \mathrm{End}(E))$,

$$h(\alpha, \beta) = \sum_{1 \leq i_1 < \cdots < i_k \leq 2n} h(\alpha(\mathrm{e}_{i_1}, \ldots, \mathrm{e}_{i_k}), \beta(J^{\mathrm{s}}\mathrm{e}_{i_1}, \ldots, J^{\mathrm{s}}\mathrm{e}_{i_k}))\,.$$

Let $\mathcal{C}(E,h)$ denote the space of all connections ∇ in E such that

$$X(h(\xi_1,\xi_2)) = h(\nabla_X\xi_1,\xi_2) + h(\xi_2,\nabla_X\xi_1) \quad \text{for} \quad \xi_1,\xi_2 \in \Gamma(E)\ .$$

$\mathcal{C}(E,h)$ is an affine space over

$$\mathcal{E}(E,h) = \left\{\theta \in \Omega^1(M,\mathrm{End}(E)) : \omega(\theta(X)\xi_1,\xi_2) = \omega(\theta(X)\xi_2,\xi_1)\right\}\ .$$

We now consider the following two functionals.

$$I_1 : \mathcal{C}(E,h) \to \mathbb{R}\ , \qquad I_1(\nabla) = \frac{1}{2}\int_M h(R^\nabla, R^\nabla)\,\omega^{(n)}\ ,$$

$$I_2 : \mathcal{C}(E,h) \to \mathbb{R}\ , \qquad I_2(\nabla) = \frac{1}{2}\int_M h(\mathrm{sRic}^\nabla, \mathrm{sRic}^\nabla)\,\omega^{(n)}\ ,$$

where

$$\omega^{(k)} = \frac{1}{k!}\omega^k\ .$$

Theorem

For $i = 1, 2$, a connection $\nabla \in \mathcal{C}(E, h)$ is a critical point of I_i iff $\nabla \mathrm{sRic}^\nabla = 0$.

Sketch of the proof:

Step 1: $\mathrm{d} I_1(\nabla)\theta = \int_M h(\mathrm{d}^\nabla \theta, R^\nabla)\, \omega^{(n)}$

Step 2: Define $* : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{2n-k}(M, \mathrm{End}(E))$ by

$$h(\alpha, \beta)\, \omega^{(n)} = h(\alpha \wedge *\beta) \quad \text{for} \quad \alpha, \beta \in \Omega^k(M, \mathrm{End}(E)) .$$

Set $\delta^\nabla = (-1)^{k+1} * \mathrm{d}^\nabla * : \Omega^{k+1}(M, \mathrm{End}(E)) \to \Omega^k(M, \mathrm{End}(E))$. Then

$$\int_M h(\mathrm{d}^\nabla \alpha, \beta)\, \omega^{(n)} = \int_M h(\alpha, \delta^\nabla \beta)\, \omega^{(n)} .$$

Thus: ∇ is critical for I_1 iff $\mathrm{d}^\nabla * R^\nabla = 0$.

Step 3: For $\nabla \in \mathcal{C}(E,h)$ and $\alpha \in \Omega^1(M, \mathrm{End}(E))$,

$$\delta^\nabla \alpha = -\mathrm{d}^\nabla \alpha(\omega) \; .$$

Step 4: Let ∇^t be a curve in $\mathcal{C}(E,h)$ with

$$\nabla^0 = \nabla \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\nabla^t\Big|_{t=0} = \theta \; .$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sRic}^{\nabla^t}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}R^{\nabla^t}(\omega)\Big|_{t=0} = \mathrm{d}^\nabla\theta(\omega) = -\delta^\nabla\theta \; .$$

Hence

$$\mathrm{d}I_2(\nabla)\theta = -\int_M h(\delta^\nabla\theta, \mathrm{sRic}^\nabla)\,\omega^{(n)} = -\int_M h(\theta, \nabla\mathrm{sRic}^\nabla)\,\omega^{(n)} \; .$$

Thus: ∇ is critical for I_2 iff $\nabla\mathrm{sRic}^\nabla = 0$.

Step 5: For $\alpha \in \Omega^1(M, \mathrm{End}(E))$ and $\beta \in \Omega^2(M, \mathrm{End}(E))$,

$$*\alpha = \alpha \wedge \omega^{(n-1)}$$

and

$$*\beta = \beta(\omega) \otimes \omega^{(n-1)} - \beta \wedge \omega^{(n-2)} .$$

Remark: Until now, we have not used that $\mathrm{d}\omega = 0$.

