On invariants of Fedosov manifolds

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We consider moduli spaces of two structures:

• Fedosov structure $\Phi = (M, \omega, \nabla)$, $\nabla\omega = 0$

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and the "baby" example:

• Symmetric connection ∇ (or Γ)

Zapiski Seminarov POMI, 2002
`math.DG/0112291`

There is **deformation quantization** on Φ

(B.Fedosov - canonical construction, and
M. De Wilde - P. B. A. Lecompte, 1980's)

Subject of **Functional Moduli** has its **origins** in local classification problems of classical analysis.

Typical outcome of a classification is a **Normal Form** depending on **m** arbitrary functions of **n** variables, so called

functional moduli.

Q.: Are **m** and **n** - *intrinsic* to the problem (not dependent on many choices made while solving it, e.g. of specific local coordinates)?

Universal approach is given by *Poincaré series*.

Classifying geometric structures at a point amounts to the following:

Consider the space $\mathcal{F}$ of germs of the structure, and take a quotient under the action of smooth coordinate changes (preserving the point).

The same action on a space of k-jets, $\mathcal{F}_k$ is more tractable. In particular, the moduli space $\mathcal{M}_k$ is finite-dimensional.

Poincaré series encodes all these dimensions, "*moduli numbers*", in a single series.

Action

$G := \text{Diff}(\mathbb{R}^n, 0)$: $\mathcal{F} \circlearrowleft$ $\mathcal{F}_k \circlearrowleft$

germs k-jets

Moduli spaces

$$\mathcal{M} = \mathcal{F}/G \qquad \mathcal{M}_k = \mathcal{F}_k/G$$

$$\dim \mathcal{M} = \infty \qquad \dim \mathcal{M}_k < \infty$$

Definition Formal power series

$$p_S(t) = \dim \mathcal{M}_0 + \sum_{k=1}^{\infty} (\dim \mathcal{M}_k - \dim \mathcal{M}_{k-1}) t^k$$

is called the *Poincaré series* of S.

Remark If S does depend on **m** arbitrary functions of **n** variables, then:

$$p_S(t) = \frac{\mathbf{m}}{(1-t)^{\mathbf{n}}} \ ,$$

a *rational* function.

Indeed, dimension of moduli spaces of k-jets is the number of monomials up to the order k in the formal power series of the m given invariants:

$$\dim \mathcal{M}_k = \mathbf{m} \binom{\mathbf{n}+k}{\mathbf{n}} \ .$$

Problem(Arnold, 1999)
Is it true that the Poincaré series in most of the local problems of analysis are *rational* functions?

In **commutative algebra** in a similar setting there is the Hilbert basis theorem (finiteness of the bases of the ideals of analytic function germs).

However this problem belongs to **differential algebra**.

A **finiteness** theorem in differential algebra (as formulated by Tresse, 1894):

For any "*natural*" differential-geometric structure, all the differential invariants are generated by a finite number of invariant functions and of invariant vector fields (as differential polynomials).

However, the proof is not rigorous, conditions are not precise.

Contributions

A.Einstein: Rigidity of scalar wave equation
Maxwell equation in vacuum
Gravity field equation

A.S. Shmelev (1990's): Riemannian structure
Kähler structure
hyper-Kähler structure

A. Vershik - V. Gershkovich (1988):
distributions in $\mathbb{R}^n$

Theorem(S.D. 2004)
For the symmetric connection and the Fedosov structure, the Poincaré series coefficients are polynomial in k, and the series are:

$$p_\Gamma(t) = (\delta_1^n + \delta_2^n - n^2)t - \delta_2^n t^2$$

$$+n \sum_{k=1}^{\infty} [\frac{n(n+1)}{2}\binom{n+k-1}{n-1} - n\binom{n+k+1}{n-1}]t^k$$

$$p_\Phi(t) = -\binom{2n+1}{2}t^2 + \delta_{2n}^2(t^2 - t^3)$$

$$+ t \sum_{k=1}^{\infty} [\binom{2n+2}{3}\binom{2n+k-1}{2n-1} - \binom{2n+k+2}{2n-1}]t^k \ .$$

Both are **rational** in t, confirming the *finiteness* assertion of Tresse.

Moduli space of symmetric connections

Goal: $p_\Gamma(t)$, main problem:

describing a stabilizer of generic k-jet.

Connections ∇ and $\tilde{\nabla}$
have the same k-jet at 0 if

$$\forall X, Y \in \Gamma(T\mathbb{R}^n),\ \forall f \in C^\infty(\mathbb{R}^n)\ ,$$

the functions $\nabla_X Y(f)$ and $\tilde{\nabla}_X Y(f)$
have the same k-jet at 0.

$j^k(\Gamma)$ will denote the k-jet of Γ.

