On invariants of Fedosov manifolds

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We consider moduli spaces of two structures:

- Fedosov structure $\Phi = (\mathcal{M}, \omega, \nabla), \nabla \omega = 0$

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math.DG/0310469

and the "baby" example:

- Symmetric connection $\nabla$ (or $\Gamma$)

*Zapiski Seminarov POMI*, 2002
math.DG/0112291

There is deformation quantization on $\Phi$

(B.Fedosov - canonical construction, and M. De Wilde - P. B. A. Lecompte, 1980’s)
Subject of **Functional Moduli** has its **origins** in local classification problems of classical analysis.

**Typical outcome** of a classification is a **Normal Form** depending on \( m \) arbitrary functions of \( n \) variables, so called

\[ \text{functional moduli}. \]

**Q.** Are \( m \) and \( n \) - **intrinsic** to the problem (not dependent on many choices made while solving it, e.g. of specific local coordinates)?

Universal approach is given by **Poincaré series**.

Classifying geometric structures at a point amounts to the following:

Consider the space \( \mathcal{F} \) of germs of the structure, and take a quotient under the action of smooth coordinate changes (preserving the point).
The same action on a space of $k$-jets, $\mathcal{F}_k$ is more tractable. In particular, the moduli space $\mathcal{M}_k$ is finite-dimensional.

**Poincaré series** encodes all these dimensions, "moduli numbers", in a single series.

**Action**

$$G := \text{Diff}(\mathbb{R}^n, 0) : \quad \mathcal{F} \bigcirc \quad \mathcal{F}_k \bigcirc$$

germs \quad \text{k-jets}

**Moduli spaces**

$$\mathcal{M} = \mathcal{F}/G \quad \mathcal{M}_k = \mathcal{F}_k/G$$

$$\text{dim } \mathcal{M} = \infty \quad \text{dim } \mathcal{M}_k < \infty$$
**Definition** Formal power series

\[ p_S(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k \]

is called the *Poincaré series* of \( S \).

**Remark** If \( S \) does depend on \( m \) arbitrary functions of \( n \) variables, then:

\[ p_S(t) = \frac{m}{(1 - t)^n}, \]

a *rational* function.

Indeed, dimension of moduli spaces of \( k \)-jets is the number of monomials up to the order \( k \) in the formal power series of the \( m \) given invariants:

\[ \dim \mathcal{M}_k = m \binom{n + k}{n}. \]
Problem (Arnold, 1999)
Is it true that the Poincaré series in most of the local problems of analysis are *rational* functions?

In *commutative algebra* in a similar setting there is the Hilbert basis theorem (finiteness of the bases of the ideals of analytic function germs).

However this problem belongs to *differential algebra*. 
A finiteness theorem in differential algebra (as formulated by Tresse, 1894):

For any “natural” differential-geometric structure, all the differential invariants are generated by a finite number of invariant functions and of invariant vector fields (as differential polynomials).

However, the proof is not rigorous, conditions are not precise.

Contributions

A. Einstein: Rigidity of scalar wave equation
Maxwell equation in vacuum
Gravity field equation

A. S. Shmelev (1990’s): Riemannian structure
Kähler structure
hyper-Kähler structure

A. Vershik - V. Gershkovich (1988):
distributions in $\mathbb{R}^n$
\textbf{Theorem (S.D. 2004)}

For the symmetric connection and the Fedosov structure, the Poincaré series coefficients are polynomial in $k$, and the series are:

\begin{align*}
p_{\Gamma}(t) &= (\delta_1^n + \delta_2^n - n^2)t - \delta_2^nt^2 \\
&\quad + n \sum_{k=1}^{\infty} \left[ \frac{n(n+1)(n+k-1)}{2} \binom{n+k}{n-1} - n \binom{n+k+1}{n-1} \right] t^k
\end{align*}

\begin{align*}
p_{\Phi}(t) &= -\binom{2n+1}{2}t^2 + \delta_2^{2n}(t^2 - t^3) \\
&\quad + t \sum_{k=1}^{\infty} \left[ \binom{2n+2}{3} \binom{2n+k-1}{2n-1} - \binom{2n+k+2}{2n-1} \right] t^k.
\end{align*}

Both are \textbf{rational} in $t$, confirming the \textit{finiteness} assertion of Tresse.
Moduli space of symmetric connections

Goal: \( p_\Gamma(t) \), main problem:

describing a stabilizer of generic \( k \)-jet.

Connections \( \nabla \) and \( \tilde{\nabla} \)

have the same \( k \)-jet at 0 if

\[
\forall X, Y \in \Gamma(T \mathbb{R}^n), \; \forall f \in C^\infty(\mathbb{R}^n),
\]

the functions \( \nabla_X Y(f) \) and \( \tilde{\nabla}_X Y(f) \)
have the same \( k \)-jet at 0.

\( j^k(\Gamma) \) will denote the \( k \)-jet of \( \Gamma \).

