G_2-geometry on solvmanifolds

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Common sector of Type II string equations

- Strominger (1986): (M, g) n-dimensional Riemannian spin manifold with a 3-form T (field strength), a spinor field Ψ (supersymmetry) and a function Φ (dilaton).

- Bosonic equations:

$$R^g_{ij} - \frac{1}{4} T_{imn} T_{jmn} + 2\nabla^{LC}_i \delta_j \phi = 0, \ \delta(e^{-2\phi} T) = 0.$$

- Fermionic equations:

$$(\nabla^{LC}_X + \frac{1}{4} i_X T) \cdot \psi = 0, \quad T \cdot \Psi = 2 d\phi \cdot \Psi.$$

It is impossible to look for solutions on a fixed manifold.

Geometric meaning of the 3-form T?

The first fermionic equation means that Ψ is parallel with respect to a new connection:

$$\nabla_X Y = \nabla^{LC}_X Y + \frac{1}{2} T(X, Y, -).$$

$\Rightarrow$ T = torsion of ∇ and the equations become:

$$\mathrm{R}ic^{\nabla} + \frac{1}{2}\delta T + 2\nabla^{LC} d\phi = 0, \quad \delta(e^{-2\phi}T) = 0$$

$$\nabla\psi = 0, \quad T \cdot \Psi = 2d\phi \cdot \Psi$$

Remarks:

• Bosonic equations generalize Einstein equations of general relativity.

• Calabi-Yau and Joyce manifolds are exact solutions with $T = 0$ and $\phi = \text{const}$.

$\Rightarrow$ If $\dim M = 7$, integrable G_2 structures are exact solutions and non-integrable G_2-structures can be studied using metric connections with skew-symmetric torsion.

• Can the same 7-manifold (M, g) carry integrable and non-integrable G_2 structures *simultaneously*?

• There are 7-manifolds with a parallel spinor for ∇^{LC} admitting a covariantly constant spinor for some other ∇?

<u>Aim</u> The answer is YES to both questions. The manifold will be a rank-one solvable extension of a 6-dimensional nilpotent Lie group ($\Rightarrow$ non compact).

Connections with skew-symmetric torsion

(M^n, g) oriented Riemannian manifold

∇: metric connection

$\nabla_X Y = \nabla^{LC}_X Y + A(X, Y)$,
with $A \in \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n) \Rightarrow A$ can be identified with the torsion T of ∇.

If $A \in \Lambda^3(\mathbb{R}^n) \Rightarrow$

$$\nabla_X Y = \nabla^{LC}_X Y + \frac{1}{2} T(X, Y, -)$$

and ∇ is called a metric connection with skew-symmetric torsion.

$\Rightarrow$ The ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

Lifting metric connections to SM

$$\nabla_X Y = \nabla_X^{LC} Y + A_X Y,$$

with $A_X = i_X T \in \mathfrak{so}(n) \cong \Lambda^2(\mathbb{R}^n) \Rightarrow$
$A_X = \sum_{i<j} \alpha_{ij} e_i \wedge e_j$.

Since the lift into $\mathfrak{spin}(n)$ of $e_i \wedge e_j$ is $\frac{e_i \cdot e_J}{2}$, A_X defines an element in $\mathfrak{spin}(n)$ ($\Rightarrow$ an endomorphism on SM).

The action of A_X is:

$A_X Y = i_Y A_X$ on vectors ($A_X \in \mathfrak{so}(n)$)

$A_X \psi = \frac{1}{2} A_X \cdot \psi$ on spinors ($A_X \in \mathfrak{spin}(n)$)

"$\cdot$ "is the Clifford product of a k-form by a spinor

$\Rightarrow$ the lift of ∇ on SM is

$$\nabla_X \psi = \nabla_X^{LC} \psi + \frac{1}{2} A_X \cdot \psi.$$

Non existence theorems

If the dilaton function $\phi = \text{const}$

- Bosonic equations: $Ric^{\nabla} = 0$, $\delta T = 0$

- Fermionic equations: $\nabla \Psi = 0$, $T \cdot \Psi = 0$

Theorem A full solution of Strominger's model with $\phi = \text{const}$ satisfies necessarily $T = 0$ or $\psi = 0$.