Action of $G := \mathrm{Diff}(\mathbb{R}^n, 0)$ on $\mathcal{F}$ and $\mathcal{F}_k$

$$\varphi : \Gamma \mapsto \varphi^*\Gamma\,, \qquad j^k\Gamma \mapsto j^k(\varphi^*\Gamma)\,,$$

where

$$(\varphi^*\nabla)_X Y = \varphi_*^{-1}(\nabla_{\varphi_* X}\varphi_* Y)$$

Consider a filtration of G by normal subgroups:

$$G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \dots ,$$

where

$$G_k = \{ \varphi \in G \mid \varphi(x) = x + O(|x|^k) \} \ .$$

G_k acts trivially on $\mathcal{F}_p$ for $k \geq p + 3$,

e.g.

$$G|_{\mathcal{F}_p} = G/G_{p+3}\Big|_{\mathcal{F}_p}$$

Consider instead infinitesimal action of

$$\mathrm{Vect}_0(\mathbb{R}^n) = Lie(G)$$

(To calculate dimensions, go into tangent space.)

Definition For $V \in \mathrm{Vect}_0(\mathbb{R}^n)$ generating a local 1-parameter subgroup g^t of $\mathrm{Diff}(\mathbb{R}^n, 0)$, *the Lie derivative* of a connection ∇ in the direction V is a (1,2)-tensor:

$$\mathcal{L}_V \nabla = \frac{d}{dt}\Big|_{t=0} g^{t*} \nabla$$

Lemma

$$(\mathcal{L}_V \nabla)(X, Y) = [V, \nabla_X Y] - \nabla_{[V,X]} Y - \nabla_X [V, Y]$$

This defines the action on germs, on jets:

$$\mathcal{L}_V(j^k \Gamma) = j^k(\mathcal{L}_V \Gamma) \ .$$

Why is it well-defined
(Right Hand Side depends on $(j^k\Gamma)$ only)?

In coordinates, $(\mathcal{L}_V \Gamma)^l_{ij} =$

$$V^k \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \Gamma^k_{ij} \frac{\partial V^l}{\partial x^k} + \Gamma^l_{kj} \frac{\partial V^k}{\partial x^i} + \Gamma^l_{ik} \frac{\partial V^k}{\partial x^j} + \frac{\partial^2 V^l}{\partial x^i \partial x^j}$$

$$= (\tilde{\mathcal{L}}_V \Gamma) + \frac{\partial^2 V}{\partial x^2}$$

Elements of k-th order and less are only contributed by $j^k\Gamma$, because $V(0) = 0$.

Action is well-defined means the following is a commutative diagram:

$$\begin{array}{ccccccccccc} j^0\mathcal{F} & \leftarrow & .. & \leftarrow & j^{k-1}\mathcal{F} & \stackrel{\pi_k}{\leftarrow} & j^k\mathcal{F} & \leftarrow & .. & \leftarrow & \mathcal{F} \\ \downarrow \mathcal{L}_V & & & & \downarrow \mathcal{L}_V & & \downarrow \mathcal{L}_V & & & & \downarrow \mathcal{L}_V \\ j^0\Pi & \leftarrow & .. & \leftarrow & j^{k-1}\Pi & \stackrel{\pi_k}{\leftarrow} & j^k\Pi & \leftarrow & .. & \leftarrow & \Pi \end{array}$$

π_k is the natural projection,
$\mathcal{F}$ and Π - spaces of germs of connections, and (1,2)-tensors at 0.

The space of orbits

$$\mathcal{M} = \mathcal{F}/\mathsf{Diff}(\mathbb{R}^n, 0)$$

is called the *moduli space* of symmetric connections at 0, and

$$\mathcal{M}_k = \mathcal{F}_k/\mathsf{Diff}(\mathbb{R}^n, 0)$$

- the moduli space of connection k-jets.

The action is algebraic, a subspace

$$\mathcal{F}_k^0 \subset \mathcal{F}_k$$

of points on generic orbits
(those of largest dimension)
is a smooth manifold, open and dense in $\mathcal{F}_k$.

$\mathcal{M}_k^0$ is the corresponding moduli space.

Define:

$$\dim \mathcal{M}_k := \dim \mathcal{M}_k^0 \ .$$

To calculate Poincaré series, we need:

$$\dim \mathcal{M}_k = \dim \mathcal{F}_k - \dim \mathcal{O}_k$$

$$\dim \mathcal{O}_k = \operatorname{codim} G_\Gamma$$

Stabilizer of a generic k-jet G_Γ

V stabilizes Γ iff:

$$\mathcal{L}_V(j^k\Gamma) = 0$$

In local coordinates, consider grading
in homogeneous components:

$$V = V_1 + V_2 + \dots$$

($V_0 = 0$, since V preserves the origin),

$$\Gamma = \Gamma_0 + \Gamma_1 + \dots$$

Then:

$$\mathcal{L}_V(j^k\Gamma) = j^k\mathcal{L}_V(\Gamma)$$

$$= j^k\mathcal{L}_{V_1+V_2+\ldots}(\Gamma_0 + \Gamma_1 + \ldots + \Gamma_k + \ldots)$$

