

Deformations of Hypercomplex  
Structures  
associated to Heisenberg  
Groups

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## HYPERCOMPLEX MANIFOLDS.

Three complex structures  $I_1, I_2$  and  $I_3$  on a smooth manifold form a hypercomplex structure if

$$I_1^2 = I_2^2 = I_3^2 = -Id, \text{ and } I_1I_2 = I_3 = -I_2I_1,$$

where  $Id$  is the identity map. Then

$$I_{\vec{a}} = a_1I_1 + a_2I_2 + a_3I_3$$

defines a complex structure for any point  $\vec{a} = (a_1, a_2, a_3)$  in the unit 2-sphere  $S^2$

The smooth manifold  $W = X \times S^2$  is called twistor space and is endowed with an integrable almost complex structure  $\mathcal{I}$  defined by  $\mathcal{I}_{(x, \vec{a})} = I_{\vec{a}} \oplus J_{S^2}$ .

## HYPERCOMPLEX STRUCTURES RELATED TO THE HEISENBERG GROUP

The real *Heisenberg group* of dimensional  $4m+1$  is the Lie group  $H_{4m+1}$  whose underlying manifold is  $\mathbf{R}^{4m} \times \mathbf{R}$  with coordinates  $(x, y, z) := (x_1, \dots, x_{2m}, y_1, \dots, y_{2m}, z)$  and whose group law is given by

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' - 2 \sum_{j=1}^{2m} (x_j y'_j - y_j x'_j))$$

The left translations of  $\{\frac{\partial}{\partial x_j}|_0, \frac{\partial}{\partial y_j}|_0, \frac{\partial}{\partial z}|_0\}$  are the following vector fields.

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial z}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}. \quad (1)$$

These vectors form a basis for the *Heisenberg algebra*  $\mathfrak{h}_{4m+1}$  of the Heisenberg group  $H_{4m+1}$ .

Let  $\mathfrak{t}_3$  be the 3-dimensional Abelian algebra. Consider the direct sum  $\mathfrak{h}_{4m+1} \oplus \mathfrak{t}_3$ , where  $\mathfrak{t}_3$  is 3-dimensional Abelian algebra and  $\mathfrak{h}_{4m+1}$  is  $4m + 1$ -dimensional Heisenberg algebra. It has generators  $\{X_1, X_2, \dots, X_{2m}, Y_1, Y_2, \dots, Y_{2m}, Z\}$  with only nonvanishing bracket  $[Y_i, X_j] = 4\delta_{ij}Z$ . The subspace  $\mathfrak{c}$  spanned by  $Z$  is the center of the Heisenberg algebra. Fix a basis  $\{E_1, E_2, E_3\}$  for  $\mathfrak{t}_3$  also. Consider the endomorphisms  $I_1, I_2$  and  $I_3$  of  $\mathfrak{h}_{4m+1} \oplus \mathfrak{t}_3$  defined by

$$\begin{aligned}
I_1 X_{2a-1} &= X_{2a}, & I_1 Y_{2a-1} &= Y_{2a}, & I_1 Z &= E_1; \\
I_2 X_{2a-1} &= Y_{2a-1}, & I_2 X_{2a} &= -Y_{2a}, & I_2 Z &= E_2; \\
I_3 X_{2a-1} &= Y_{2a}, & I_3 X_{2a} &= Y_{2a-1}, & I_3 Z &= E_3; \\
I_2 E_1 &= -E_3, & I_3 E_1 &= E_2, & I_1 E_2 &= E_3;
\end{aligned}$$

Through left translations, these endomorphisms define almost complex structures on the product of the Heisenberg group and the three-dimensional additive group  $H_{4m+1} \times \mathbf{R}^3 \cong \mathbf{R}^{4m+4}$ .

As  $[I_a X, I_a Y] = [X, Y]$  for any left-invariant vector fields  $X$  and  $Y$  and  $1 \leq a \leq 3$ , these complex structures are integrable.

Let  $\Gamma$  be the subgroup  $(\mathbf{Z}^{4m+1}, *)$  of the Heisenberg group where  $*$  is the product in  $H_{4m+1}$ . Then  $\Gamma$  is a lattice and the quotient space  $(\Gamma \backslash H_{4m+1}) \times (\mathbf{R}^3 / \mathbf{Z}^3)$  is a compact manifold. Since  $I_a$  are invariant it is equipped with hypercomplex structure. It admits a hypercomplex projection

$$\phi : \Gamma \backslash H_{4m+1} \times (\mathbf{R}^3 / \mathbf{Z}^3) \rightarrow T^{4m}.$$

We denote this compact hypercomplex manifold by  $X$ .