[Compact case, Agricola.
General case, Agricola, Friedrich, Nagy, Puhle.]

Vanishing Theorem [Agricola-Friedrich]
(M^n, g, T) compact spin with $Scal^g \leq 0$. If dT acts on spinors as a non positive endomorphism and $\exists \psi \neq 0$ solution of

$$\nabla_X \psi = \nabla_X^{LC} \psi + (i_X T) \cdot \psi = 0$$

then $T = 0 = Scal^g$ and $\nabla^{LC} \psi = 0$.

Corollary On a Calabi-Yau or Joyce manifold a torsion connection such that $dT = 0$ can have parallel spinor only for $T = 0$.

$\Rightarrow$ rigidity of vacuum solutions under deformation of the connection.

The Vanishing Theorem applies also to nilmanifolds $\Gamma \backslash G$.

We will give 7-dimensional examples for which the rigidity theorem does not hold.

Starting point There exist (incomplete) metrics of holonomy G_2 with 2-step nilpotent isometry group N^6 acting on orbits of codim 1 [Gibbons–Lü–Pope–Stelle].

Such metrics are 'scale invariant' meaning that there is a homothetic Killing vector field X ($L_X g = c\, g$) and locally conformally equivalent to homogeneous metrics on rank-one solvable extensions of N^6 [Chiossi, F].

$\Rightarrow$ classification of 6-dimensional nilpotent Lie algebras endowed with an $SU(3)$-structure and a "Hermitian" derivation D such that the G_2-structure on the solvable extension is conformally parallel (integrable).

SU(3) structures

Let N^6 be a (real) 6-dimensional manifold. An $SU(3)$ reduction is given by

$$(J,\,\omega,\,h)$$

plus a choice of

$$\eta = \eta^+ + i\eta^- \in \Lambda^{3,0}$$

One can choose an orthonormal basis of 1-forms such that

$$\omega = e^{14} - e^{23} + e^{56},$$
$$\eta = (e^1 + ie^4) \wedge (e^2 - ie^3) \wedge (e^5 + ie^6)$$

with $\eta^\pm \wedge \omega = 0$.

Note η^+ determines J and $\eta^- = J\eta^+$.

Intrinsic torsion for SU(3)

$$\text{Hol}(N^6, h) \subseteq SU(3) \iff \begin{cases} d\omega = 0, \\ d\eta^+ = 0, \\ d\eta^- = 0. \end{cases}$$

The failure of this condition is measured by the intrinsic torsion (tensor) τ belonging to the space

$$\begin{aligned} T^* \otimes \mathfrak{su}(3)^\perp &\cong T^* \otimes ([\![\Lambda^{2,0}]\!] \oplus \mathbb{R}) \\ &= \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \end{aligned}$$

where $\mathcal{W}_j$ are the so-called 'Gray–Hervella classes' for $j \leq 4$.

If $d\omega^2 = d\eta^+ = 0 \Rightarrow$ the $SU(3)$ structure is *half-flat*.

G_2 structures

Given M^7 such that

$$T_m^* M^7 = T_n^* N^6 \oplus \mathbb{R}$$

(e.g. a Riemannian product $M^7 = N^6 \times \mathbb{R}$) and an $SU(3)$-structure (ω, η^+), define

$$\begin{aligned} \varphi &= \omega \wedge e^7 + \eta^+, \\ *\varphi &= \eta^- \wedge e^7 + \tfrac{1}{2}\omega^2, \end{aligned}$$

where $e^7 = dt$ (t a coordinate on $\mathbb{R}$).

The associated metric g has an orthonormal basis for which

$$\varphi = e^{147} - e^{237} + e^{567} + e^{125} + e^{136} + e^{246} - e^{345}$$

has isotropy $G_2 \subseteq SO(7)$.