Action of \( G := \text{Diff}(\mathbb{R}^n, 0) \) on \( \mathcal{F} \) and \( \mathcal{F}_k \)

\[
\varphi: \Gamma \mapsto \varphi^* \Gamma, \quad j^k \Gamma \mapsto j^k(\varphi^* \Gamma),
\]

where

\[
(\varphi^* \nabla)_X Y = \varphi^{-1}(\nabla_{\varphi^* X} \varphi^* Y)
\]
Consider a filtration of $G$ by normal subgroups:

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \ldots ,$$

where

$$G_k = \{ \varphi \in G \mid \varphi(x) = x + O(|x|^k) \} .$$

$G_k$ acts trivially on $F_p$ for $k \geq p + 3$, e.g.

$$G|_{F_p} = G/G_{p+3}|_{F_p}$$

Consider instead infinitesimal action of

$$\text{Vect}_0(\mathbb{R}^n) = \text{Lie}(G)$$

(To calculate dimensions, go into tangent space.)
**Definition** For $V \in \text{Vect}_0(\mathbb{R}^n)$ generating a local 1-parameter subgroup $g^t$ of $\text{Diff}(\mathbb{R}^n, 0)$, the Lie derivative of a connection $\nabla$ in the direction $V$ is a $(1,2)$-tensor:

$$\mathcal{L}_V \nabla = \frac{d}{dt} \bigg|_{t=0} g^{t*} \nabla$$

**Lemma**

$$(\mathcal{L}_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V, X]} Y - \nabla_X [V, Y]$$

This defines the action on germs, on jets:

$$\mathcal{L}_V (j^k \Gamma) = j^k(\mathcal{L}_V \Gamma).$$

Why is it well-defined
(Right Hand Side depends on $(j^k \Gamma)$ only)?
In coordinates, $(\mathcal{L}_V \Gamma)^l_{ij} =$

$$V^k \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \Gamma^k_{ij} \frac{\partial V^l}{\partial x^k} + \Gamma^l_{kj} \frac{\partial V^k}{\partial x^i} + \Gamma^l_{ik} \frac{\partial V^k}{\partial x^j} + \frac{\partial^2 V^l}{\partial x^i \partial x^j}$$

$$= (\tilde{\mathcal{L}}_V \Gamma) + \frac{\partial^2 V}{\partial x^2}$$

Elements of $k$-th order and less are only contributed by $j^k \Gamma$, because $V(0) = 0$.

Action is well-defined means the following is a commutative diagram:

$$
\begin{array}{cccc}
\cdots & \cdots & j^0 \mathcal{F} & \cdots \\
\downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V \\
j^0 \Pi & \cdots & j^{k-1} \Pi & \cdots \\
\end{array}
\begin{array}{cccc}
\cdots & \cdots & j^k \mathcal{F} & \cdots \\
\downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V & \downarrow \mathcal{L}_V \\
\cdots & \cdots & j^k \Pi & \cdots \\
\end{array}
\begin{array}{cccc}
\Pi & \cdots & \cdots & \Pi \\
\end{array}
$$

$\pi_k$ is the natural projection, $\mathcal{F}$ and $\Pi$ - spaces of germs of connections, and $(1,2)$-tensors at 0.
The space of orbits

\[ \mathcal{M} = \mathcal{F}/\text{Diff}(\mathbb{R}^n, 0) \]

is called the \textit{moduli space} of symmetric connections at 0, and

\[ \mathcal{M}_k = \mathcal{F}_k/\text{Diff}(\mathbb{R}^n, 0) \]

- the moduli space of connection \( k \)-jets.

The action is algebraic, a subspace

\[ \mathcal{F}^0_k \subset \mathcal{F}_k \]

of points on generic orbits (those of largest dimension) is a smooth manifold, open and dense in \( \mathcal{F}_k \).

\( \mathcal{M}^0_k \) is the corresponding moduli space.

Define:

\[ \dim \mathcal{M}_k := \dim \mathcal{M}^0_k . \]
To calculate Poincaré series, we need:
\[ \dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k \]
\[ \dim \mathcal{O}_k = \text{codim } G_{\Gamma} \]

**Stabilizer of a generic k-jet** \( G_{\Gamma} \)

\( V \) stabilizes \( \Gamma \) iff:
\[ \mathcal{L}_V(j^k \Gamma) = 0 \]

In local coordinates, consider grading in homogeneous components:
\[ V = V_1 + V_2 + \ldots \]

( \( V_0 = 0 \), since \( V \) preserves the origin),

\[ \Gamma = \Gamma_0 + \Gamma_1 + \ldots \]
Then:

\[ \mathcal{L}_V(j^k \Gamma) = j^k \mathcal{L}_V(\Gamma) \]
\[ = j^k \mathcal{L}_{V_1+V_2+\ldots}(\Gamma_0 + \Gamma_1 + \ldots + \Gamma_k + \ldots) \]