# AUTOMORPHISMS AND DEFORMATIONS OF THE HYPERCOMPLEX STRUCTURE ON $X$ ARISING FROM DEFORMATIONS OF THE LATTICE.

**Theorem 1** *The hypercomplex structure on  $H_{4m+1} \times \mathbf{R}^3$  constructed above is equivalent to the standard one on  $\mathbf{H}^{m+1}$ .*

This follows from the fact that the structure is Obata flat. In fact one can find explicit global hypercomplex coordinates.

It is known that any deformed lattice  $\Gamma'$  in nilpotent Lie group is obtained from  $\Gamma$  by applying an automorphism of the group. So if we want to consider the set of hypercomplex structures arising from deformations of the lattice in  $X$ , then the automorphism group of the Lie algebra acts transitively on this set. So in order to find its dimension we need:

**Lemma 1** *Let  $\{Z, E_1, E_2, E_3, X_{2a-1}, X_{2a}, Y_{2a-1}, Y_{2a}\}$  be an ordered basis for  $\mathfrak{h}_{4m+1} \times \mathbf{R}^3$ . The automorphism group of  $\mathfrak{h}_{4m+1} \oplus \mathfrak{t}^3$  consists of elements  $\Upsilon$  leaving the center invariant of the form:*

$$\Upsilon = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

*where  $A$  is in  $\text{End}_{\mathfrak{c}} \oplus \text{Hom}(\mathfrak{t}^3, \mathfrak{c} \oplus \mathfrak{t}^3)$ , and  $B$  is in  $\text{Hom}(\mathfrak{h}_{4m+1} \oplus \mathfrak{t}^3, \mathfrak{c} \oplus \mathfrak{t}^3)$ . Moreover,  $C$  is a matrix preserving a symplectic form in  $\mathbf{R}^{4m}$  up to a constant.*

If  $V = (z, e_j, x_i, y_i)$  and  $V' = (z', e'_j, x'_i, y'_i)$  in the given basis, then

$$[V, V'] = -2\omega(V, V')Z$$

for the skew form  $\omega$  with  $\text{Ker}\omega = \text{span}\{Z, E_1, E_2, E_3\}$  and  $\omega(V, V') = 2(-y_i x'_i + x_i y'_i)$ . Since the center is preserved,  $\omega(\Upsilon(V), \Upsilon(V')) = s\omega(V, V')$  which gives the proof. Denote the set of  $V \in \mathbf{R}^{4m}$  with this property  $CSp(2m, \mathbf{R})$ .

The next step is to identify the automorphisms preserving the hypercomplex structure.

**Proposition 1** *The hypercomplex spaces  $(H_{4n+1} \times \mathbf{R}^3)/\Gamma$  and  $(H_{4n+1} \times \mathbf{R}^3)/\overline{\Upsilon}(\Gamma)$ , where  $\overline{\Upsilon} = \exp(\Upsilon)$  are equivalent if and only if:*

$$A = sI, s \neq 0$$

$$B \in CSp(2m, \mathbf{R}) \cap GL(m, \mathbf{H}) = CSp(m)$$

$$C = (C_1 C_2 \dots C_n) = \begin{pmatrix} \alpha_{2i-1} & \alpha_{2i} & \beta_{2i-1} & \beta_{2i} \\ -b_i & a_i & -d_i & c_i \\ -c_i & d_i & a_i & -b_i \\ -d_i & -c_i & b_i & a_i \end{pmatrix}$$

*In particular the dimension of the group of hypercomplex automorphisms arising in this way is  $1 + 9m + 2m^2$ .*

The proposition follows from the fact that all hypercomplex automorphisms of  $\mathbf{H}^n$  are affine.

As a result we have:

**Theorem 2** *The dimension of the automorphism group of the Lie algebra  $\mathfrak{h}_{4m+1} \oplus \mathfrak{t}_3$ , is  $13 + 18m + 8m^2$ . Its subgroup which also preserves the hypercomplex structure on  $H_{4m+1} \times \mathbf{R}^3$  has dimension  $1 + 9m + 2m^2$ .*

**Corollary 1** *The dimension of the deformations of the hypercomplex structure on  $X = (\Gamma \backslash H_{4m+1}) \times (\mathbf{R}^3 / \mathbf{Z}^3)$  arising from a deformation of the lattice is  $12 + 9m + 6m^2$*

## TWISTOR SPACE AND HYPERCOMPLEX DEFORMATIONS

The twistor space  $W$  has the following properties:

- The projection  $p : W \rightarrow S^2$  is holomorphic. with fiber  $p^{-1}(\vec{\alpha})$  isomorphic to the complex manifold  $(X, I_{\vec{\alpha}})$ .
- The map is also *real* in the sense that there is an anti-holomorphic involution  $\tau$  on  $W$  such that  $p \circ \tau = \rho \circ p$  where  $\rho$  is the anti-podal map on the 2-sphere.
- The submanifolds  $\{x\} \times S^2$  form a family of holomorphic sections of  $p$  with normal bundle identified with the vertical bundle of  $p$  given by  $\mathbb{C}^m \otimes p^* \mathcal{O}(1)$  .

Then there is a correspondence:

- (hypercomplex structures on  $X$ )  $\overset{1}{\leftarrow} \overset{1}{\leftrightarrow}$  (quadruples  $(W, \mathcal{I}, p, \tau)$  as above

Based on the above correspondance the deformations of hypercomplex structures are identified to deformations of the real map  $p : W \rightarrow \mathbf{CP}^1$  (H.Pedersen Y.S.Poon). Consider the exact sequence:

$$0 \rightarrow \mathcal{D}_W \rightarrow \Theta_W \xrightarrow{dp} p^* \Theta_{\mathbf{CP}^1} \rightarrow 0$$

Infinitesimal deformations of the map  $p$  are described by the cohomology spaces  $H^k(W, \mathcal{D}_W)$ . The real part of  $H^1(W, \mathcal{D}_W)$  contains the deformations of the hypercomplex structures with  $H^2(W, \mathcal{D}_W)$  being the obstruction space. Similarly, the real part of the cohomology spaces  $H^k(W, \Theta_W)$  contains the deformation theory of the quaternionic structures (Salamon). Using this theory H.Pedersen and Y.S.Poon previously have determined the deformations of the compact homogeneous hypercomplex manifolds.

## COMPLEX AND HYPERCOMPLEX DEFORMATIONS ON $4m$ -TORUS

It is known that every small deformation of  $4m$ -dimensional torus  $T^{4m} = \mathbf{R}^{4m}/\Gamma$  arises from a small deformation of the defining lattice  $\Gamma$ . This leads to identification of the moduli space component with the set

$$SL(4m, \mathbf{Z}) \backslash GL(4m, \mathbf{R}) / GL(2m, \mathbf{C})$$

Similarly we obtain that every local deformation of the standard hypercomplex structure on  $T^{4m}$  is generated by a deformation of  $\Gamma = \mathbf{Z}^{4m}$ . In particular the connected component of the moduli space is the topological space

$$SL(4m, \mathbf{Z}) \backslash GL(4m, \mathbf{R}) / GL(m, \mathbf{H})$$

Similar result holds for the quaternionic deformations.

## MAIN RESULTS

- **Theorem 3** *The real dimension of the virtual parameter space of deformations of hypercomplex structures on the  $(4m + 4)$ -dimensional manifold  $X$  is equal to  $6m^2 + 11m + 12$ .*
- **Theorem 4** *Every point in the virtual parameter space is an infinitesimal deformation of an integrable deformation.*
- **Theorem 5** *There are deformations of the hypercomplex structure on  $X$  which do not arise from a deformation of the lattice.*

## OUTLINE OF THE PROOF

First consider the twistor space  $Z$  of the tori  $T^{4m}$  with the projection  $p : Z \rightarrow \mathbf{CP}^1$ . Then in a local coordinate  $\mu$  of  $\mathbf{CP}^1$  the local coordinates of  $Z$  are  $w_1^a = \mu z_1^a - \bar{z}_2^a$ ,  $w_2^a = \mu z_2^a + \bar{z}_1^a$  and  $\mu$ .

So there exists explicit local trivialization of  $(1,0)$  and  $(0,1)$  tangent and cotangent spaces of  $Z$ . If  $U_1 \cup U_2 = \mathbf{CP}^1$  is the standard covering of  $\mathbf{CP}^1$  then the forms

$$\bar{\sigma}_1^a = \frac{\bar{\mu} d\bar{z}_1^a - dz_2^a}{1 + |\mu|^2}, \quad \bar{\sigma}_2^a = \frac{\bar{\mu} d\bar{z}_2^a + dz_1^a}{1 + |\mu|^2}$$

determine trivialization for  $R^1 p_* \mathcal{O}(U_1)$ . They form basis of  $H^1(p^{-1}(\lambda), \mathcal{O})$  for each  $\lambda \in U_1 \subset \mathbf{CP}^1$ . Then one can determine the direct image sheaves

$$R^q p_* \mathcal{O} = \wedge^q (\mathfrak{t}^{*(0,1)} \otimes \mathcal{O}(1))$$