The intrinsic torsion of a G_2 structure can be identified with $\nabla^{LC}\varphi$ and is encoded into the exterior derivatives $d\varphi$, $d*\varphi$ by

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \\ d*\varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi \end{aligned}$$

where τ_i is an intrinsic torsion i-form.

$$\begin{aligned} \nabla\varphi \in T^* \otimes \mathfrak{g}_2^\perp &\cong \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \\ &\cong \mathbb{R} \oplus \mathfrak{g}_2 \oplus S_0(\mathbb{R}^7) \oplus \mathbb{R}^7 \end{aligned}$$

[Fernández-Gray]

class	type	conditions
$\mathcal{X}_4$	conformally parallel	$\tau_0 = \tau_2 = \tau_3 = 0$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	cocalibrated	$\tau_1 = \tau_2 = 0$
$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	G_2T	$\tau_2 = 0$

Theorem [Friedrich-Ivanov]

$\tau_2 = 0 \iff \exists!$ a torsion connection ∇ such that $\nabla\varphi = 0$

$\iff \exists$ a vector field X such that $\delta\varphi = -i_X\varphi$.

The resulting 3-form torsion is

$$T = \frac{7}{6}\tau_0\varphi - *d\varphi + *(4\tau_1 \wedge \varphi)$$

and ∇ admits (at least) one parallel spinor.

$T \in \Lambda^3(\mathbb{R}^7) = \mathbb{R} \oplus S_0(\mathbb{R}^7) \oplus \mathbb{R}^7$.

(M^7, φ) is locally conformal parallel iff

$$\begin{aligned} d\varphi &= 3\tau_1 \wedge \varphi, \\ d * \varphi &= 4\tau_1 \wedge *\varphi. \end{aligned}$$

In this case, $T = *(\tau_1 \wedge \varphi)$.

G_2 can be lifted to a subgroup of $Spin(7)$.

Δ_7: complex vector space of all 7-dimensional spinors $\Rightarrow$ Δ_7 is a complex $Spin(7)$-representation and the complexification of a real representation.

$\Delta_7 \cong \mathbb{R} \oplus \mathbb{R}^7$ (under G_2).

We will use the real representation obtained by

$$
\begin{aligned}
e_1 &= +E_{18} + E_{27} - E_{36} - E_{45},\\
e_2 &= -E_{17} + E_{28} + E_{35} - E_{46},\\
e_3 &= -E_{16} + E_{25} - E_{38} + E_{47},\\
e_4 &= -E_{15} - E_{26} - E_{37} - E_{48},\\
e_5 &= -E_{13} - E_{24} + E_{57} + E_{68},\\
e_6 &= +E_{14} - E_{23} - E_{58} + E_{67},\\
e_7 &= +E_{12} - E_{34} - E_{56} + E_{78},
\end{aligned}
$$

E_{ij} standard basis of $\mathfrak{so}(7)$.

Solvable Extensions

Let $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \langle\cdot,\cdot\rangle')$ be a metric nilpotent Lie algebra.

Then $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, \langle\cdot,\cdot\rangle)$ is a metric solvable extension of $(\mathfrak{n}, \langle\cdot,\cdot\rangle')$ if $[\cdot,\cdot]$ restricted to $\mathfrak{n}$ coincides with $[\cdot,\cdot]_{\mathfrak{n}}$ and $\langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}} = \langle\cdot,\cdot\rangle'$.

One says that $\mathfrak{s}$ is 'standard' if $\mathfrak{a} = (\mathfrak{s}^1)^\perp$ is abelian, and $\dim\mathfrak{a}$ is called the rank.

If the rank is 1 with $\mathfrak{a} = \langle A\rangle$, the extension is *of Iwasawa type* if

(i) $\mathrm{ad}_A \neq 0$ is self-adjoint with respect to $\langle\, ,\, \rangle$,

(ii) $\mathrm{ad}_A|_{\mathfrak{n}}$ is positive-definite.

The study of standard solvable Lie algebras with Einstein metrics reduces to rank-one metric solvable extensions

$$(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H,\ \langle \cdot, \cdot \rangle)$$

with $\langle H, \mathfrak{n} \rangle = 0$, $||H|| = 1$ and Lie bracket

$$\begin{cases} [H, X] = DX, \\ [X, Y] = [X, Y]_{\mathfrak{n}} \end{cases}$$

for some $\mathfrak{n}$ and $D \in \mathrm{Der}(\mathfrak{n})$ [Heber].

All nilpotent Lie groups of dim ≤ 6 admit a rank-one Einstein solvable extension [Lauret, Will].

Classification theorem [Chiossi–F]

Let N be a nilpotent Lie group of dimension 6 with an invariant $SU(3)$ structure (ω, η^+). Suppose that there is a derivation D of the Lie algebra $\mathfrak{n}$ such that $(DJ)^2 = (JD)^2$.