We arrive at the stabilizer system (*):

\[
\begin{align*}
\mathcal{L}_{V_1} \Gamma_0 + \frac{\partial^2 V_2}{\partial x^2} &= 0 \\
\mathcal{L}_{V_1} \Gamma_1 + \tilde{\mathcal{L}}_{V_2} \Gamma_0 + \frac{\partial^2 V_3}{\partial x^2} &= 0 \\
& \vdots \\
\tilde{\mathcal{L}}_{V_{k+1}} \Gamma_0 + \tilde{\mathcal{L}}_{V_k} \Gamma_1 + \ldots + \mathcal{L}_{V_1} \Gamma_k + \frac{\partial^2 V_{k+2}}{\partial x^2} &= 0
\end{align*}
\]

Need to find all \((V_1, V_2, \ldots, V_{k+2})\) solving this for a generic \(\Gamma\). Assuming \(V_1\) - arbitrary, find \(V_2\) using the following PDE lemma.
**PDE Lemma** Given a family \( \{f_{ij}\}_{1 \leq i, j \leq n} \) of smooth functions, solution \( u \) for the system:

\[
\begin{cases}
u_{,kl} = f_{kl} \\
1 \leq k, l \leq n
\end{cases}
\]

exists iff

\[
\begin{cases}f_{ij} = f_{ji} \\
f_{ij,k} = f_{kj,i}
\end{cases}
\]

If \( f_{ij} \) are homogeneous polynomials of degree \( s \geq 0 \), then \( u \) can be uniquely chosen as a polynomial of degree \( s + 2 \).

If we treat highest-order \( V_k \) in each equation of (*) as an unknown, then PDE Lemma imposes compatibility conditions of the sort:

\[
(L\nu_{,\Gamma})^{l}_{ij,p} = (L\nu_{,\Gamma})^{l}_{pj,i}
\]

These are trivial for the 1st equation in (*)

\[\Rightarrow \exists! V_2\]
However, to find $V_3$ from the next equation, we must have:

$$(\mathcal{L}_{V_1} \Gamma_1 + \tilde{\mathcal{L}}_{V_2} \Gamma_0)_{ij,p} = (\mathcal{L}_{V_1} \Gamma_1 + \tilde{\mathcal{L}}_{V_2} \Gamma_0)_{pj,i} \quad (**)$$

These are $\frac{n^3(n-1)}{2}$ equations on $n^2$ variables.

**Proposition** For $n \geq 3$, $(**)$ $\Rightarrow V = 0$.

Idea of proof. Consider $(**)$ as a homogeneous linear system on components of $V_1$, and show it’s non-degenerate in general position. We construct a specific connection $\Gamma$, and a suitable minor of the system, that is non-degenerate. Since it is an open condition, it would be generically true.

This allows to write the Poincaré series.
**Special coordinates**

In affine normal coordinates, the stabilizer system (*) is considerably simplified:

\[
\begin{cases}
L_{V_1} \Gamma_0 = 0 \\
L_{V_1} \Gamma_1 = 0 \\
& \vdots \\
L_{V_1} \Gamma_{k-1} = 0.
\end{cases}
\]

As a tradeoff, there are extra symmetry conditions on Christoffel symbols $\Gamma$. Nonetheless, the proof is much simpler.

The idea was suggested by A.Vlassov.

For Fedosov structure $\Phi$, the special coordinates are Darboux coordinates of $\omega$.

In constructing $p_\Phi(t)$, I relied on results in

"Fedosov Manifolds"
Gelfand, Retakh and Shubin, 1997.
Interpreting results

**Fact** If the coefficient $a(k)$ of the series
\[
\sum_{k=0}^{\infty} a(k) t^k \quad (***)
\]
is polynomial in $k$, then the series is a rational function.

Indeed, denote
\[
\varphi_m(t) = \sum_{k=0}^{\infty} k^m t^k, \quad m \in \mathbb{Z}_+,\]
then
\[
\varphi_m(t) = \left( t \frac{d}{dt} \right) \varphi_{m-1}(t) \quad \text{for } m \in \mathbb{N}.
\]
Thus
\[
\varphi_m(t) = \left( t \frac{d}{dt} \right)^m \varphi_0(t) = \left( t \frac{d}{dt} \right)^m \left( \frac{1}{1-t} \right).
\]

The recipe for converting (***) is:
“Switch $k \rightarrow t \frac{d}{dt}$, and drop the $\Sigma$!”
Poincaré series - explicit formulas

\[ p_\Gamma(t) = (\delta_1^n - n^2)t + (t - t^2)\delta_2^n \]
\[ + n \left( \frac{\binom{n+1}{2} - 1}{(1 - t)^n} - \frac{2}{(1 - t)^{n-1}} - \cdots - \frac{n}{(1 - t)} \right) \]

\[ p_\Phi(t) = -n(2n + 1)t^2 + \delta_{2n}^2(t^2 - t^3) \]
\[ + \frac{\binom{2n+2}{3} - 1}{(1 - t)^{2n}} - \frac{\binom{2n+2}{3} + 2}{(1 - t)^{2n-1}} - \frac{3}{(1 - t)^{2n-2}} \]
\[ \cdots - \frac{k + 1}{(1 - t)^{2n-k}} - \cdots - \frac{2n}{(1 - t)} + n(2n + 1) \]

Thus the Poincaré series are not just any rational functions, but of the form required by the Tresse finiteness claim (poles exclusively at \( t = 1 \)).