From here and the projection formula one obtains:

$$\begin{aligned}
R^q p_* p^* \mathcal{O}(\ell) &= R^q p_* \mathcal{O} \otimes \mathcal{O}(\ell) \\
&= \mathcal{O}(\ell) \otimes \wedge^q (\mathfrak{t}^{*(0,1)} \otimes \mathcal{O}(1)) \\
&= \mathcal{O}(\ell + q) \otimes \mathfrak{t}^{*(0,q)}.
\end{aligned}$$

The Lerray spectral sequence for the map  $p : Z \rightarrow \mathbf{CP}^1$  gives  $E_2^{p,q} = H^p(\mathbf{CP}^1, R^q p_* p^* \mathcal{O}(\ell))$ , and  $E_\infty^{p,q} \Rightarrow H^{p+q}(Z, p^* \mathcal{O}(\ell))$ . When  $\ell \geq -1$ ,  $E_2^{p,q} = 0$  for all  $p \geq 1$  and  $q \geq 0$ . Therefore, the spectral sequence degenerates at  $E_2$ , and

$$\begin{aligned}
H^k(Z, p^* \mathcal{O}(\ell)) &= \bigoplus_{p+q=k} E_2^{p,q} = E_2^{0,k} \\
&= H^0(\mathbf{CP}^1, R^k p_* p^* \mathcal{O}(\ell)) \\
&= H^0(\mathbf{CP}^1, \mathcal{O}(\ell + k) \otimes \mathfrak{t}^{*(0,k)}) \\
&= \mathfrak{t}^{*(0,k)} \otimes S^{\ell+k} \mathbf{C}^2.
\end{aligned}$$

Now using the fact that  $\text{Ker}(dp) = \mathcal{D}_Z = \mathfrak{t}^{1,0} \otimes \mathcal{O}(1)$  we have:

**Lemma 2** *There is a natural isomorphism*

$$H^k(Z, \mathcal{D}_Z) = \mathfrak{t}^{1,0} \otimes \mathfrak{t}^{*(0,k)} \otimes S^{k+1}\mathbf{C}^2.$$

We observe that

$$\dim H^1(Z, D_Z) = \dim(GL(4m, \mathbf{R})/GL(m, \mathbf{H}))$$

In order to obtain the result about the local moduli space on the tori we need to check the integrability of each element in  $\dim H^1(Z, D_Z)$  taking into account the fact that the obstruction space is not zero. We use the Kodaira-Spencer method which states that  $\Phi_1 \in H^1(Z, D_Z)$  is integrable if there is a (locally) convergent power series  $\Phi(t) = \Phi_1 t + \Phi_2 t^2 + \dots$ , where  $\Phi_k$  are  $(0,1)$  vector-valued forms, such that

$$\bar{\partial}\Phi(t) + \frac{1}{2}\{\Phi(t), \Phi(t)\} = 0.$$

This is equivalent to

$$\bar{\partial}\Phi_{n+1} = -1/2 \sum_{i=1}^n \{\Phi_i, \Phi_{n+1-i}\}$$

for each  $n$ . In case of tori  $\{\Phi_1, \Phi_2\} = 0$  so the series has only the first term.

The hypercomplex map  $r : X \rightarrow T^{4m}$  gives rise to a holomorphic map  $\psi : W \rightarrow Z$  of the corresponding twistor spaces. Again there are explicit trivializations of the tangent and cotangent spaces of  $W$  which leads to the identification of  $H^k(W, D_W)$ .

Since the obstruction space is not zero, the integrability in Theorem 2 is again investigated using the Kodaira-Spencer method. We construct a convergent power series defining one parameter family of deformations related to every element in  $H^1(W, D_W)$  and. Finally the relation with quaternionic deformations is provided by the coboundary map for the cohomology sequence induced by the map  $p$ . It appears that the coboundary map is injective which gives the dimension of the groups  $H^k(W, \Theta_W)$ . The convergence is similar.

Similar methods provide the space of quaternionic deformations on  $X$

- **Theorem 6** *Every quaternionic deformation of the hypercomplex structure on  $X$  is a hypercomplex deformation.*
- **Theorem 7** *The real dimension of the parameter space of deformations of quaternionic structures on the  $(4m+4)$ -dimensional manifold  $X$  is equal to  $6m^2 + 11m + 9$ .*