Then on the solvable extension

$$\mathfrak{s} = \mathfrak{n} \oplus \langle e_7 \rangle$$

with $\mathrm{ad}_{e_7} = D$, the G_2-structure

$$\varphi = \omega \wedge e^7 + \eta^+$$

is conformally parallel iff $\mathfrak{n}$ is
(i) either $\mathbb{R}^6$ or 2-step nilpotent, and
(ii) not isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$.

Sketch of the proof

• Since the derivation $D = \mathrm{ad}_{e_7}$ preserves the orthogonal splitting $\mathfrak{n}^1 \oplus (\mathfrak{n}^1)^\perp$, one can suppose that D is diagonalizable by a unitary basis $\{e_1, \ldots, e_6\}$ with $e_1 \in (\mathfrak{n}^1)^\perp$. The Maurer-Cartan equations of $\mathfrak{s}$ are

$$\begin{cases} de^j = \hat{d}e^j + C_j e^{j7}, & 1 \leq j \leq 6 \\ de^7 = 0 \end{cases}$$

where $\hat{d} = d|_{\Lambda^* \mathbb{R}^6}$, $\hat{d}e^1 = 0$, and $C_j \neq 0$.

• φ is conformally parallel iff $\exists m \neq 0$ such that

$$\begin{aligned} d\omega \wedge e^7 + d\eta^+ &= -3m\eta^+ \wedge e^7, \\ d\eta^- \wedge e^7 + \omega \wedge d\omega &= 2m\omega^2 \wedge e^7. \end{aligned} \qquad (*)$$

In this case, (ω, η^+) is necessarily half-flat.

Consequences

If $\mathfrak{n}$ is not abelian $\Rightarrow$

- J cannot be integrable

- $d\omega \neq 0$.

[Conti-Tomassini studied symplectic half-flat structures on 6-dimensional nilmanifolds N]

$dT \neq 0$ unless $m = 0$.

[Chiossi–Swann proved that if J is integrable and $N \times S^1$ is G_2T, then N is balanced]

Corollary Any solvmanifold corresponding to a 6-dimensional 2-step nilpotent Lie algebra, except $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, admits a pair of Einstein metrics (one homogeneous and one Ricci-flat).

nilpotent Lie algebra	eigenvalues of ad_{e_7}
$(0, 0, e^{15}, 0, 0, 0)$	$-(\frac{2m}{3}, m, \frac{4m}{3}, m, \frac{2m}{3}, m)$
$(0, 0, e^{15}, e^{25}, 0, e^{12})$	$-(\frac{3m}{5}, \frac{3m}{5}, \frac{6m}{5}, \frac{6m}{5}, \frac{3m}{5}, \frac{6m}{5})$
$(0, 0, e^{15} - e^{46}, 0, 0, 0)$	$-(\frac{3m}{4}, m, \frac{3m}{2}, \frac{3m}{4}, \frac{3m}{4}, \frac{3m}{4})$
$(0, e^{45}, -e^{15} - e^{46}, 0, 0, 0)$	$-(\frac{4m}{5}, \frac{6m}{5}, \frac{7m}{5}, \frac{3m}{5}, \frac{3m}{5}, \frac{4m}{5})$
$(0, e^{45}, e^{46}, 0, 0, 0)$	$-(m, \frac{5m}{4}, \frac{5m}{4}, \frac{m}{2}, \frac{3m}{4}, \frac{3m}{4})$
$(0, e^{16} + e^{45}, e^{15} - e^{46}, 0, 0, 0)$	$-(\frac{2m}{3}, \frac{4m}{3}, \frac{4m}{3}, \frac{2m}{3}, \frac{2m}{3}, \frac{2m}{3})$

Corresponding Ricci-flat metrics

If (S, φ, g) is conformally parallel, the change $\tilde{g} = e^{2f} g$ with $df = me^7$ produces a Ricci-flat metric in terms of coordinates $(x_1, \ldots, x_6, t)$ where $e^7 = dt$.

$\tilde{g}$ is Ricci-flat and admits LC-parallel spinors.