We arrive at the stabilizer system (*):

$$\begin{cases} \mathcal{L}_{V_1}\Gamma_0 + \dfrac{\partial^2 V_2}{\partial x^2} = 0 \\ \mathcal{L}_{V_1}\Gamma_1 + \tilde{\mathcal{L}}_{V_2}\Gamma_0 + \dfrac{\partial^2 V_3}{\partial x^2} = 0 \\ \qquad\quad \vdots \\ \tilde{\mathcal{L}}_{V_{k+1}}\Gamma_0 + \tilde{\mathcal{L}}_{V_k}\Gamma_1 + \ldots + \mathcal{L}_{V_1}\Gamma_k + \dfrac{\partial^2 V_{k+2}}{\partial x^2} = 0 \end{cases}$$

Need to find all $(V_1, V_2, \ldots, V_{k+2})$ solving this for a generic Γ. Assuming V_1 - arbitrary, find V_2 using the following PDE lemma.

PDE Lemma Given a family $\{f_{ij}\}_{1\le i,j\le n}$ of smooth functions, solution u for the system:

$$\begin{cases} u_{,kl} = f_{kl} \\ 1 \le k,l \le n \end{cases}$$

exists iff

$$\begin{cases} f_{ij} = f_{ji} \\ f_{ij,k} = f_{kj,i} \end{cases}$$

If f_{ij} are homogeneous polynomials of degree $s \ge 0$, then u can be uniquely chosen as a polynomial of degree $s+2$.

If we treat highest-order V_k in each equation of (*) as an unknown, then PDE Lemma imposes compatibility conditions of the sort:

$$(\mathcal{L}_V \Gamma)^l_{ij,p} = (\mathcal{L}_V \Gamma)^l_{pj,i}$$

These are trivial for the 1st equation in (*) $\Rightarrow \exists! V_2$

However, to find V_3 from the next equation, we must have:

$$(\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{ij,p} = (\mathcal{L}_{V_1}\Gamma_1 + \widetilde{\mathcal{L}}_{V_2}\Gamma_0)_{pj,i} \qquad (**)$$

These are $\dfrac{n^3(n-1)}{2}$ equations on n^2 variables.

Proposition For $n \geq 3$, $(**) \Rightarrow V = 0$.

Idea of **proof**.
Consider (**) as a homogeneous linear system on components of V_1, and show it's non-degenerate in general position. We construct a specific connection Γ, and a suitable minor of the system, that is non-degenerate. Since it is an open condition, it would be generically true.

This allows to write the Poincaré series.

Special coordinates

In affine normal coordinates, the stabilizer system (*) is considerably simplified:

$$\begin{cases} \mathcal{L}_{V_1}\Gamma_0 = 0 \\ \mathcal{L}_{V_1}\Gamma_1 = 0 \\ \qquad \vdots \\ \mathcal{L}_{V_1}\Gamma_{k-1} = 0\ . \end{cases}$$

As a tradeoff, there are extra symmetry conditions on Christoffel symbols Γ. Nonetheless, the proof is much simpler.

The idea was suggested by A.Vlassov.

For Fedosov structure Φ, the special coordinates are Darboux coordinates of ω.

In constructing $p_\Phi(t)$, I relied on results in

"Fedosov Manifolds"
Gelfand, Retakh and Shubin, 1997.

Interpreting results

Fact If the coefficient $a(k)$ of the series

$$\sum_{k=0}^{\infty} a(k)t^k \qquad (***)$$

is polynomial in k, then the series is a rational function.

Indeed, denote

$$\varphi_m(t) = \sum_{k=0}^{\infty} k^m t^k \ , \qquad m \in \mathbb{Z}_+ \ ,$$

then

$$\varphi_m(t) = \left(t\frac{d}{dt}\right)\varphi_{m-1}(t) \quad \text{for } m \in \mathbb{N} \, .$$

Thus

$$\varphi_m(t) = \left(t\frac{d}{dt}\right)^m \varphi_0(t) = \left(t\frac{d}{dt}\right)^m \left(\frac{1}{1-t}\right) \, .$$

The recipe for converting (***) is:
"Switch $k \to t\frac{d}{dt}$, and drop the Σ !"

Poincaré series - explicit formulas

$$p_\Gamma(t) = (\delta_1^n - n^2)t + (t - t^2)\delta_2^n$$

$$+n\left(\frac{\binom{n+1}{2} - 1}{(1-t)^n} - \frac{2}{(1-t)^{n-1}} - \dots \frac{n}{(1-t)}\right)$$

$$p_\Phi(t) = -n(2n+1)t^2 + \delta_{2n}^2(t^2 - t^3)$$

$$+\frac{\binom{2n+2}{3} - 1}{(1-t)^{2n}} - \frac{\binom{2n+2}{3} + 2}{(1-t)^{2n-1}} - \frac{3}{(1-t)^{2n-2}}$$

$$\dots - \frac{k+1}{(1-t)^{2n-k}} - \dots - \frac{2n}{(1-t)} + n(2n+1)$$

Thus the Poincaré series are not just any rational functions, but of the form required by the Tresse finiteness claim (poles exclusively at $t = 1$).