Step 6: One gets

$$\begin{aligned}
\mathrm{d}^{\nabla} * R^{\nabla} &= \mathrm{d}^{\nabla}\left(R^{\nabla}(\omega) \otimes \omega^{(n-1)} - R^{\nabla} \wedge \omega^{(n-2)}\right) \\
&= \mathrm{d}^{\nabla}\left(\mathrm{sRic}^{\nabla} \otimes \omega^{(n-1)}\right) - \mathrm{d}^{\nabla} R^{\nabla} \wedge \omega^{(n-2)} \\
&\quad - R^{\nabla} \wedge \mathrm{d}\omega^{(n-2)} \\
&= \nabla \mathrm{sRic}^{\nabla} \wedge \omega^{(n-1)} \\
&= *\nabla \mathrm{sRic}^{\nabla} \, .
\end{aligned}$$

Hence $\mathrm{d}^{\nabla} * R^{\nabla} = 0$ iff

$$\nabla \mathrm{sRic}^{\nabla} = 0 \, . \qquad \text{(sYM)}$$

□

Remarks:

- One can show that $l_2 - l_1 = \mathrm{const.}$
- In

  D. Yan: Hodge structure on symplectic manifolds.
  Adv. Math. 120, 142-154 (1996)

  it is proven that, for any $\alpha \in \Omega^k(M, \mathrm{End}(E))$,

  $$\delta^\nabla \alpha = \mathrm{d}^\nabla(\omega \lrcorner \alpha) - \omega \lrcorner \mathrm{d}^\nabla \alpha \ .$$

  (cp. Steps 3 and 6)
- There is no analog of the Hermitian Yang-Mills equation, since

  $$\mathrm{sRic}^\nabla = \lambda \, \mathrm{id}_E$$

  for some $\lambda \in \mathbb{R}$ implies $\mathrm{sRic}^\nabla = 0$.

- Find solutions of (sYM).
- Which symplectic manifolds (M, ω) admit a solution $\nabla \in \mathcal{C}(M, \omega)$ of (sYM). By results of Cahen, Gutt et al., the existence of torsion-free solutions restricts the topology.
- Study the corresponding moduli space.
- What about a **symplectic Yang-Mills flow**?
- Study **self-dual** and **anti-self-dual** connections in dimension 4.

Proposition

If $\dim M = 4$, then

- $*R^\nabla = R^\nabla$ iff $R^\nabla = L \otimes \omega$ for a $L \in \Gamma(\mathrm{End}(E))$.
- $*R^\nabla = -R^\nabla$ iff $\mathrm{sRic}^\nabla = 0$.

Proof: This follows from

$$*R^\nabla = R^\nabla(\omega) \otimes \omega - R^\nabla = \mathrm{sRic}^\nabla \otimes \omega - R^\nabla .$$

□

Remark: $R^\nabla = L \otimes \omega$ and $\mathrm{sRic}^\nabla = 0$ yield solutions of (sYM) also in arbitrary dimensions.

Let $\mathcal{J}(M, \omega)$ be the space of all almost complex structures on M that are ω**-compatible**, i.e.

$$g(X, Y) := \omega(X, JY)$$

is a Riemannian metric on M. For $J \in \mathcal{J}(M, \omega)$, set

$$\mathcal{C}(M, \omega, J) = \{\nabla \in \mathcal{C}(M, \omega) : \nabla J = 0\}$$

and consider the functional

$$I_{2,J} : \mathcal{C}(M, \omega, J) \to \mathbb{R} , \qquad I_{2,J} = \frac{1}{2} \int_M \omega(\mathrm{sRic}^\nabla, \mathrm{sRic}^\nabla)\, \omega^{(n)} .$$

Proposition

$\nabla \in \mathcal{C}(M,\omega,J)$ is a critical point of $I_{2,J}$ iff $\nabla \mathrm{sRic}^{\nabla} = 0$.

Proof: This follows from the theorem above and the fact that

$$\nabla \in \mathcal{C}(M,\omega,J) \mapsto \nabla \mathrm{sRic}^{\nabla} \in \Omega^1(M, \mathrm{End}(TM))$$

is a vector field on $\mathcal{C}(M,\omega,J)$. □

Fixing $J \in \mathcal{J}(M,\omega)$, we can define a **symplectic scalar curvature** by

$$\mathrm{sscal}^{\nabla,J} = \mathrm{Tr}\left(J \circ \mathrm{sRic}^{\nabla}\right) = \mathrm{Tr}_g\left(\mathrm{sric}^{\nabla}\right) .$$

Let

$$\mathcal{B}(M,\omega) = \{(\nabla, J) \in \mathcal{C}(M,\omega) \times \mathcal{J}(M,\omega) : \nabla J = 0\}$$

and consider the functionals

$$I_3 : \mathcal{B}(M,\omega,J) \to \mathbb{R}\,, \quad I_3(\nabla, J) = \int_M \mathrm{sscal}^{\nabla,J}\,\omega^{(n)}\,,$$

$$I_4 : \mathcal{B}(M,\omega,J) \to \mathbb{R}\,, \quad I_4(\nabla, J) = \int_M \omega(\mathrm{sRic}^{\nabla}, J \circ \mathrm{sRic}^{\nabla})\,\omega^{(n)}\,.$$

Proposition

The functionals I_3 and I_4 are constant. In particular,

$$I_3(\nabla, J) = 4\pi\, c_1(M,\omega) \cap [\omega]^{n-1}[M]$$

for any $(\nabla, J) \in \mathcal{B}(M,\omega)$.