φ defines a LC-parallel spinor Ψ by

$$\varphi(X, Y, Z) = \frac{1}{4} < X \cdot Y \cdot Z \cdot \Psi, \Psi >,$$

where "$\cdot$ " is the Clifford multiplication and $<,>$ is the scalar product on the spinor bundle.

In terms of the spin representation Δ_7 one has $\Psi = (0, 0, 0, 0, 1, 1, -1, 1)$.

Comparing the metrics with the ones found by GLPS

nilpotent Lie algebra	Hol	LC-parallel spinors
$(0,0,e^{15},e^{25},0,e^{12})$	G_2	1
$(0,e^{45},-e^{15}-e^{46},0,0,0)$	G_2	1
$(0,e^{16}+e^{45},e^{15}-e^{46},0,0,0,)$	G_2	1
$(0,0,e^{15}-e^{46},0,0,0)$	$SU(3)$	2
$(0,e^{45},e^{46},0,0,0)$	$SU(3)$	2
$(0,0,e^{15},0,0,0)$	$SU(2)$	4

E.g. The metric corresponding to $(0,0,e^{15},e^{25},0,e^{12})$ is locally isometric to the scale-invariant metric on $M^6 \times \mathbb{R}$, where

$$M^6 \xrightarrow{T^3} T^3.$$

Example The metric

$$\begin{aligned}\tilde{g} \;=\; & e^{-\frac{2}{5}mt}(dx_1^2 + dx_6^2) + e^{-\frac{4}{5}mt}(dx_4^2 + dx_5^2) \\ & + \tfrac{9}{25}m^2 e^{\frac{4}{5}mt}(dx_3 - \tfrac{2}{3}x_1 dx_5 + \tfrac{2}{3}x_4 dx_6)^2 \\ & + \tfrac{9}{25}m^2 e^{\frac{2}{5}mt}(dx_2 + \tfrac{2}{3}x_4 dx_5)^2 + e^{-2mt}dt^2\end{aligned}$$

on the solvmanifold corresponding to

$$(0, e^{45}, -e^{15} - e^{46}, 0, 0, 0)$$

is new!

It is scale invariant with symmetry generated by the homothetic Killing field

$$\begin{aligned}Z = & -\tfrac{5}{m}\tfrac{\partial}{\partial t} + 4x_1\tfrac{\partial}{\partial x_1} + 4x_6\tfrac{\partial}{\partial x_6} + 3x_4\tfrac{\partial}{\partial x_4} + 3x_5\tfrac{\partial}{\partial x_5} \\ & + \tfrac{21}{5}mx_3\tfrac{\partial}{\partial x_3} + \tfrac{18}{5}mx_2\tfrac{\partial}{\partial x_2}.\end{aligned}$$

Evolution of SU(3)-structures

If the G_2-structure

$$\varphi = \omega \wedge e^7 + \eta^+$$

on $\mathfrak{s}$ is conformally parallel, (ω, η^+) is half-flat.

Theorem [Hitchin]
If a compact manifold N^6 has a half-flat $SU(3)$-structure, $\exists$ a metric with Hol $\subseteq G_2$ on $N^6 \times I$ for some interval I.

Proposition [Chiossi-F]
Any of the Ricci-flat metrics $\tilde{g}$ on the solvable Lie group S can be obtained evolving the $SU(3)$ structure on the 2-step nilmanifold $\Gamma\backslash N^6$.

Questions related to spinors:

• Is $\tilde{g}$ induced by another (not integrable) G_2-structure?

• Which solvable Lie groups $(S, \tilde{g})$ admit a parallel spinor for another torsion connection?

We will assume T has the form

$T \in \Lambda^3_{11} =<$ simple forms appearing in $\eta^+, \eta^-, \omega \wedge e^7 >$.

The Clifford multiplication by $i_{e_i}T$ has, as an endomorphism, the block structure

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

The Levi-Civita connection $\tilde{\nabla}^{LC}$ is given by:

$\tilde{\nabla}^{LC}_X Y = \nabla^{LC}_X Y + df(X)Y + df(Y)X - g(X,Y)\text{grad} f,$

with $df = me^7 \Rightarrow \tilde{\nabla}^{LC}_{e_7} X = 0, \forall X \in \mathfrak{n} \Rightarrow$

$\text{Ker}(\tilde{\nabla}^{LC}_{e_7} + i_{e_7}T) = \text{Ker}(i_{e_7}T)$.

Parallel Spinors

Reduction Theorem [Agricola, Chiossi, F]

A non-trivial element in the kernel of $i_{e_7}T$ for $T \neq 0 \in \Lambda^3_{11}(Y)$ is a linear combination of upper-block forms:

- $\Psi = (a, b, c, d, 0, 0, 0, 0)$ with

$T_{567} = 0$, $T_{147} = -T_{237}$;

- $\Psi = (a, b, \mp a, \pm b, 0, 0, 0, 0)$ and

$T_{147} = -T_{237} + \epsilon T_{567}$

or lower-block forms:

- $\Psi = (0, 0, 0, 0, e, f, g, h)$ with

$T_{567} = 0$, $T_{147} = T_{237}$.

- $\Psi = (0, 0, 0, 0, e, f, \pm e, \mp f)$ and

$T_{147} = T_{237} + \epsilon T_{567}$.

$(S, \tilde{g})$: solvmanifold corresponding to

$$(0, 0, e^{15}, e^{25}, 0, e^{12}).$$

$\Psi = (0, 0, 0, 0, 1, 1, -1, 1)$: unique LC-parallel spinor

<u>Theorem</u> [Agricola, Chiossi, F]
The equation $\nabla_X \Psi = 0$ admits precisely 7 solutions for some $T \neq 0$, namely:
a) An $\mathbb{RP}^1$-parameter family of pairs $(T_{r,s}, \Psi_{r,s})$ such that $\nabla^{r,s}\Psi_{r,s} = 0$;
for $r = s$, one has $T_{r,s} = 0$ and $\Psi_{r,s}$ is a multiple of Ψ.
b) Six isolated solutions in pairs $(T_i^\epsilon, \Psi_i^\epsilon)$ for $i = 1, 2, 3$ and $\epsilon = \pm$.
The G_2 structures admit precisely one parallel spinor, and are of general type $\mathbb{R} \oplus S_0(\mathbb{R}^7) \oplus \mathbb{R}^7$, except for $r = s$ ($\varphi_{r,r}$ is parallel) and $r = -s$ (no $\mathbb{R}$-part).

$(S, \tilde{g})$: solvmanifold corresponding to

$$(0, e^{45}, -e^{15} - e^{46}, 0, 0, 0).$$

<u>Theorem</u> [Agricola, Chiossi, F]

If $\exists\ \psi \neq 0$ solution of $\nabla\psi = 0$, then:

(a) ψ is a multiple of

$(1 + 2i\epsilon\sqrt{2},\ 3,\ 1 + 2i\epsilon\sqrt{2},\ -3, 0, 0, 0, 0)$ and

$T = \frac{2}{3}\,[-2e^{126} + e^{135} - 4e^{245} + e^{346}] + i\epsilon\sqrt{2}\,[e^{125} + e^{136} + e^{246} + e^{345}] + \frac{2}{3}i\epsilon\sqrt{2}\,[-e^{147} - e^{567} + 2e^{237}]$ or

(b) ψ is a multiple of

$(3,\ -1 + 2i\epsilon\sqrt{2},\ -3,\ -1 + 2i\epsilon\sqrt{2}, 0, 0, 0, 0)$ and

$T = \frac{2}{3}\,[e_{126} - e_{135} + 4e_{245} - 2e_{346}] + i\epsilon\sqrt{2}\,[-e_{125} + e_{136} + e_{246} - e_{345}] + \frac{2}{3}\,i\epsilon\sqrt{2}\,[-e_{147} + e_{567} + 2e_{237}]$ or

(c) ψ is a multiple of

$(0, 0, 0, 0,\ 1 + 2i\epsilon\sqrt{2},\ 3,\ 1 + 2i\epsilon\sqrt{2},\ -3)$ and

$T = \frac{2}{3}\,[e_{126} - 2e_{135} + 4e_{245} - e_{346}] + i\epsilon\sqrt{2}\,[e_{125} + e_{136} - e_{246} - e_{345}] + \frac{2}{3}\,i\epsilon\sqrt{2}\,[e_{147} - e_{567} + 2e_{237}]$.

<u>Question</u> Any physical interpretation of the complex solutions?